A solution is demonstrated to an infinite-horizon, discrete-time utility model describing the consumption and cutting behavior of a nonindustrial private landowner who is managing a multiple age-class forest and who values both consumption derived from harvesting the trees and amenity derived from the standing trees. A policy rule is derived to attain a normal forest from any initial age-class distribution. It is demonstrated that a noncyclical forest allowing a constant periodic harvest is typically not a normal forest. Therefore, an even-flow timber harvesting is not tied to the existence of a normal forest structure.

Key words: age-class models, amenities, dynamic analysis, normal forest, timber supply.

Forest economics literature has a long tradition of looking at the optimal management of one single stand. This tradition goes back to Faustmann’s work in the nineteenth century (Faustmann). However, a characteristic feature in forest management is the existence of and possible interdependencies between multiple age-classes, stands. These interdependencies become especially difficult to ignore when one is dealing with multiple-use forest management problems. Forest economics literature recognizing the feature of multiple stands has two main strains. One has been interested in landowner timber and non-timber decisions in the presence of spatial interactions of two or more adjacent stands (Swallow and Wear; Swallow, Talukdar, and Wear; Koskela and Ollikainen). The other has incorporated stand or age-class dynamics into models describing landowner or timber market behavior. This latter literature has modeled optimal behavior either based on timber benefits (Berk 1976, Berk 1979, Johansson and Löfgren, Lyon and Sedjo, Sedjo and Lyon, Mitra and Wan 1985, Mitra and Wan 1986, Salo and Tahvonen 2002, 2003, Sohngen, Mendelsohn, and Sedjo) or both timber and nontimber benefits (Berk and Bible, Bowes and Krutilla 1985, Bowes and Krutilla 1989, Tahvonen).

The line of models that recognize both timber and amenity values of forests goes back to Hartman. However, the Hartman model is a single-stand model and thus ignores the interfaces of multiple age-classes. In their work, Bowes and Krutilla (1985, 1989) made the amenity values of a forest dependent on the mix of stand ages. This essentially makes the optimal decisions concerning one single stand dependent on the entire forest structure. Bowes and Krutilla (1989) illustrated how this feature of the model allows a considerably richer description of the optimal harvesting policies as compared to a single stand case. However, as noted by Bowes and Krutilla (1989) their analytical and numerical analysis was to some extent limited by the complexity of the problem. In this paper, by combining the multiple-use and multiple-stand forestry problem, rules are derived to characterize the optimal harvesting scheme. This scheme generalizes the harvesting rules of the Hartman model. We focus on the case of an individual landowner who owns both forest and other assets and we show that steady-state forests exist and demonstrate the optimal harvesting schemes toward the steady state.

In the resource economics literature, there has been a steady interest in the multiple age-class forestry model since the 1970s. Johansson and Löfgren presented a primal-dual characterization of a linear-forest model based on...

A multiple-stand forest structure has also been introduced to the two-period harvest-scheduling model (Uusivuori and Kuuluvainen). However, the two-period model typically ignores the land value inherent in infinite-horizon Faustmann model and its extensions. In this paper, we show that a multiple-period generalization of the two-period age-class model will make the model equivalent with the Faustmann model and its extensions.

A normal forest structure means that forestland is evenly distributed over age-classes. When normal forest structure is maintained in perpetuity, it provides a constant flow of harvest over time, a feature that is often considered ideal. Mitra and Wan (1985) proved that if the discount rate is zero and periodic utility is concave, a normal forest structure with maximum sustainable yield (MSY) rotation is the optimal long-run steady state given any initial age-class structure. This steady state is characterized by periodic harvesting in which the oldest age-class fulfilling the maturity criterion is entirely harvested. Salo and Tahvonen (2002, 2003) proved more generally that a steady state may also occur within an age-class structure that is not a normal forest. In this type of steady state, it is optimal to maintain a forest structure in which the stands are unevenly allocated over the age-classes giving rise to a cyclical harvesting scheme where periodic harvest levels reoccur in cycles.

Salo and Tahvonen (2003) showed that age-class cycles disappear when the period length approaches zero, when optimal rotation is not unique, or when the discount rate approaches zero. They concluded that the argument concerning normal forest convergence may depend on whether the use of forest resources is characterized by some seasonality that would give rise to a natural period length. Salo and Tahvonen (2002) and Tahvonen showed that cycles vanish with zero discounting but that this is not typically associated with normal forest structure. In this paper, we show that with positive discounting age-class cycles may disappear without a normal forest convergence when age-classes are allowed to be partially harvested. This means that an even-flow timber harvesting is not tied to the existence of a normal forest structure.

Previous age-class forest models have used "price supports" with shadow price vectors (Mitra and Wan 1985), dual formulation of a linear forest model (Johansson and Löfgren), optimal control formulation with Hamiltonian functions (Lyon and Sedjo, Sedjo and Lyon), and Lagrangian functions (Salo and Tahvonen 2003, Tahvonen). Viitala uses this approach to nonindustrial private forest owner’s case in a similar setting to what is presented in this study. All these formulations typically lead to a characterization of the optimal conditions with time-dependent shadow prices attached to the age-classes. Instead, in this paper we solve the model by direct substitution of the laws of motion and the decision variables into the objective function. We introduce a specific decision-variable matrix, which we expect to be a useful description of the age-class dynamics in other, especially, more policy-oriented contexts as well. For example, the way dynamics is specified in the current model allows us to derive a straightforward rule to attain a normal forest structure from any initial age-class distribution. This policy rule should be a useful contribution, considering the central place that the concept of normal forest has had in the forest economics literature now for nearly two centuries.

In the present model, optimal cutting rates depend on the initial age-class distribution of the forest. Land value is determined by the optimal progression of the harvests, while the optimal harvests depend on the land value causing inseparability of consumption and harvesting decisions. Initially, the optimal harvest scheme evolves in time. However, there exists a long-run equilibrium for the harvesting scheme, toward which numeric computations always converge. This equilibrium may be cyclical or noncyclical. If the equilibrium is noncyclical, the long-run age-class distribution as well as the periodic timber harvesting levels are constant. In the noncyclical solution, the age-class distribution will not typically be even. The noncyclical solution may imply old-growth preservation combined with timber harvesting. If the long-run equilibrium is cyclical, the long-run age-class distribution and the periodic timber harvesting levels occur in stable cycles.
The model and the optimal solution are presented in the next section. The section also contains a proof for the existence of an optimal long-run cyclical or noncyclical steady state forest. After this optimal convergence paths toward equilibria along with some comparative statics results are demonstrated with numerical examples. The article concludes by examining implications of the present study for future theoretical and empirical work.

The Model of a Utility-Maximizing Landowner

The Model

The landowner manages a forest consisting of stands representing \( n \) age classes. The initial age class distribution is given by the \( n \times 1 \) vector \( x_0 = (x_{01}, x_{02}, \ldots, x_{0n}) \), where the elements give the initial land areas in hectares under each of the age classes, \( 1 \) indexing the youngest and \( n \) the oldest one. The age class indexes refer to the periodic ages of the stands for age classes \( 1, \ldots, n - 1 \), while for the age class \( n \) the index refers to the minimum age of forest in this class, so that stands in the oldest age class are at least \( n \) periods old.\(^1\) The aggregate growth of the forest is described by the \( 1 \times n \) vector \( q = (q_1, q_2, \ldots, q_n) \), consisting of the per hectare timber volumes in each of the \( n \) age-classes such that \( q_i > q_{i-1} \) for \( i = 2, \ldots, n \).\(^2\)

By choosing at the beginning of time period \( t \) the shares to be harvested of each of the \( n \) age classes, \( a_{it} \), the landowner maximizes utility from periodic consumption \( c_t \) and from the amenity values of the standing forest volume \( Q_t \) over an infinite time horizon according to an additively separable utility function:

\[
V(w_0, x_0) = \max_{(a_{it}, w_{it+1})_{t=0}^\infty} \sum_{t=0}^\infty \beta^t [u(c_t) + M(Q_t)]
\]

\[
c_0 = \sum_{i=1}^n a_{0i} x_{0i} (pq_i - k) + w_0 - w_1
\]

\[
c_t = \sum_{i=1}^n a_{it} x_{ti} (pq_i - k) + w_t (1+r) - w_{t+1} \quad t \geq 1
\]

where \( u() \) and \( M() \) are the strictly concave periodic consumption and amenity subutility functions, respectively. For each age-class and time period \( t \) a decision parameter \( a_{it} \) is attached with respect to which the objective function is optimized. The decision parameter gives the share (between 0 and 1) of the land in each age-class that will be harvested. For example, if \( a_{it} = 1 \), the entire age-class is cut down, and when \( a_{it} = 0 \), the entire age-class is left standing. Further in (1), \( p \) is the unit timber price, assumed constant over time and over the stands, \( k \) is the replanting cost per hectare, \( w_0 \) is initial nonforest wealth, and \( r \) is the market interest rate. The periodic harvesting shares are collected in the matrix \( A_t \) which defines the age-class allocation of the land as described in the age-class dynamics section.

The constraints \( a_{it} \geq 0 \) imply “the cross vintage bound,” that is, the size of the land area of age class \( i + 1 \) in period \( t + 1 \) cannot exceed the land area of age class \( i \) in period \( t \) (Mitra and Wan 1985). The constraints \( 1 - a_{it} \geq 0 \) guarantee, with the initial non-negativity condition of the age classes, that the land areas of the age classes cannot take negative values in the consecutive periods.

The initial financial assets \( w_0 \) are adjusted to the level of \( w_1 \) in the beginning of the first period. The discount factor is given by \( \beta = \frac{1}{1+r} \), where \( r \) is the subjective time preference rate, assumed to be non-negative.

In Bowes and Krutilla (1985, 1989) the amenity value of the forest is a function of the acreage of land in the different age-classes. In the present model, the amenity function \( M(Q_t) \) is more specific. The amenity value of a single stand depends on the acreage through the biomass volume contained in the stand and the amenity value of the total forest depends on the age-class distribution of the stands.

Age-Class Dynamics

To describe the stand dynamics, we specify a transition matrix \( A_t \). This matrix collects the decision variables, the \( a \)‘s, introduced above. The age-class dynamics is given by

\[
x_{t+1} = A_t x_t
\]

\(^1\) Thus the model assumes a given number of age-classes where the oldest age-class is exogenously determined, and the youngest age-class is one period old after seeding.

\(^2\) Note that when a stand reaches the oldest age-class it also reaches the highest per ha standing timber volume. Thus the oldest stand represents the climax age-class.
where

$$A_t = \begin{bmatrix} a_{t1} & a_{t2} & \cdots & \cdots & a_{tn} \\ 1 - a_{t1} & 0 & \cdots & \cdots & 0 \\ 0 & 1 - a_{t2} & 0 & \cdots & \cdots \\ \vdots & \vdots & \ddots & 0 & \cdots \\ 0 & \cdots & \cdots & 1 - a_{tn-1} & 1 - a_{tn} \end{bmatrix}$$

and $x_t = (x_{t1}, x_{t2}, \ldots, x_{tn})'$.

The $A_t$ matrix above determines the periodic relations between the different age classes. The first row in $A_t$ gives the periodic harvest shares of each of the $n$ age classes. The following rows give the uncut shares of subsequent age classes with the last row collecting the shares left uncut in period $t$ of the oldest and the second oldest age classes. The above formulation of the stand dynamics is potentially a useful tool in studying the effects of various public policy measures, such as those related to forest conservation and carbon sequestration policies. In the present context, we will not pursue these issues further, but limit ourselves to prove the following proposition to demonstrate the usefulness of the $A_t$ matrix in policy contexts.

**PROPOSITION 1.** A constant harvesting policy where at least one age-class is harvested partially ($0 < a_i < 1$ for at least one $i$) leads to an equilibrium noncyclical age-class distribution. The equilibrium distribution is independent of the initial distribution. In particular, by following a policy such that $a_n + a_{n-1} = 1$ and $a_i = 0$ ($i = 1, \ldots, n - 2$) the equilibrium forest will be normal where forestland is evenly allocated among the age-classes.

The proof is given in Appendix A. According to the proposition a steady-state noncyclical age class distribution is connected to a constant, unchanging harvesting policy where at least one stand is always cut in part, that is, for which it holds that $0 < a_i < 1$. The proposition implies that given any initial forest structure a policymaker can reach a target-forest by setting the harvesting parameters appropriately. The equilibrium is typically a forest where the stands are unevenly distributed between the age classes. However, a simple management rule dictating that setting the harvesting shares of the two oldest age-classes so that their sum equals one while always abstaining from cutting any of the younger stands will lead to a normal forest structure. The simplicity of the rule is surprising considering its strong implication and in view of how central the concept of a normal forest in forest management practice and literature has been now for almost two centuries.

**Optimal Consumption and Cutting**

Starting from the first period, a decision is made on the $a$'s for each age-class based on the marginal effects of allocating land between two consecutive age-classes representing two consecutive and infinitely repeated rotation lengths. For example, in period 1, the $a$ for the first class is determined based on utility flows generated by one period rotation versus utility flows generated by a two-period rotation. The $a$ for the second class is determined based on utility flows generated by a two period rotation versus utility flows generated by a three period rotation. Consecutively, all the $a$ parameters are determined for the first period. This is followed by determining the $a$'s in a similar way for the second and all future periods. In this way the periodic solutions are interdependent in the optimal plan.

First-order conditions (Appendix B) imply the following:

$$\frac{u'(c_i)}{u'(c_{i+1})} = \frac{1 + r}{1 + \rho}$$  \hspace{1cm} (3)

$$\left[ (pq_i - k)(1 + r) \frac{(1 + r)^j}{(1 + r)^j - 1} - (pq_{i+1} - k) \frac{(1 + r)^{j+1}}{(1 + r)^{j+1} - 1} \right] + (1 + r) \sum_{h=0}^{\infty} \left( \frac{1}{1 + r} \right)^h i + j \sum_{j=1}^{i} \left( \frac{1}{1 + r} \right)^j \frac{M'(Q_{i+h+i+1})}{u'(c_{i+h+i+1})} q_j - \sum_{h=0}^{\infty} \left( \frac{1}{1 + r} \right)^{h(i+1)} \frac{M'(Q_{i+h(i+1)+1})}{u'(c_{i+h(i+1)+1})} q_{i+1}$$  \hspace{1cm} (4)

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3 Thus the inner product between the first line in $A_t$ and $x_t$ gives the land area in the first age class in period $t+1$. The inner products between the following lines in $A_t$ and $x_t$ give the land areas in $t+1$ in the remaining age classes.
Expression (3) gives the standard rule for optimal consumption path. Consumption is constant in case \( p \) equals \( r \) and nonconstant in case \( p \) differs from \( r \). Expression (4) contains the optimal cutting policy and generalizes the Hartman rule of optimal harvesting in the one stand case (Appendix C). The first two rows in (4) collect the marginal consumption (monetary) benefits, \((pq_i - k)(1 + r)\frac{1 + r}{1 + r - r}\), of following rotation \( i \) infinitely and the corresponding marginal costs, \((pq_{i+1} - k)\frac{1 + r}{1 + r - r+1}\). The next two rows in (4) represent the marginal amenity (nonmonetary) benefits of following rotation \( i \) infinitely, while the last two rows in (4) give the corresponding marginal costs. The summation terms running from 1 to \( i \) (by index \( j \)) collect the marginal amenity benefits and costs generated over the entire rotation period of a stand and they are the counterparts of the integral term in Hartman model. The summation terms running from 0 to infinity (by index \( h \)) capture the infinite amenity flows.

According to (4), the stand is left uncut \((a_i = 0)\) when the total marginal benefits of cutting the stand are smaller than the total marginal benefits of leaving the stand uncut. In the reverse case the stand is cut \((a_i = 1)\). Importantly, the rule may also imply a solution in which the stand is only harvested partially \((0 < a_i < 1)\).

In this case, part of land is allocated to forestry while part is allocated to forestry following rotation \( i + 1 \). As a comparison, in a standard utility maximization problem without the amenity valuation the optimal solution occurs at point where stands representing the Faustmann rotation age or older age classes are harvested entirely.

In (4), the \( M'(Q_i) \) terms measure the relative value of marginal amenity and consumption utilities, and can be referred to as the marginal trade-off between \textit{in situ} and consumption utilities (Kuuluvainen and Tahvonen) or the marginal rate of substitution between amenity and consumption utilities (Amacher, Koskela, and Ollikainen). The marginal amenity benefits following rotation \( i \) (second row in [4]) are smaller than those following rotation \( i + 1 \) (third row in [4]) implying that an optimal cutting policy tends to consist of a larger timber stock the more amenities are valued relative to the consumption of goods and services. In fact, in case no utility is derived from amenities the amenity terms in (4) drop out and it can be shown that the optimal condition is consistent with the Faustmann rotation rule (Appendix C).

It is obvious that the optimal harvesting policy is typically not constant over time in case the subjective time preference rate and the interest rate differ, that is, \( p \) differs from \( r \).

In this case, the consumption path and the \( u'(c_i) \) terms in (4) evolve over time and the landowner adjusts his harvesting policy accordingly until a possible constant or cyclical steady state has been reached. It is less obvious that the optimal harvesting policy will typically not be constant initially even in the case when \( p \) equals \( r \) and consumption is constant. The reason for this is that the optimal cutting policy depends on the underlying land distribution between the age classes. Unless the initial age class distribution is exactly that corresponding to the optimal harvesting chosen as the first period solution, the harvesting decisions will change the age-class distribution, and the harvesting policy itself evolves in time.

An optimal harvesting scheme \( \{A_t\}_{t=0}^{\infty} \) maximizes utility in such a way that the following budget constraint is met:

\[
\sum_{t=0}^{\infty} \beta^t c_t = w_0 + \sum_{t=0}^{\infty} \beta^t \times \sum_{i=1}^{n} a_{i} x_{ti}(pq_i - k).
\]

There can only be one such age class if the expression in (4) increases monotonically as a function of the stand age. In this case (4) gives the necessary and sufficient conditions for optimal cutting sequence assuming that the budget constraint (5) holds. Further, in this case, there is only one such sequence so that if we find a sequence converging to a steady state we have found the optimal solution.

In case amenities are not included in the model, the optimal condition is constant over time. This is reflected in the fact that in expression (4) time index \( t \) does not show up in the marginal monetary utility terms.

In the numerical analysis we will demonstrate the case of differing subjective time preference rate and market interest rate, with the resulting consequences for the equilibrium forest.
In a numerical analysis, the first period consumption for any harvesting program can be found using (5) and the consumption path can be calculated using (3). The inseparability of the consumption and harvesting decisions and the utility maximizing consumption and harvesting schemes can numerically be circumvented by using iterative methods.8

**Long-Run Steady State**

In a steady state where consumption and timber volume are fixed at a certain level or cycle ($c^*$ and $Q^*$, respectively) the constant harvesting decisions can be expressed as follows (Appendix B):

$$\begin{cases} 0 > 0 & a_i = 1 \\ 0 < 0 & a_i = 0 \end{cases} \Rightarrow 0 < a_i < 1$$

where

$$\varphi(i) \equiv \left[ (pq_i - k) + \frac{M'(Q^*)}{u'(c^*)} \sum_{j=1}^{i} q_j \frac{q_{j+1} - k}{(1 + r)^{j+1}} \right] \times \left[ \frac{(1 + r)^i}{(1 + r)^i - 1} \right] - \left[ (pq_{i+1} - k) + \frac{M'(Q^*)}{u'(c^*)} \sum_{j=1}^{i} q_{j+1} \frac{q_j}{(1 + r)^{j+1}} \right] \times \left[ \frac{(1 + r)^{i+1}}{(1 + r)^{i+1} - 1} \right].$$

The proof of the following proposition is given in Appendix D.

**PROPOSITION 2.** A steady-state forest exists which is either a cyclical forest where the optimal age-class distribution, harvests, and standing timber volume move in stationary cycles or a noncyclical forest where the optimal age-class distribution, harvests, and standing timber volume are constant and which typically is not a normal forest. If $\varphi(i)$ is strictly increasing in $i$, the steady state is unique.

The proposition implies that a cyclical distribution is connected to a harvesting schedule in which the stands are left standing until they reach a certain age-class and are then entirely cut down. A steady-state noncyclical age-class distribution is connected to a constant harvesting policy where one stand is always cut in part, that is, for which it holds that $0 < a_i < 1$. The noncyclical forest contrasts the case of a cyclical forest in which a land distribution reoccurs in stationary cycles. In the noncyclical forest, the land is typically unevenly allocated among the age-classes. Thus we have shown that with nonzero discounting constant timber-flow and constant standing timber stock do not need to be linked to a normal forest. Earlier literature has linked constant timber-flow forest to a non-normal stand structure in case of zero discounting (Mitra and Wan 1985, Salo and Tahvonen 2002).

Under the condition that $\varphi(i)$ is strictly increasing in $i$ the harvesting schemes following the necessary conditions of optimality and converging to a steady state are optimal. To find convergent harvesting schemes we use numerical analysis. In the numerical analysis, the steady-state forest is derived by first using the optimal harvesting conditions in the steady state. The convergence paths are then solved using an iterative algorithm.

**Numerical Examples**

Figures 1–7 demonstrate stand structure dynamics during twenty periods with logarithmic consumption and amenity utility functions9 and selected parameter values for timber price, planting costs, initial external and forest assets, initial age-class structure, and the growth function. The numerical examples are constructed in such a way that the Faustmann rotation coincides with the second oldest age-class (i.e., three periods in the hypothetical case of four age-classes). This means that given a model specification without the amenity utility the optimal policy would follow a rotation age of three periods leading to cyclical harvesting and age-class pattern. We first assume that $p$ equals $r$.

**The Effect of Initial Age-Class Structure**

Examples 1 and 2 illustrate the impact of the initial age-class distribution on the optimal harvesting scheme. In example 1 (figures 1(a)–(c)), the case is demonstrated where—starting from an initial age-class distribution where land is evenly allocated between the

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8 In a numerical solution of the model the equilibrium value of the forestland can be found by iteration where starting from a given initial forestland value the optimal sequence of the $a$ variables is solved at each time of the iteration and the forestland value is recalculated from the present value of timber harvesting pattern implied by the sequence of the $a$ variables.

9 Logarithmic utilities are safe in this case since both consumption and timber volume take positive values larger than one.
Figure 1. (a) Optimal sequences of standing stock and harvested timber. (b) Optimal sequences of harvested shares of the oldest and second oldest age-class. (c) Optimal land allocation among the age-classes. $U(c) = \ln c$, $M(Q) = \ln Q$. Growth function: $q_1 = 1$, $q_2 = 4$, $q_3 = 16$, $q_4 = 22$. Initial land allocation: 25% in each age-class, $w_0 = 80$, $r = 1$, $\rho = 1$, $p = 1$, $k = 3$. Periodic consumption at equilibrium = 56.9. Period-beginning standing stock at equilibrium = 46.9, harvest at equilibrium = 20.1.

Figure 2. (a) Optimal sequences of standing stock and harvested timber. (b) Optimal sequences of harvested shares of the oldest and second oldest age-class. (c) Optimal land allocation among the age-classes. $U(c) = \ln c$, $M(Q) = \ln Q$. Growth function: $q_1 = 1$, $q_2 = 4$, $q_3 = 16$, $q_4 = 22$. Initial land allocation: 1/3 in age-classes 1–3, 0 in age-class 4, $w_0 = 80$, $r = 1$, $\rho = 1$, $p = 1$, $k = 3$. Periodic consumption at equilibrium = 50.9. Standing stock at equilibrium: cyclical, harvest at equilibrium: cyclical.

The long-run standing timber volumes are larger while the harvested timber volumes are smaller in figure 1(a) as compared to the average volumes in figure 2(a).\textsuperscript{10} The landowner in example 1 saves part of the oldest age-class—this case thus represents a possible combined old-growth preservation and harvesting—and

\textsuperscript{10}Because the maximum sustained yield rotation in the case is 4 periods, the wealthier landowner in figure 1 harvests less on average than the less wealthier landowner in figure 2.
maintains a higher standing timber volume than the landowner in example 2. This is natural because the initial age-class distribution implies that the landowner in example 1 is wealthier than the landowner in example 2. While the nonforest assets are equal for both landowners, the forestland in the first case is more valuable than in the second case as some of it is initially allocated to age-class 4, the most valuable age-class. As the first landowner receives smaller marginal utility from consumption than the second one, he prefers larger standing timber than the second landowner.\footnote{Bowes and Krutilla (1989, p. 107) describe the behavior at the limit: “... when poor timber stands are inherited with older growth already supplying considerable amenity value, harvesting may not be desirable at all.”}

**Comparative Statics**

As demonstrated above, if forest is valued \textit{in situ}, larger wealth tends to lead to higher equilibrium standing timber stocks. This impact is unambiguous since higher initial wealth will always—\textit{ceteris paribus}—imply higher consumption levels, which according to the optimal conditions (4) will lower marginal consumption utility $u'(c)$. This tends to lead to larger preferred standing timber stock. However, by (4) and (6), it is also evident that the effects on equilibrium stock of timber price, replanting costs, or interest rate/subjective time preference rate are ambiguous due to substitution and income effects working into opposite directions. The substitution effect will work through intertemporal optimal trade-off of consecutive age-classes into one direction, while the income effect works through altered consumption levels into the other direction. This is in line with results obtained with two-period models and with rotation models including amenity
Figure 5. (a) Total effects of a timber price change from \( p = 1 \) to \( p = 2.5 \) on the optimal sequences of standing stock and harvested timber. (b) Total effects of a timber price change from \( p = 1 \) to \( p = 2.5 \) on the optimal sequences of harvested shares of the oldest and second oldest age-class. (c) Total effects of a timber price change from \( p = 1 \) to \( p = 2.5 \) on the optimal land allocation among the age-classes.

In figures 3 and 4, the substitution and income effects, and in figure 5 the total effect of a timber price increase are illustrated. The initial age-class distribution is the long-run equilibrium steady-state forest reached in the case illustrated in example 1. First, in figure 3 we see that as the timber price is increased, the substitution effect works toward a smaller standing stock at equilibrium, harvest at equilibrium cyclical.

The algorithm behind the numerical analysis first drives the system into a long-run equilibrium. The substitution effect is obtained by introducing a price change while retaining consumption at its old-equilibrium level. The income effect is obtained by calculating first the land value change implied by a timber price change and introducing this to the system and finding a new equilibrium by iteration.
timber stock. The income effect works to the opposite direction (figure 4). With the income effect working alone the optimal stock would be adjusted upward to a level implied by an age-class distribution where some 80% of the total land is old growth.

The total effect of a timber price increase is depicted in figure 5. The substitution effect would separately have implied a new noncyclical equilibrium in which part of the third (and the entire fourth) age-class is being harvested, while the income effect would separately have implied a new noncyclical equilibrium in which part of the fourth age-class is being harvested. However, as a result of their combined effects the system will reach a new cyclical equilibrium where the entire fourth age-class is being harvested but none of the third age-class. The new equilibrium in figure 5 can be compared to the old equilibrium in figure 1. In the new equilibrium after the price increase, the average of the long-run timber supply will be only moderately higher than in the old equilibrium as the old equilibrium was beyond MSY rotation. In the short-run, however, there is a very large timber supply increase as a result of the timber price increase.

When Consumption Path Is Not Constant

Above we set the landowner’s subjective time preference rate equal to the market interest rate. This made the optimal consumption schedule constant. When the subjective time preference rate is allowed to differ from the interest rate, the optimal consumption path is no longer constant. In case the time preference rate is smaller than the interest rate, the landowner’s marginal consumption utility declines with time (see expression 3), and he chooses an increasing consumption path because the marginal costs of postponing consumption are lower than the market rate of interest. This will be reflected in the optimal harvesting scheduling as well (figure 6).

From the optimal condition governing the periodic harvesting it is evident that in case the marginal consumption utility declines and as the consumption increases in time the landowner’s marginal benefit of harvesting decreases. He reacts by adjusting the total timber volume upwards—although not monotonically because of “jumps” in the age-class distribution—and by “consuming” more of the amenity values of the standing timber. Eventually, the standing timber stock approaches the maximum possible implied by the situation where all of the stands are in the oldest age-class.

In the opposite case when the landowner’s time preference rate is larger than the interest rate, consumption decreases over time because the marginal costs of postponing consumption are higher than the market rate of interest. Marginal consumption utility increases leading

\[ U(c) = \ln c, \quad M(Q) = \ln Q. \]

Growth function: \( q_1 = 1, q_2 = 4, q_3 = 16, q_4 = 22 \). Initial land allocation: 1/3 in age-classes 1–3, 0 in age-class 4, \( u_0 = 80, \ r = 0.75, \ p = 1, k = 3 \). Periodic consumption: decreasing. Standing stock at equilibrium: cyclical, harvest at equilibrium: cyclical.
to increasing marginal benefit of harvesting (figure 7). The landowner reacts by “consuming” less of the amenities of the standing timber, and eventually brings the forest assets to a point where they maximize the present value of consumption utility. This occurs when the Faustmann rotation (in this case three periods) is followed. Because the Faustmann solution is always an all-or-nothing type of solution in terms of the harvested shares of the stands, the long-run equilibrium is a cyclical forest. The above results are in line with those obtained previously with a rotation model (Tahvonen and Salo).

Conclusions

In this article, we have demonstrated a solution to an infinite-horizon, discrete-time utility model describing the behavior of a nonindustrial private landowner who is managing a multiple age-class forest and who values both consumption derived from harvesting the trees and amenity derived from the standing trees. The landowner discounts the time with a positive rate. We showed that there exists a steady-state forest that is either cyclical or noncyclical. The noncyclical forest structure is typically a nonnormal forest. Earlier literature has linked the disappearance of cycles to a normal forest structure in case of positive discounting. The presented model allowed an equilibrium to occur where part of the forest is preserved as old growth while part is being harvested for timber.

The noncyclical long-run forest was connected to optimal harvesting schedule where one of the forest stands is partially cut. In this sense the model generalizes forest rotation models where stands are always either allowed to grow or they are cut down entirely. The model is a generalization of the Hartman single-stand linear-utility harvesting model. The possibility of the partial-harvesting-type solution arises because of the presence of the amenity subutility function in the objective function. It is easy to guess other model variants where a similar type of solution should be a possible outcome. Based on experience with two-period and rotation models, one can conjecture that introducing the assumption of imperfect capital markets or the assumption of uncertain timber prices with nonconstant risk aversion will lead to a parallel type of solution.

In solving the model we introduced a decision-variable matrix collecting the harvesting shares of all the land classes. The solution of the model is readily comparable to the much-used two-period models in resource economics. The introduced decision-variable matrix could be a useful tool in other contexts than the one illustrated here. The matrix can be used in defining a harvesting policy aiming at a specific forest structure and to trace the dynamic path of the forest structure to the equilibrium forest. For example, in view of carbon sequestration policies both the structure of the equilibrium forest and the transition toward it are relevant issues. Studying the regulations or incentives that the government can employ to enhance carbon intake by forests should be addressed by future research.

Our analytic and numeric results have implications on the empirical research of timber markets. The fact that the optimal adjustment of the forest structure was dependent on the initial age-class distribution of the land managed by the landowner suggests that variables that are able to capture this effect should be present in econometric models. The same applies to variables describing the landowners’ other-than-forest assets and income.

[Received May 2003; accepted March 2004.]

References


Appendix A: Existence of an Equilibrium Forest and Its Independence of the Initial Age-Class Structure

Define the dynamics for a constant harvesting policy as follows:

\[ x_{t+1} = Ax_t \]

where

\[ A = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & \cdots & a_n \\ 1 - a_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 - a_2 & 0 & \cdots & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 - a_{n-1} & 1 - a_n \end{bmatrix} \]

and \( x_t = (x_{t1}, x_{t2}, \ldots, x_{tn})' \)

and \( 0 < a_i < 1, \) for at least one stand \( i = 1, \ldots, n, \) and \( a_i \neq 1 \) for \( i = 1, \ldots, n. \)
The solution to (A.1) is \( \mathbf{x}_t = \mathbf{A}^t \mathbf{x}_0 \). \( \mathbf{A} \) has an analogous structure to the probability matrices in Markovian chains. \( \mathbf{A} \) is a non-negative matrix with column sums all equal to unity, and the modulus of the maximum eigenvalue of \( \mathbf{A} \) is unity. Because \( \mathbf{A} \) is constant, there exists a limit for the sequence of the age-class distribution \( \lim_{t \to \infty} \mathbf{x}_t = \mathbf{x} \) (Cox and Miller, p. 80). This means that cycles in the age-class distribution disappear in the limit, and the forest will reach an equilibrium where each age-class is maintained in a particular size in perpetuity.

Next we write the relationship between the harvesting parameters and the equilibrium age-classes as follows:

\[
\begin{align*}
(A.2) \quad x_1 &= a_1 x_1 + \cdots + a_n x_n \\
x_2 &= (1 - a_1) x_1 \\
x_3 &= (1 - a_2) x_2 \\
& \vdots \\
x_{n-1} &= (1 - a_{n-2}) x_{n-2} \\
x_n &= (1 - a_{n-1}) x_{n-1} + (1 - a_n) x_n
\end{align*}
\]

where \( x_i \) and \( a_i \) are now the equilibrium levels of the elements of the \( \mathbf{x} \) vector and the matrix \( \mathbf{A} \). From the last row, \( x_n \) can be written as follows:

\[
(A.3) \quad x_n = \frac{1 - a_{n-1}}{a_n} x_{n-1}.
\]

By recursive substitution, each \( x_i \) can be expressed with \( x_1 \). Then the age-classes can be summed up to get the total forestland area:

\[
(A.4) \quad x_1 + (1 - a_1) x_1 + (1 - a_2) (1 - a_1) x_1 \\
+ (1 - a_2)(1 - a_1) x_1 \\
+ \cdots + (1 - a_{n-2})( \cdots (1 - a_2)(1 - a_1) x_1 \\
+ (1 - a_{n-1})( \cdots (1 - a_2)(1 - a_1) x_1 = X
\]

where \( X \) is the total land area under forestry. From above, \( x_1 \) can be expressed with the \( a_i \) and the \( X \) alone:

\[
(A.5) \quad x_1 = X \left[ 1 + (1 - a_1) + (1 - a_2)(1 - a_1) \\
+ \cdots + (1 - a_{n-2})( \cdots (1 - a_2)(1 - a_1) \\
+ (1 - a_{n-1})( \cdots (1 - a_2)(1 - a_1) \right] / a_n
\]

This is the equilibrium size of the first age-class expressed as a function of \( X \) and the \( a \) parameters. By substituting recursively back into (A.1), the equilibrium values for the whole forest structure become

\[
(A.6) \quad x_1 = X \left[ \frac{1 + (1 - a_1) + (1 - a_2)}{(1 - a_1) + \cdots + (1 - a_{n-2})( \cdots (1 - a_1) \\
\times ( \cdots (1 - a_1) \\
+ (1 - a_{n-1})( \cdots (1 - a_1) \right] / a_n
\]

\[
x_2 = (1 - a_1) X \left[ \frac{1 + (1 - a_1) + (1 - a_2)}{(1 - a_1) + \cdots + (1 - a_{n-2})( \cdots (1 - a_1) \\
\times ( \cdots (1 - a_1) \\
+ (1 - a_{n-1})( \cdots (1 - a_1) \right] / a_n
\]

\[
x_3 = (1 - a_2)(1 - a_1) X \left[ \frac{1 + (1 - a_1)}{(1 - a_1) + \cdots + (1 - a_{n-2})( \cdots (1 - a_1) \\
\times ( \cdots (1 - a_1) \\
+ (1 - a_{n-1})( \cdots (1 - a_1) \right] / a_n
\]

\[
\vdots
\]

\[
x_{n-1} = \prod_{i=1}^{n-2} (1 - a_i) X \left[ \frac{1 + (1 - a_1)}{(1 - a_1) + \cdots + (1 - a_{n-2})( \cdots (1 - a_1) \\
\times ( \cdots (1 - a_1) \\
+ (1 - a_{n-1})( \cdots (1 - a_1) \right] / a_n
\]

\[
x_n = \frac{1 - a_{n-1}}{a_n} \prod_{i=1}^{n-2} (1 - a_i) X \left[ \frac{1 + (1 - a_1)}{(1 - a_1) + \cdots + (1 - a_{n-2})( \cdots (1 - a_1) \\
\times ( \cdots (1 - a_1) \\
+ (1 - a_{n-1})( \cdots (1 - a_1) \right] / a_n
\]

Above the equilibrium age-class distribution is expressed with only the harvesting parameters and the forestland area. Thus, the equilibrium forest
structure is a function of the constant harvesting policy and total forest area only, and is independent of the original age-class distribution.

To see that the policy \( a_n + a_{n-1} = 1 \) and \( a_i = 0 (i = 1, \ldots, n - 2) \) will lead to a normal forest structure set \( x_n = x_{n-1} \) in (A.3) to get \( a_n = 1 - a_{n-1} \), and set \( x_i = x_{i-1} \) for \( i = 2, \ldots, n - 1 \) in (A.2) to get \( a_i = 0 (i = 1, \ldots, n - 2) \). Note that both \( a_n \) and \( a_{n-1} \) are required to be larger than zero and smaller than one.

### Appendix B: Optimal Conditions for the Harvesting Decisions

Substituting the constraints of (1) into the objective function leads to the following recursive form:

\[
(A.7) \quad u \left[ \sum_{i=1}^{n} a_i x_i (pq_i - k) + w_0 - w_f \right] + M(Q_0) + \sum_{i=1}^{\infty} \beta^{i} u \left[ \sum_{i=1}^{n} a_i a_{i-1,i} x_i (pq_i - k) \right. \\
+ \sum_{i=2}^{n-1} a_i (1 - a_{i-1,i}) x_i (pq_i - k) + \sum_{i=n+1}^{\infty} a_n (1 - a_{n-1,i}) x_i (pq_i - k) \\
+ w_t (1 + r) - w_f \left. + \sum_{i=1}^{\infty} \beta^{i} M[q A_{i-1} x_i - 1]. \right]
\]

Taking the derivatives with respect to \( w_t \)'s leads to the standard intertemporal first-order condition (3) for the consumption. To derive the harvesting rules we work with the net marginal benefits of harvesting the share of \( a_i \) from age class \( i \). Taking the derivative wrt \( a_i \) of (A.7) leads to

\[
(A.8) \quad u'(c_i) x_i (pq_i - k) + \left( \frac{1}{1 + \rho} \right)^i u'(c_{i+1}) x_i (pq_i - k) \\
+ \left( \frac{1}{1 + \rho} \right)^{2i} u'(c_{i+2}) x_i (pq_i - k) + \cdots \\
- \left( \frac{1}{1 + \rho} \right)^{(i+1)+1} u'(c_{i+1}) x_i (pq_i - k) \\
- \left( \frac{1}{1 + \rho} \right)^{2(i+1)+1} u'(c_{i+2(i+1)+1}) x_i (pq_i - k) - \cdots \\
+ \sum_{j=1}^{i} \left( \frac{1}{1 + \rho} \right)^j M'(Q_{i+j}) x_j + \left( \frac{1}{1 + \rho} \right)^i \\
\times \left[ \sum_{j=i}^{\infty} \left( \frac{1}{1 + \rho} \right)^j M'(Q_{i+j}) x_j q_j \right] + \\
\left( \frac{1}{1 + \rho} \right)^{2i} \left[ \sum_{j=i}^{\infty} \left( \frac{1}{1 + \rho} \right)^j M'(Q_{i+2j}) x_j q_j \right] + \cdots \\
- \frac{1}{1 + \rho} M'(Q_{i+1}) x_i q_{i+1} \\
- \left( \frac{1}{1 + \rho} \right)^{(i+1)+1} M'(Q_{i+1(i+1)+1}) x_i q_{i+1} \\
- \left( \frac{1}{1 + \rho} \right)^{2(i+1)+1} M'(Q_{i+2(i+1)+1}) x_i q_{i+1} \\
- \cdots - \frac{1}{1 + \rho} \sum_{j=1}^{i} \left( \frac{1}{1 + \rho} \right)^j x_j q_j \\
\times M'(Q_{i+j+1}) x_j q_j - \left( \frac{1}{1 + \rho} \right)^{(i+1)+1} \\
\times \sum_{j=1}^{i} \left( \frac{1}{1 + \rho} \right)^j M'(Q_{i+j+1}) x_j q_j - \cdots -
\]

By recursive substitution, the optimal condition for consumption \( u'(c_i) = \frac{1 + r}{1 + \rho} u'(c_{i+1}) \) leads to \( \left( \frac{1}{1 + \rho} \right)^m = \left( \frac{1}{1 + r} \right)^m \frac{u'(c_i)}{u'(c_{i+1})} \). Substituting these into (A.8) and dividing by \( u'(c_i) \) leads to (A.9):

\[
(A.9) \quad (pq_i - k) + \left( \frac{1}{1 + r} \right)^i (pq_i - k) \\
+ \left( \frac{1}{1 + r} \right)^{2i} (pq_i - k) + \cdots - \frac{1}{1 + r} \\
\times (pq_{i+1} - k) - \left( \frac{1}{1 + r} \right)^{(i+1)+1} (pq_{i+1} - k) \\
- \left( \frac{1}{1 + r} \right)^{2(i+1)+1} (pq_{i+1} - k) - \cdots \\
+ \sum_{j=1}^{i} \left( \frac{1}{1 + r} \right)^j M'(Q_{i+j}) x_j q_j + \left( \frac{1}{1 + r} \right)^i \\
\times \left[ \sum_{j=i}^{\infty} \left( \frac{1}{1 + r} \right)^j M'(Q_{i+2j}) x_j q_j \right] + \left( \frac{1}{1 + r} \right)^{2i} \\
\times \left[ \sum_{j=i}^{\infty} \left( \frac{1}{1 + r} \right)^j M'(Q_{i+2(j+1)}) x_j q_j \right] + \cdots 
\]
Using infinite-series closed-form expression for the consumption utility terms in (A.9) and collecting the infinite series expressions for the amenity utility terms, the net marginal benefits of harvesting from age-class \( i \) \( \varphi(t, i) \) can be expressed as

\[
(A.10) \\
\varphi(t, i) = \\
\left( p_{qi} - k \right)(1 + r)^i \frac{(1 + r)^{i+1}}{(1 + r)^{i+1} - 1} \\
- \left( p_{qi+1} - k \right) \frac{(1 + r)^{i+1}}{(1 + r)^{i+1} - 1} \\
+ (1 + r) \sum_{h=0}^{\infty} \frac{1}{1 + r} (h_i)^i \frac{1}{1 + r} \\
\times \frac{M'(Q + h_i i + j)}{w'(c + h_i i + j)} q_j \\
- \sum_{h=0}^{\infty} (1 + r)^{h(i+1)} \left[ \frac{M'(Q + h_i i + j + 1)}{w'(c + h_i i + j + 1)} q_{i+1} \right] \\
+ \sum_{j=1}^{i} \frac{1}{1 + r} \left[ \frac{M'(Q + h_i i + j + 1)}{w'(c + h_i i + j + 1)} q_j \right].
\]

The first-order conditions of optimal harvesting can be expressed as \( \varphi(t, i) < 0, a_0 = 0; \varphi(t, i) > 0, a_0 = 1; \varphi(t, i) = 0, 0 < a_0 < 1. \) These conditions imply the rules in (4). With the strict concavity of the utility subfunctions, and imposing that \( \varphi(t, i) \) is strictly increasing in \( i \) the first-order conditions are sufficient and unique.

In a steady state, the period beginning timber volume \( Q \) and the consumption \( c^* \) are constant (\( Q \) cyclically or noncyclically) and \( \varphi(t, i) \) can be written as

\[
(A.11) \\
\varphi(i) = (pq_i - k)(1 + r) \left[ 1 + \left( \frac{1}{1 + r} \right)^i \right] \\
+ \left( \frac{1}{1 + r} \right)^{2i} + \cdots \\
- (pq_{i+1} - k) \\
\times \left[ 1 + \left( \frac{1}{1 + r} \right)^{i+1} + \left( \frac{1}{1 + r} \right)^{2(i+1)} + \cdots \right] \\
+ (1 + r) \frac{M'(Q^*)}{w'(c^*)} \sum_{j=1}^{i} \frac{1}{1 + r} q_j \\
\times \left[ 1 + \left( \frac{1}{1 + r} \right)^i + \left( \frac{1}{1 + r} \right)^{2i} + \cdots \right] \\
- M'(Q_{i+1}) \frac{1}{w'(c^*)} [q_{i+1} + \sum_{j=1}^{i} \frac{1}{1 + r} q_j] \\
\times \left[ 1 + \left( \frac{1}{1 + r} \right)^{i+1} + \left( \frac{1}{1 + r} \right)^{2(i+1)} + \cdots \right]
\]

where \( Q^* \) and the consumption \( c^* \) are the steady-state timber volumes and consumption. Using the closed-form expressions for the infinite series (A.11) leads to expression (6) in text. Imposing the condition that \( \varphi(i) \) is strictly increasing in \( i \) rules out a case where multiple harvesting schemes would satisfy the necessary conditions for optimality in the equilibrium.

**Appendix C: Equivalency with the Hartman Model**

In Hartman’s model, the utility structure is linear. Applying this in expression (6) in text we can write:

\[
(A.12) \\
\left( pq_i - k \right)(1 + r) \frac{(1 + r)^i}{(1 + r)^{i+1} - 1} \\
- \left( pq_{i+1} - k \right) \frac{(1 + r)^{i+1}}{(1 + r)^{i+1} - 1} \\
+ (1 + r) \sum_{j=1}^{i} \frac{1}{1 + r} q_j \frac{(1 + r)^i}{(1 + r)^{i+1} - 1} \\
- q_{i+1} + \sum_{j=1}^{i} \frac{1}{1 + r} q_j (1 + r)^{i+1} \frac{1}{(1 + r)^{i+1} - 1} \\
\geq 0.
\]
Dividing by \((1+r)^t\), rearranging, and taking 1 + \(r\) out of the brackets on the left-hand side:

\[
(A.13) \quad \left[ (pq_i - k) + \sum_{j=1}^{i} \left( \frac{1}{1+r} \right)^j q_j \right] \\
\times \left( \frac{1}{1+r} \right)^{i+1} - 1 > (pq_{i+1} - k) \\
+ q_{i+1} + \sum_{j=1}^{i} \left( \frac{1}{1+r} \right)^j q_j.
\]

Subtracting \((pq_i - k) + \sum_{j=1}^{i} \left( \frac{1}{1+r} \right)^j q_j\) from both sides and rewriting:

\[
(A.14) \quad \left[ (pq_i - k) + \sum_{j=1}^{i} \left( \frac{1}{1+r} \right)^j q_j \right] \\
\times \left( \frac{1}{1+r} \right)^{i+1} - 1 \leq (pq_{i+1} - k) \\
- pq_i - k + q_{i+1}.
\]

Dividing both sides by \(pq_i - k\) and rewriting the interest rate term on the left-hand side and after rearranging:

\[
(A.15) \quad \frac{pq_{i+1} - pq_i}{pq_i - k} > \frac{1}{1 \left( \frac{1}{1+r} \right)} + \sum_{j=1}^{i} \left( \frac{1}{1+r} \right)^j q_j \\
- \frac{q_{i+1}}{pq_i - k}.
\]

Expression (A.15) is the discrete-time counterpart of expression (10) in Hartman. In case amenities are not included (A.15) reduces to

\[
(A.16) \quad \frac{pq_{i+1} - pq_i}{pq_i - k} > \frac{1}{1 \left( \frac{1}{1+r} \right)}.
\]

Expression (A.16) is the discrete-time version of Faustmann formula (e.g., expression 8.5 in Clark, p. 259).

Appendix D: Existence of a Long-Run Optimal Steady-State Forest

In this Appendix, we show that in case the age-class structure converges to a cyclical or noncyclical steady state, it is always possible to find values for the decision variables \(a_i\), which satisfy the first-order conditions defined in (4). For a cyclical steady state, we first state a general temporal relationship between two age-class distributions representing different time periods. In general, the following relationship holds between the age-class distributions at time \(t\) and at time \(t + j\):

\[
(A.17) \quad x_{t+j} = A^j x_t
\]

where \(A\) is the transition matrix constant in the steady state. Assume that there exists an equilibrium cycle whose length is \(n\) periods, where \(n\) is the number of age-classes. This means that the same age-class distribution is repeated every \(n\) periods. Then \(x_{tn} = x_t\). To have such an equilibrium cycle, matrix \(A\) has to have the property that \(A^n = I\). Such a matrix exists because the \(n \times n\) matrix

\[
(A.18) \quad A = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}
\]

satisfies the condition \(A^n = I\).

In \(A\) above, the \(n\)th column represents the youngest age-class, \(n\), to be cut. There will be no older age-classes. An \(n\) can be found so that a harvesting regime (A.18) satisfies the optimal conditions stated in expression (4). Then a cyclical steady state exists. Assuming that \(\varphi(i)\) in (A.11) is strictly increasing over the age-classes, the steady state is unique.

We continue with the noncyclical steady-state forest. Note first that the above-type of harvesting schedule where one stand is always entirely cut over can also exhibit a noncyclical forest. However, this occurs only if the equilibrium forest happens to be normal, so that an equal size of land is allocated for all age-classes.

Consider next a noncyclical forest where at least one stand is always harvested in part. Equation (A.6) in Appendix A characterizes such a noncyclical steady state, because both the harvesting policy and the age-class distribution are constant in a forest described by (A.6). Expressions in (A.6) can be substituted into (4) and optimal values for \(a_i\'s\) can be obtained. Then an optimal noncyclical steady state must exist. For the uniqueness of the steady state we assume that \(\varphi(i)\) in (A.11) strictly increases over the age-classes. In this case, the first-order conditions are necessary and sufficient, and solution satisfying them is unique. Therefore, a steady state satisfying (4) must also be unique.