



1 real-valued with an exponential type of decay at infinity. Under the assumptions  
2 that the operator has no spectral or non-spectral singularities and no negative  
3 eigenvalues he was able to provide a uniqueness theorem for the inverse problem  
4 when given the so-called reflection and connection coefficients  $R_+$  and  $C_+$  for all  
5  $k \neq 0$  in the rays  $\arg(k) = 0$  and  $\arg(k) = \frac{\pi}{4}$ , respectively. More recently, in 2008  
6 Aktosun and Papanicolaou [9] studied the operator  $L_4$  with time-evolving coeffi-  
7 cients. In that work the time-dependence in several related scattering coefficients  
8 was discussed. This linear operator  $L_4$  has also been studied in higher dimensions  
9 [11, 12]. In [10] the operator  $L_4$  was generalized by adding a second-order perturba-  
10 tion. The method of Born approximation was studied for this linear operator and it  
11 was proved that Born approximation can be used to recover the local singularities  
12 of the coefficients.

13 The motivation to study non-linear operators of order four can be found, for  
14 instance, in theory of vibrations of beams and the study of elasticity. For example,  
15 by looking for the time-harmonic solutions  $U(x, t) = u(x)e^{-i\omega t}$  to the non-linear  
16 beam equation (cf. [1])

$$\partial_t^2 U + \partial_x^4 U + mU + |U|^{p-1}U = 0, \quad m > 0,$$

17 we arrive at the equation

$$u^{(4)} + (m + |u|^{p-1})u = \omega^2 u,$$

where the left-hand-side is of type (1) with  $q_1 \equiv 0$ . Albeit real-valued, the following  
simple model of a suspension bridge from [4] also uses a fourth-order non-linear  
equation to model the downward deflection  $u(x, t)$ . The bridge is understood as a  
beam of length  $L$  with hinged ends. The deflection  $u(x, t)$  is subject to three forces  
with the homogeneous Navier boundary conditions

$$\begin{aligned} \partial_t^2 u + \gamma \partial_x^4 u &= -ku^+ + W + f(x, t), \\ u(0, t) = u(L, t) &= \partial_x^2 u(0, t) = \partial_x^2 u(L, t) = 0, \end{aligned}$$

18 where  $\gamma, k$  and  $W$  are constants called Young's modulus, the spring constant and  
19 forcing constant, respectively. The function  $f(x, t)$  is an external forcing term and  
20  $u^+ = \max\{0, u\}$ .

The present paper concerns scattering problems for the operator  $Q_4$  and we  
study only one special set of solutions to the equation  $Q_4 u = k^4 u$ . More precisely,  
the direct scattering problem for  $Q_4$  can be formulated by the equation

$$Q_4 u = k^4 u, \quad u = u_0 + u_{\text{sc}}, \quad u_0(x, k) = e^{ikx}, \quad k \in \mathbb{R}, \quad (2)$$

where  $u_{\text{sc}}$  must be outgoing in some sense. Instead of studying directly the equation  
(2) we apply the standard theory of ordinary differential equations to obtain an

integral equation. For  $k > 0$  the kernel of the integral operator

$$\left(\frac{d^4}{dx^4} - k^4 - i0\right)^{-1} = \frac{1}{2k^2} \left( \left(-\frac{d^2}{dx^2} - k^2 - i0\right)^{-1} - \left(-\frac{d^2}{dx^2} + k^2\right)^{-1} \right) \quad (3)$$

1 (or a fundamental solution to operator  $L_0 := \frac{d^4}{dx^4} - k^4$ ) is given by

$$G_k^+(|x|) := \frac{1}{4k^3} (ie^{ik|x|} - e^{-k|x|}).$$

2 This kernel is outgoing in the sense that it satisfies the radiation condition

$$\left(\frac{\partial}{\partial|x|} - ik\right) G_k^+(|x|) = o(1), \quad |x| \rightarrow \infty.$$

Note that due to (3) our operator inherits some Schrödinger-like properties and this allows us to use similar techniques as those used to study the non-linear Schrödinger operator [13, 14]. By convolving this fundamental solution formally with (2) we obtain the integral equation

$$u(x, k) = u_0(x, k) - \int_{-\infty}^{\infty} G_k^+(|x-y|)(q_1(y, |u(y, k)|)u'(y, k) + q_0(y, |u(y, k)|)u(y, k))dy. \quad (4)$$

As it turns out (see Section 3), the solution to this integral equation has the asymptotic representation

$$\begin{aligned} u(x, k) &= a(k)u_0(x, k) + o(1), \quad x \rightarrow +\infty, \\ u(x, k) &= u_0(x, k) + b(k)e^{-ikx} + o(1), \quad x \rightarrow -\infty, \end{aligned}$$

3 where  $a(k)$  and  $b(k)$  are called the transmission and reflection coefficients, respec-  
4 tively. They are defined by

$$a(k) := 1 - \frac{i}{4k^3} \int_{-\infty}^{\infty} e^{-iky} (q_1(y, |u|)u' + q_0(y, |u|)u) dy$$

5 and

$$b(k) := -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{iky} (q_1(y, |u|)u' + q_0(y, |u|)u) dy$$

6 for sufficiently large  $k > 0$ . This asymptotic representation can now be regarded  
7 as a radiating solution to (4). It turns out that for our purposes it is enough to  
8 study only the reflection coefficient for sufficiently large  $k > 0$ .

1 In this paper we consider the inverse problem of recovering the potential func-  
2 tions  $q_1$  and  $q_0$  by the Born approximation method. The reflection coefficient  $b(k)$   
3 is used to define the inverse Born approximation  $h_B$  as the Fourier transform of  
4  $\frac{ik^3}{2\sqrt{2\pi}}b(k/2)$ . We show that this approximation recovers the jumps and local sin-  
5 gularities in the combination  $h(x) := -\frac{1}{2}q_1'(x, 1) + q_0(x, 1)$ , where the derivative  
6 is taken with respect to  $x$ . Our main result is that the difference  $h_B - \text{Re}(h)$  de-  
7 fines a continuous function (if the imaginary part is smooth enough) and hence the  
8 singularities of  $h_B$  coincide with those in  $h$ .

9 The following notations are used throughout this text. We use  $C > 0$  as a  
10 generic constant when it is not important to keep track of its precise value. The  
11 symbol  $L^p(\Omega)$  is used to denote the  $p$ -based Lebesgue space over the set  $\Omega \subset \mathbb{R}$ .  
12 The space of continuous functions vanishing at infinity is denoted by  $\dot{C}(\mathbb{R})$  and the  
13 Sobolev spaces  $W_p^k(\mathbb{R})$ , for  $k \in \mathbb{N}$ , are the spaces of those functions whose weak  
14 derivatives up to order  $k$  are in  $L^p(\mathbb{R})$ . In the case where  $p = 2$  and  $s \in \mathbb{R}$  we say  
15 that  $f \in H^s(\mathbb{R})$  if the norm

$$\|f\|_{H^s(\mathbb{R})} := \left( \int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

is finite. Here the Fourier transform pair of  $f$  is defined by the formulae

$$\begin{aligned} \widehat{f}(\xi) &\equiv F(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx, \\ F^{-1}(f)(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} f(\xi) d\xi. \end{aligned}$$

16 This paper is organized as follows. In Section 2 we establish the existence of  
17 solutions  $u = u_0 + u_{sc}$  to (4) with the property that  $u_{sc} \in W_{\infty}^1(\mathbb{R})$ . In Section  
18 3 we study the asymptotic behaviour of  $u$  at  $x \rightarrow -\infty$  and define the reflection  
19 coefficient  $b(k)$ . We also motivate the definition of the inverse Born approximation  
20  $h_B$ . Finally, Section 4 concerns the inverse problem of finding the jumps and  
21 singularities of  $h$ . We also present a numerical example to demonstrate the Born  
22 approximation visually.

## 23 **2 Existence and uniqueness**

24 In this section we prove that under quite general assumptions the integral equation  
25 (4) has a unique solution  $u$ , when  $k > 0$  is large enough. In the linear case  
26 [10] a Neumann-type series could be used to construct the solution, but the non-  
27 linearity disables this approach in the case of operator  $Q_4$ . Instead, we apply

1 Banach fixed point theorem to prove the existence and uniqueness of the direct  
 2 scattering problem. Our first assumption<sup>1</sup> is similar to that in [14].

3 **Assumption 2.1.** *Let us assume that the coefficients  $q_j$ ,  $j = 0, 1$ , have the follow-*  
 4 *ing properties*

5 1. *There exists functions  $\alpha_j \in L^1(\mathbb{R})$  such that for all  $a > 0$  we find  $C_j(a) > 0$*   
 6 *with the property that  $|q_j(x, s)| \leq C_j(a)\alpha_j(x)$ , for all  $0 \leq s \leq a$ .*

7 2. *The coefficients  $q_j$  have the Lipschitz property in the second variable, that is,*  
 8 *there exists  $\beta_j \in L^1(\mathbb{R})$  such that for all  $a > 0$  we find  $C'_j(a) > 0$  with the*  
 9 *property that  $|q_j(x, s_1) - q_j(x, s_2)| \leq C'_j(a)\beta_j(x)|s_1 - s_2|$  for all  $0 \leq s_1, s_2 \leq$*   
 10  *$1 + a$ .*

Suppose  $u$  is a solution to (4) and denote

$$G'_k(x) := (G_k^+)'(|x|) = -\frac{\operatorname{sgn}(x)}{4k^2} (e^{ik|x|} - e^{-k|x|}). \quad (5)$$

Then

$$u_{\text{sc}}(x, k) = -\int_{-\infty}^{\infty} G_k^+(|x - y|) (q_1(y, |u_{\text{sc}} + u_0|)(u_{\text{sc}} + u_0)' + q_0(y, |u_{\text{sc}} + u_0|)(u_{\text{sc}} + u_0)) dy \quad (6)$$

and by the Leibniz integral rule

$$u'_{\text{sc}}(x, k) = -\int_{-\infty}^{\infty} G'_k(x - y) (q_1(y, |u_{\text{sc}} + u_0|)(u_{\text{sc}} + u_0)' + q_0(y, |u_{\text{sc}} + u_0|)(u_{\text{sc}} + u_0)) dy.$$

11 Now solving (6) for  $u_{\text{sc}}$  is equivalent to solving (4). To do this, pick any  $\rho > 0$  and  
 12 consider the closed ball  $B_\rho(0) := \{f \in W_\infty^1(\mathbb{R}) \mid \|f\|_\infty + \|f'\|_\infty \leq \rho\}$ , where  $\|\cdot\|_p$   
 13 denotes the usual  $L^p$ -norm on the line.

14 **Theorem 2.2.** *Under Assumption 2.1, for any  $\rho > 0$  there exists  $k_0 > 0$  such that*  
 15 *the integral equation (6) has a unique solution  $u_{\text{sc}}$  in  $B_\rho(0) \subset W_\infty^1(\mathbb{R})$  uniformly in*  
 16  *$k \geq k_0$ .*

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<sup>1</sup>We implicitly assume that the coefficients  $q_j(x, |u(x, k)|)$  are measurable (which is the case, for instance, when  $q_j$  are measurable in the first variable and continuous in the second variable), so that it makes sense to talk about integrability.

1 *Proof.* Let us denote

$$T(\tilde{u})(x) := - \int_{-\infty}^{\infty} G_k^+(|x-y|) (q_1(y, |\tilde{u}+u_0|)(\tilde{u}+u_0)' + q_0(y, |\tilde{u}+u_0|)(\tilde{u}+u_0)) dy$$

2 for any  $\tilde{u} \in W_{\infty}^1(\mathbb{R})$ . Correspondingly,

$$(T(\tilde{u}))'(x) = - \int_{-\infty}^{\infty} G_k'(x-y) (q_1(y, |\tilde{u}+u_0|)(\tilde{u}+u_0)' + q_0(y, |\tilde{u}+u_0|)(\tilde{u}+u_0)) dy.$$

3 We will use Banach fixed-point theorem (see for example [2]) to prove this  
4 theorem. Since  $B_{\rho}(0)$  is a closed subspace in the complete metric space  $W_{\infty}^1(\mathbb{R})$ , it  
5 suffices to show that  $T : B_{\rho}(0) \rightarrow B_{\rho}(0)$  and that  $T$  is a contraction.

6 Let  $\tilde{u} \in B_{\rho}(0)$ . From Assumption 2.1 we get

$$|q_j(y, |\tilde{u}+u_0|)| \leq C_j(1+\rho)\alpha_j(y), \quad j = 0, 1.$$

Using this fact we may estimate

$$\begin{aligned} |T(\tilde{u})(x)| &\leq \frac{1}{2k^3} \int_{-\infty}^{\infty} (|q_1(y, |\tilde{u}+u_0|)| |\tilde{u}' + iku_0| + |q_0(y, |\tilde{u}+u_0|)| |\tilde{u}+u_0|) dy \\ &\leq \frac{C_1(1+\rho)\|\alpha_1\|_1}{2k^3} (k + \|\tilde{u}'\|_{\infty}) + \frac{C_0(1+\rho)\|\alpha_0\|_1}{2k^3} (1 + \|\tilde{u}\|_{\infty}) \\ &\leq \frac{C_1(1+\rho)\|\alpha_1\|_1 + C_0(1+\rho)\|\alpha_0\|_1}{2k^2} + \frac{\tilde{C}_{\rho}}{2k^3} \rho, \end{aligned}$$

7 when  $k \geq 1$  and where we denote  $\tilde{C}_{\rho} := \max\{C_1(1+\rho)\|\alpha_1\|_1, C_0(1+\rho)\|\alpha_0\|_1\}$ .

8 Similarly, for the derivative of  $T(u)$  we get

$$|(T(\tilde{u}))'(x)| \leq \frac{C_1(1+\rho)\|\alpha_1\|_1 + C_0(1+\rho)\|\alpha_0\|_1}{2k} + \frac{\tilde{C}_{\rho}}{2k^2} \rho,$$

when  $k \geq 1$ . Adding the above inequalities and taking the supremum yields

$$\begin{aligned} \|T(\tilde{u})\|_{W_{\infty}^1(\mathbb{R})} &\leq \frac{C_1(1+\rho)\|\alpha_1\|_1 + C_0(1+\rho)\|\alpha_0\|_1}{2k} \\ &\quad + \frac{C_1(1+\rho)\|\alpha_1\|_1 + C_0(1+\rho)\|\alpha_0\|_1}{2k^2} + \tilde{C}_{\rho} \left( \frac{1}{2k^2} + \frac{1}{2k^3} \right) \rho \\ &\leq \frac{\tilde{C}_{\rho}}{k} (2 + \rho), \end{aligned}$$

9 when  $k \geq 1$ . To conclude that  $T : B_{\rho}(0) \rightarrow B_{\rho}(0)$  we need only to show that  
10  $\|T(\tilde{u})\|_{W_{\infty}^1(\mathbb{R})} \leq \rho$ . But this happens when  $k \geq \tilde{C}_{\rho} \left(1 + \frac{2}{\rho}\right)$ .

It remains to show that  $T$  is a contraction. Let  $\tilde{u}, \tilde{v} \in W_\infty^1(\mathbb{R})$ . Then we estimate

$$\begin{aligned}
|T(\tilde{u}) - T(\tilde{v})| &\leq \frac{1}{2k^3} \int_{-\infty}^{\infty} |q_1(y, |\tilde{u} + u_0|)(\tilde{u}' + ik u_0) - q_1(y, |\tilde{v} + u_0|)(\tilde{v}' + ik u_0)| dy \\
&\quad + \frac{1}{2k^3} \int_{-\infty}^{\infty} |q_0(y, |\tilde{u} + u_0|)(\tilde{u} + u_0) - q_0(y, |\tilde{v} + u_0|)(\tilde{v} + u_0)| dy \\
&\leq \frac{1}{2k^3} \int_{-\infty}^{\infty} (|q_1(y, |\tilde{u} + u_0|)\tilde{u}' - q_1(y, |\tilde{v} + u_0|)\tilde{v}'| \\
&\quad + k|q_1(y, |\tilde{u} + u_0|) - q_1(y, |\tilde{v} + u_0|)|) dy \\
&\quad + \frac{1}{2k^3} \int_{-\infty}^{\infty} (|q_0(y, |\tilde{u} + u_0|)\tilde{u} - q_0(y, |\tilde{v} + u_0|)\tilde{v}| \\
&\quad + |q_0(y, |\tilde{u} + u_0|) - q_0(y, |\tilde{v} + u_0|)|) dy
\end{aligned}$$

and by Assumption 2.1

$$\begin{aligned}
|T(\tilde{u}) - T(\tilde{v})| &\leq \frac{C'_1(\rho)\|\beta_1\|_1}{2k^2} \|\tilde{u} - \tilde{v}\|_\infty + \frac{C_1(1+\rho)\|\alpha_1\|_1}{2k^3} \|\tilde{u}' - \tilde{v}'\|_\infty \\
&\quad + \frac{C'_1(\rho)\|\beta_1\|_1}{2k^3} \|\tilde{v}'\|_\infty \|\tilde{u} - \tilde{v}\|_\infty \\
&\quad + \frac{C'_0(\rho)\|\beta_0\|_1}{2k^3} \|\tilde{u} - \tilde{v}\|_\infty + \frac{C_0(1+\rho)\|\alpha_0\|_1}{2k^3} \|\tilde{u} - \tilde{v}\|_\infty \\
&\quad + \frac{C'_0(\rho)\|\beta_0\|_1}{2k^3} \|\tilde{v}'\|_\infty \|\tilde{u} - \tilde{v}\|_\infty \\
&\leq \frac{\tilde{C}'_\rho}{2k^2} \|\tilde{u} - \tilde{v}\|_{W_\infty^1(\mathbb{R})}, \tag{7}
\end{aligned}$$

1 where

$$\tilde{C}'_\rho := \max \{C_0(1+\rho)\|\alpha_0\|_1 + (1+\rho)(C'_0(\rho)\|\beta_0\|_1 + C'_1(\rho)\|\beta_1\|_1), C_1(1+\rho)\|\alpha_1\|_1\}.$$

2 Derivatives are estimated similarly, and we obtain the estimate

$$|T(\tilde{u})' - T(\tilde{v})'| \leq \frac{\tilde{C}'_\rho}{2k} \|\tilde{u} - \tilde{v}\|_{W_\infty^1(\mathbb{R})}.$$

It means that

$$\|T(\tilde{u}) - T(\tilde{v})\|_{W_\infty^1(\mathbb{R})} \leq \frac{\tilde{C}'_\rho}{k} \|\tilde{u} - \tilde{v}\|_{W_\infty^1(\mathbb{R})}. \tag{8}$$

3 Finally, fixing some  $k_0 > \max \left\{ 1, \tilde{C}'_\rho, \tilde{C}_\rho \left( 1 + \frac{2}{\rho} \right) \right\}$  gives that  $T : B_\rho(0) \rightarrow B_\rho(0)$   
4 and is a contraction when  $k \geq k_0$ .  $\square$

1 *Remark 2.3.* In the sequel we will let  $\rho > 0$  denote a fixed constant, as in Theorem  
2 2.2. In addition to the existence and the uniqueness of the solution, the proof of  
3 Banach's fixed-point theorem (see e.g., [2]) gives us an estimate in terms of the  
4 first iteration. Starting the iteration scheme from  $u_{\text{sc}}^{(0)} := 0 \in B_\rho(0)$ , the definition  
5  $u_{\text{sc}}^{(j)} = T(u_{\text{sc}}^{(j-1)})$  for  $j = 1, 2, \dots$  yields

$$\|u_{\text{sc}} - u_{\text{sc}}^{(j)}\|_{W_\infty^1(\mathbb{R})} \leq \frac{\left(\frac{\tilde{C}'_\rho}{k}\right)^j}{1 - \frac{\tilde{C}'_\rho}{k}} \|u_{\text{sc}}^{(1)}\|_{W_\infty^1(\mathbb{R})} \leq \frac{C_1(1)\|\alpha_1\|_1 + C_0(1)\|\alpha_0\|_1}{1 - \frac{\tilde{C}'_\rho}{k_0}} \times \frac{(\tilde{C}'_\rho)^j}{k^{j+1}}$$

for all  $j = 0, 1, \dots$ , when  $k \geq k_0$ . The second inequality above follows from the definition of the operator  $T$  since we have

$$\begin{aligned} |u_{\text{sc}}^{(1)}(x, k)| &= |T(0)| = \left| \int_{-\infty}^{\infty} G_k^+(|x-y|)(ikq_1(y, 1) + q_0(y, 1))e^{iky} dy \right| \\ &\leq \frac{C_1(1)\|\alpha_1\|_1 + C_0(1)\|\alpha_0\|_1}{2k^2} \end{aligned}$$

6 and similarly for the derivatives.

**Corollary 2.4.** *Let us denote  $u_j := u_0 + u_{\text{sc}}^{(j)}$ . Under Assumption 2.1 the estimates*

$$\begin{aligned} \|u - u_j\|_\infty &\leq \frac{C_1(1)\|\alpha_1\|_1 + C_0(1)\|\alpha_0\|_1}{1 - \frac{\tilde{C}'_\rho}{k_0}} \times \frac{(\tilde{C}'_\rho)^j}{2k^{j+2}}, \\ \|u' - u'_j\|_\infty &\leq \frac{C_1(1)\|\alpha_1\|_1 + C_0(1)\|\alpha_0\|_1}{1 - \frac{\tilde{C}'_\rho}{k_0}} \times \frac{(\tilde{C}'_\rho)^j}{2k^{j+1}} \end{aligned}$$

hold uniformly in  $k \geq k_0$  for  $j = 0, 1, \dots$ . In particular,

$$\|u_{\text{sc}}\|_\infty \leq \frac{C_1(1)\|\alpha_1\|_1 + C_0(1)\|\alpha_0\|_1}{k^2},$$

7 when  $k \geq 2k_0$ .

8 *Proof.* The estimates follow from inequalities (7) and (8) and Remark 2.3.  $\square$

### 9 **3 Asymptotic behaviour of $u$ at $-\infty$**

Having obtained a solution  $u$  to (4) with  $u_{\text{sc}} \in W_\infty^1(\mathbb{R})$ , we study its asymptotic behaviour. Let us assume here that  $q_1(x, 1) \in W_1^1(\mathbb{R})$ . To motivate the definition of the inverse Born approximation in Section 4 we first formally define

$$u(x, k) := \overline{u(x, -k)} \quad \text{and} \quad u'(x, k) := \overline{u'(x, -k)}, \quad (9)$$

1 when  $k \leq -k_0$ . These functions are the unique solutions to

$$u = u_0 - \int_{-\infty}^{\infty} \overline{G_{-k}^+(|x-y|)} \left( \overline{q_1(y, |u|)} u' + \overline{q_0(y, |u|)} u \right) dy$$

2 for  $k \leq -k_0$ . Here the principal part of the above integral operator has the same  
3 behaviour as the Lippmann-Schwinger equation does.

4 Then as  $x \rightarrow -\infty$  we find that  $u$  has the asymptotic representation

$$u(x, k) = u_0(x, k) + b(k)e^{-ikx} + o(1),$$

5 where  $b(k)$  is called the reflection coefficient and is defined by

$$b(k) := -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{iky} (q_1(y, |u|)u' + q_0(y, |u|)u) dy,$$

when  $k \geq k_0$  and due to (9)

$$\begin{aligned} b(k) &:= -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{iky} \left( \overline{q_1(y, |u(y, -k)|)} u'(y, k) + \overline{q_0(y, |u(y, -k)|)} u(y, k) \right) dy \\ &= -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{iky} \left( \overline{q_1(y, |u|)} u' + \overline{q_0(y, |u|)} u \right) dy, \end{aligned}$$

6 when  $k \leq -k_0$ . For simplicity we extend  $b(k) = 0$  if  $-k_0 < k < k_0$ . The property  
7 (9) implies that  $b(k) = \overline{b(-k)}$ .

8 Due to Corollary 2.4 it is reasonable to approximate  $u \approx u_0$  and  $u' \approx ik u_0$  for  
9 large values of  $k > 0$ . Correspondingly, Assumption 2.1 gives for both  $j = 0, 1$  that

$$|q_j(y, |u_{sc} + u_0|) - q_j(y, 1)| \leq C'_j(\rho) \beta_j(y) |u_{sc}|,$$

so, roughly speaking,  $q_j(y, |u|) \approx q_j(y, 1)$  for large  $k > 0$ . By integrating by parts  
we can approximate

$$\begin{aligned} b(k) &\approx -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{2iky} (ikq_1(y, 1) + q_0(y, 1)) dy \\ &= \frac{\sqrt{2\pi}}{4ik^3} F^{-1} \left( q_0(\cdot, 1) - \frac{1}{2} q'_1(\cdot, 1) \right) (2k) \end{aligned}$$

10 for large  $k > 0$ .

## 11 4 Inverse problem

12 We are now ready to turn our attention to the inverse problem of finding the jump  
13 discontinuities and local singularities of the combination  $q_0(x, 1) - \frac{1}{2} q'_1(x, 1)$ . By

1 using Fourier inversion and our formal equality  $b(k) = \overline{b(-k)}$  for  $k < 0$  on the  
 2 approximation

$$b(k) \approx \frac{\sqrt{2\pi}}{4ik^3} F^{-1} \left( q_0(\cdot, 1) - \frac{1}{2} q_1'(\cdot, 1) \right) (2k)$$

3 we are motivated to propose the following definition.

4 **Definition 4.1.** The inverse Born approximation of  $h(x) := q_0(x, 1) - \frac{1}{2} q_1'(x, 1)$  is  
 5 defined by

$$h_B(x) = 2 \operatorname{Re} \left( \frac{i}{4\pi} \int_{2k_0}^{\infty} e^{-iky} k^3 b \left( \frac{k}{2} \right) dk \right)$$

6 in the sense of tempered distributions.

7 It is worth noting that this definition only uses positive values of  $k$ . Recalling  
 8 the iterations  $u_j := u_0 + u_{sc}^{(j)}$  we may define a new sequence  $b_j$  by

$$b_j(k) := -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{iky} (q_1(y, |u_j|) u_j' + q_0(y, |u_j|) u_j) dy,$$

9 when  $k \geq k_0$  and extend  $b_j$  by zero onto the interval  $[0, k_0]$ .

10 **Lemma 4.2.** *Under Assumption 2.1 we have*

$$|b(k) - b_j(k)| \leq C \frac{(\tilde{C}'_\rho)^j}{k^{j+4}}, \quad j = 0, 1, \dots$$

11 for some  $C > 0$  (depending on  $\rho$  and the coefficients) which is independent of  $k$   
 12 and  $j$  when  $k \geq k_0$ .

*Proof.* We start by estimating

$$\begin{aligned} |b(k) - b_j(k)| &\leq \frac{1}{4k^3} \int_{-\infty}^{\infty} |q_0(y, |u|) u - q_0(y, |u_j|) u_j| dy \\ &\quad + \frac{1}{4k^3} \int_{-\infty}^{\infty} |q_1(y, |u|) u' - q_1(y, |u_j|) u_j'| dy =: I_0 + I_1. \end{aligned}$$

By Assumption 2.1 we have

$$\begin{aligned} I_0 &\leq \frac{1}{4k^3} \int_{-\infty}^{\infty} (C'_0(\rho) |\beta_0(y)| |u - u_j| |u| + |q_0(y, |u_j|)| |u - u_j|) dy \\ &\leq \frac{1}{4k^3} (C'_0(\rho) \|\beta_0\|_1 (1 + \rho) + C_0 (1 + \rho) \|\alpha_0\|_1) \|u - u_j\|_\infty \leq C \frac{(\tilde{C}'_\rho)^j}{k^{j+5}}. \end{aligned}$$

In the same manner, by using Assumption 2.1 and Corollary 2.4 we can say that

$$\begin{aligned} I_1 &\leq \frac{1}{4k^3} \int_{-\infty}^{\infty} (C'_1(\rho) |\beta_1(y)| |u - u_j| |u'| + |q_1(y, |u_j|)| |u' - u'_j|) dy \\ &\leq \frac{1}{4k^3} (C'_1(\rho) \|\beta_1\|_1 (k + \rho) \|u - u_j\|_{\infty} + C_1(1 + \rho) \|\alpha_0\|_1 \|u' - u'_j\|_{\infty}) \leq C \frac{(\tilde{C}'_{\rho})^j}{k^{j+4}} \end{aligned}$$

1 for some constant  $C > 0$ . Combining the estimates for  $I_0$  and  $I_1$  gives the claim.  $\square$

2 **Corollary 4.3.** *The reflection coefficient can be obtained as the limit  $b(k) =$*   
 3  *$\lim_{j \rightarrow \infty} b_j(k)$  uniformly in  $k \geq k_0$ .*

4 Now the Born approximation is equal to the limit

$$h_B(\xi) = \lim_{j \rightarrow \infty} h_{B,j}(\xi)$$

5 in the sense of distributions, where

$$h_{B,j}(x) = 2 \operatorname{Re} \left( \frac{i}{4\pi} \int_{2k_0}^{\infty} e^{-iky} k^3 b_j \left( \frac{k}{2} \right) dk \right)$$

6 is called the inverse Born sequence.

7 **Lemma 4.4.** *Under Assumption 2.1, if  $q_1(x, 1) \in W_1^1(\mathbb{R})$ , the first term in the*  
 8 *inverse Born sequence is*

$$h_{B,0}(x) = \operatorname{Re}(h)(x) + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\operatorname{Im}(h)(y)}{x - y} dy \quad (\text{mod } \dot{C}(\mathbb{R})).$$

9 *Proof.* By integrating by parts in the definition of  $b_0(k/2)$ , we obtain for  $k \geq 2k_0$   
 10 that

$$b_0(k/2) = -\frac{2i}{k^3} \int_{-\infty}^{\infty} e^{iky} \left( \frac{ik}{2} q_1(y, 1) + q_0(y, 1) \right) dy = \frac{2\sqrt{2\pi}}{ik^3} F^{-1}(h)(k).$$

Let  $\chi_{[a,b]}(k)$  denote the characteristic function of the interval  $[a, b]$ . By using Definition 4.1 we obtain

$$\begin{aligned} h_{B,0}(x) &= 2 \operatorname{Re} \left( \frac{i}{4\pi} \int_{2k_0}^{\infty} e^{-ikx} k^3 b_0 \left( \frac{k}{2} \right) dk \right) \\ &= 2 \operatorname{Re} \left[ F \left( \chi_{[0, \infty[} F^{-1}(h) \right) (x) \right] + \tilde{h}(x), \end{aligned}$$

11 where

$$\tilde{h}(x) := -2 \operatorname{Re} \left[ F \left( \chi_{[0, 2k_0]} F^{-1}(h) \right) (x) \right] \in C^{\infty}(\mathbb{R})$$

- 1 as the Fourier transform of a compactly supported distribution. Furthermore,  
 2  $h \in L^1(\mathbb{R})$  gives that  $F^{-1}(h) \in L^\infty(\mathbb{R})$  and  $\chi_{[0,2k_0]}F^{-1}(h) \in L^1(\mathbb{R})$ . Invoking the  
 3 Riemann-Lebesgue lemma shows that  $\tilde{h}$  must vanish at infinity.

Next, recall that

$$\widehat{\chi_{[0,\infty[}}(x) = \sqrt{\frac{\pi}{2}}\delta_0 - \frac{i}{\sqrt{2\pi}}\text{p.v.}\frac{1}{x} \quad (10)$$

- 4 in the sense of distributions [8], where  $\chi_{[0,\infty[}$  is the Heaviside function,  $\text{p.v.}\frac{1}{x}$  is  
 5 the principal value distribution of  $\frac{1}{x}$  and  $\delta_0$  is the delta distribution. The Fourier  
 6 transform of the product  $\chi_{[0,\infty[}F^{-1}(h)$  equals

$$F(\chi_{[0,\infty[}F^{-1}(h))(x) = \frac{1}{\sqrt{2\pi}}(\widehat{\chi_{[0,\infty[}} * h)(x).$$

- 7 Finally, calculating the above convolution by using (10) gives

$$h_{\text{B},0}(x) = 2 \operatorname{Re} \left( \frac{1}{2}h - \frac{i}{2\pi}\text{p.v.}\frac{1}{x} * h \right) + \tilde{h}(x),$$

- 8 which simplifies to the claimed representation. □

- 9 It follows from Lemma 4.2 that

$$b(k) = b_j(k) + O\left(\frac{1}{k^{j+4}}\right), \quad j = 0, 1, \dots$$

as  $k \rightarrow +\infty$ . Since  $b_j$  is extended by zero onto the interval  $[0, k_0]$  then after multiplying by  $k^3$  the remainder term of  $b$  satisfies

$$\int_{2k_0}^{\infty} (1+k^2)^s \left( k^3 O\left(\frac{1}{k^{j+4}}\right) \right)^2 dk = \int_{2k_0}^{\infty} (1+k^2)^s O\left(\frac{1}{k^{2j+2}}\right) dk < \infty \quad (11)$$

- 10 if and only if  $s < j + \frac{1}{2}$ . It is well known that  $\hat{f} \in H^s(\mathbb{R})$  if and only if

$$\int_{-\infty}^{\infty} (1+|x|^2)^s |f(x)|^2 dx < \infty$$

- 11 and due to the Sobolev embedding theorem  $H^\alpha(\mathbb{R}) \subset \dot{C}(\mathbb{R})$ , when  $\alpha > \frac{1}{2}$ . The  
 12 above calculations mean that any  $O(k^{-5})$ -terms in  $b_j(k)$  correspond to continuous  
 13 terms in  $h_{\text{B},j}$  for all  $j = 1, 2, \dots$ . We have now obtained the following result.

- 14 **Lemma 4.5.** *Under Assumption 2.1 we have that  $h_{\text{B}} - h_{\text{B},j} \in H^s(\mathbb{R})$  for all  $s <$   
 15  $j + \frac{1}{2}$  and  $j = 0, 1, \dots$*

1 Let us add more regularity to the assumption about the non-linearities.

**Assumption 4.6.** Denote  $h_1(y) := q_1(y, 1)$  and  $h_0(y) := q_0(y, 1)$ . Suppose that the coefficients  $q_0$  and  $q_1$  have the following representations:

$$\begin{aligned} q_0(x, 1 + s) &= h_0(x) + q_0^*(x, s_0^*)s, \\ q_1(x, 1 + s) &= h_1(x) + q_1^*(x, 1)s + q_1^{**}(x, s_1^*)\frac{s^2}{2}, \end{aligned}$$

2 where  $|s_0^*|, |s_1^*| < |s|$ . Here we assume that  $h_1 \in W_1^1(\mathbb{R})$ ,  $q_1^*(x, 1) \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$   
3 for some  $p > 1$  and  $|q_0^*(x, s_0^*)| \leq h_0^*(x)$ ,  $|q_1^{**}(x, s_1^*)| \leq h_1^{**}(x)$  uniformly in  $|s| < s_0$   
4 for some  $0 < s_0 \leq 1$  and for some  $h_0^*, h_1^{**} \in L^1(\mathbb{R})$ .

Due to Corollary 2.4 the Maclaurin expansion  $(1 + \omega)^r = 1 + r\omega + O(\omega^2)$  shows that we may write  $|u_0 + u_{sc}| = 1 + \frac{1}{2}(u_0\overline{u_{sc}} + \overline{u_0}u_{sc}) + O(|u_{sc}|^2)$ . Combining Assumption 4.6 with this expansion allows us to linearize the coefficients as

$$\begin{aligned} q_0(x, |u_0 + u_{sc}|) &= h_0(x) + q_0^*(x, s_0^*)O(|u_{sc}|), \\ q_1(x, |u_0 + u_{sc}|) &= h_1(x) + \frac{1}{2}q_1^*(x, 1)(u_0\overline{u_{sc}} + \overline{u_0}u_{sc}) + \tilde{q}_1(x)O(|u_{sc}|^2), \end{aligned} \quad (12)$$

5 where  $\tilde{q}_1(x) = q_1^*(x, 1) + q_1^{**}(x, s_1^*)$ .

6 We now state a helpful result, which will be used in the proof of Lemma 4.8.

**Lemma 4.7.** Let  $\Phi \in L^1(\mathbb{R})$ . Then the functions defined by

$$x \mapsto e^{-kx} \int_{-\infty}^x e^{ky} \Phi(y) dy \quad \text{and} \quad x \mapsto e^{kx} \int_x^{\infty} e^{-ky} \Phi(y) dy \quad (13)$$

7 are bounded and vanish at  $|x| \rightarrow +\infty$  uniformly in  $k \geq k_0 > 0$ .

*Proof.* We consider only the left integral of (13), the right one can be analysed similarly. We start with large  $x > 0$  and split the integral as

$$\begin{aligned} \left| e^{-kx} \int_{-\infty}^x e^{ky} \Phi(y) dy \right| &= \left| e^{-kx} \int_{-\infty}^{\frac{1}{2}x} e^{ky} \Phi(y) dy + e^{-kx} \int_{\frac{1}{2}x}^x e^{ky} \Phi(y) dy \right| \\ &\leq e^{-\frac{1}{2}kx} \int_{-\infty}^{\frac{1}{2}x} |\Phi(y)| dy + \int_{\frac{1}{2}x}^x |\Phi(y)| dy. \end{aligned}$$

8 Since  $\Phi$  is integrable both of these terms vanish as  $x \rightarrow +\infty$  uniformly in  $k \geq k_0$ .

9 On the other hand, if  $x < 0$  is large in absolute value, then we have directly that

$$\left| e^{-kx} \int_{-\infty}^x e^{ky} \Phi(y) dy \right| \leq \int_{-\infty}^x |\Phi(y)| dy$$

10 and this integral vanishes as  $x \rightarrow -\infty$ . □

1 Only the integrability of  $\Phi$  uniformly in  $k \geq k_0$  is used in the above proof. This  
 2 means that we can let  $\Phi$  depend on  $k \geq k_0$  as long as it is dominated by some  
 3 integrable function which does not depend on  $k$ .

4 Now we are ready to prove our main result, that is, the precise calculation  
 5 of the second term of the inverse Born sequence. Heuristically, our aim is to  
 6 show that most parts of  $b_1(k/2)$  for  $k \geq 2k_0$  are  $O(k^{-5})$ , which by (11) shows  
 7 that the parts of  $h_{B,1}$  corresponding to them belong to  $H^s(\mathbb{R}) \subset \dot{C}(\mathbb{R})$  for all  
 8  $\frac{1}{2} < s < \frac{3}{2}$ . Additionally, the use of Assumption 4.6 results in a linearization that  
 9 yields precisely  $h_{B,0}$  and some additional terms that can be considered separately  
 10 by using properties of the Fourier transform.

11 **Lemma 4.8.** *Under Assumptions 2.1 and 4.6 the second term of the inverse Born*  
 12 *sequence is of the form*

$$h_{B,1}(x) = \operatorname{Re}(h)(x) + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\operatorname{Im}(h)(y)}{x-y} dy \pmod{\dot{C}(\mathbb{R})}.$$

*Proof.* When  $k \geq 2k_0$  we can write

$$\begin{aligned} b_1(k/2) &= -\frac{2i}{k^3} \int_{-\infty}^{\infty} e^{i\frac{k}{2}y} (q_1(y, |u_1|)u_1' + q_0(y, |u_1|)u_1) dy \\ &= -\frac{2i}{k^3} \int_{-\infty}^{\infty} e^{iky} \left( \frac{ikq_1(y, |u_1|)}{2} + q_0(y, |u_1|) \right) dy \\ &\quad - \frac{2i}{k^3} \int_{-\infty}^{\infty} e^{i\frac{k}{2}y} (q_1(y, |u_1|)(u_{\text{sc}}^{(1)})' + q_0(y, |u_1|)u_{\text{sc}}^{(1)}) dy =: I_1 + I_2, \end{aligned}$$

13 where (see Remark 2.3)

$$u_{\text{sc}}^{(1)}(y, k) = - \int_{-\infty}^{\infty} G_k^+(|y-z|) (q_1(z, 1)u_0' + q_0(z, 1)u_0) dz$$

14 and

$$(u_{\text{sc}}^{(1)})'(y, k) = \int_{-\infty}^{\infty} G_k'(y-z) (q_1(z, 1)u_0' + q_0(z, 1)u_0) dz.$$

15 It is easy to see that

$$\|u_{\text{sc}}^{(1)}\|_{\infty} \leq \frac{C}{k^2} \quad \text{and} \quad \|(u_{\text{sc}}^{(1)})'\|_{\infty} \leq \frac{C}{k}.$$

16 Consider first the integral  $I_2$ . By definition

$$I_2 = -\frac{2i}{k^3} \int_{-\infty}^{\infty} e^{i\frac{k}{2}y} q_1(y, |u_1|)(u_{\text{sc}}^{(1)})' dy - \frac{2i}{k^3} \int_{-\infty}^{\infty} e^{i\frac{k}{2}y} q_0(y, |u_1|)u_{\text{sc}}^{(1)} dy =: I_2' + I_2''.$$

1 Here  $I_2''$  is estimated by

$$|I_2''| \leq \frac{2}{k^3} \int_{-\infty}^{\infty} |q_0(y, |u_1|)| |u_{\text{sc}}^{(1)}| dy \leq \frac{C}{k^5}$$

for some  $C > 0$ . Then by using the explicit form of  $(u_{\text{sc}}^{(1)})'$  we write

$$\begin{aligned} I_2' &= \frac{1}{k^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{k}{2}(y+z)} G'_{\frac{k}{2}}(y-z) q_1(y, |u_1|) h_1(z) dy dz \\ &\quad - \frac{2i}{k^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{k}{2}(y+z)} G'_{\frac{k}{2}}(y-z) q_1(y, |u_1|) h_0(z) dy dz =: H_1 + H_2. \end{aligned}$$

Since  $|q_1(y, |u_1|)| \leq C_1(1 + \rho)\alpha_1(y)$  then the second integral  $H_2 = O(k^{-5})$ . Next we apply (12) to expand

$$\begin{aligned} H_1 &= \frac{1}{k^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{k}{2}(y+z)} G'_{\frac{k}{2}}(y-z) h_1(y) h_1(z) dy dz \\ &\quad + \frac{1}{2k^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{k}{2}(y+z)} G'_{\frac{k}{2}}(y-z) q_1^*(y, 1) h_1(z) \left( u_0 \overline{u_{\text{sc}}^{(1)}} + \overline{u_0} u_{\text{sc}}^{(1)} \right) dy dz \\ &\quad + \frac{1}{k^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{k}{2}(y+z)} G'_{\frac{k}{2}}(y-z) \tilde{q}_1(y) h_1(z) O(|u_{\text{sc}}^{(1)}|^2) dy dz =: H_1' + O(k^{-5}), \end{aligned}$$

2 where  $H_1'$  denotes the first integral and the rest are  $O(k^{-5})$ . The integral  $H_1'$   
3 contains some convenient symmetry. Let us denote

$$Q_k(y, z) := \frac{1}{k^2} G'_{\frac{k}{2}}(y-z) e^{i\frac{k}{2}(y+z)} h_1(y) h_1(z).$$

4 Then  $Q_k$  is (due to  $G'_k$ ) antisymmetric i.e.  $Q_k(y, z) = -Q_k(z, y)$  and is dominated  
5 by some function in  $L^1(\mathbb{R} \times \mathbb{R})$  uniformly in  $k \geq 2k_0$ . This allows us to conclude  
6 that

$$H_1' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_k(y, z) dy dz = 0.$$

It remains to consider the integral  $I_1$ . We use (12) to expand

$$\begin{aligned}
I_1 &= -\frac{2i}{k^3} \int_{-\infty}^{\infty} e^{iky} \left( \frac{ikq_1(y, |u_1|)}{2} + q_0(y, |u_1|) \right) dy \\
&= -\frac{2i}{k^3} \int_{-\infty}^{\infty} e^{iky} \left( \frac{ik}{2} h_1(y) + h_0(y) \right) dy \\
&\quad + \frac{1}{2k^2} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) \left( \overline{u_0 u_{sc}^{(1)}} + \overline{u_0} u_{sc}^{(1)} \right) dy \\
&\quad - \frac{2i}{k^3} \int_{-\infty}^{\infty} e^{iky} \left( q_0^*(y, s_0^*) O(|u_{sc}^{(1)}|) + \frac{ik\tilde{q}_1(y)}{2} O(|u_{sc}^{(1)}|^2) \right) dy \\
&=: b_0 \left( \frac{k}{2} \right) + J + O(k^{-5}),
\end{aligned}$$

- 1 where  $J$  denotes the second integral of this expansion. As in the proof of Lemma  
2 4.4 we have

$$b_0 \left( \frac{k}{2} \right) = \frac{2\sqrt{2\pi}}{ik^3} F^{-1}(h)(k),$$

- 3 when  $k \geq 2k_0$ . To conclude that  $J$  yields a continuous term in  $h_{B,1}$  we split the  
4 integral into two parts

$$J = \frac{1}{2k^2} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) u_0 \overline{u_{sc}^{(1)}} dy + \frac{1}{2k^2} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) \overline{u_0} u_{sc}^{(1)} dy =: J_1 + J_2.$$

Then  $J_1$  can further be split as

$$\begin{aligned}
J_1 &= \frac{i}{2k^4} \int_{-\infty}^{\infty} e^{i\frac{3k}{2}y} q_1^*(y, 1) \overline{\int_{-\infty}^{\infty} \left( ie^{i\frac{k}{2}|y-z|} - e^{-\frac{k}{2}|y-z|} \right) q_1(z, 1) e^{i\frac{k}{2}z} dz} dy \\
&\quad - \frac{1}{2k^2} \int_{-\infty}^{\infty} e^{i\frac{3k}{2}y} q_1^*(y, 1) \overline{\int_{-\infty}^{\infty} G_{\frac{k}{2}}^+(|y-z|) q_0(z, 1) e^{i\frac{k}{2}z} dz} dy \\
&= \frac{1}{2k^4} \int_{-\infty}^{\infty} e^{i\frac{3k}{2}y} q_1^*(y, 1) \int_{-\infty}^{\infty} e^{-i\frac{k}{2}|y-z|} \overline{h_1(z)} e^{-i\frac{k}{2}z} dz dy \\
&\quad - \frac{i}{2k^4} \int_{-\infty}^{\infty} e^{i\frac{3k}{2}y} q_1^*(y, 1) \int_{-\infty}^{\infty} e^{-\frac{k}{2}|y-z|} \overline{h_1(z)} e^{-i\frac{k}{2}z} dz dy + O(k^{-5}) \\
&=: J'_1 + J''_1 + O(k^{-5}).
\end{aligned}$$

Next the inner integral in  $J'_1$  is divided into two parts with respect to the region of integration on  $] -\infty, y]$  and  $]y, \infty[$  and then integrated by parts with respect to  $z$

to get

$$J'_1 = \frac{1}{2k^4} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) \overline{Q(y)} dy + \frac{1}{2ik^5} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) \overline{h_1(y)} dy \\ + \frac{1}{2ik^5} \int_{-\infty}^{\infty} e^{2iky} q_1^*(y, 1) \int_y^{\infty} e^{-ikz} \overline{h'_1(z)} dz dy =: \tilde{J}'_1(k) + O(k^{-5}),$$

1 where  $\tilde{J}'_1$  denotes the first integral and

$$Q(y) := \int_{-\infty}^y h_1(z) dz.$$

In the last step we used the fact that  $h_1 \in W_1^1(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  to conclude that the last two integrals are  $O(k^{-5})$ . Because  $Q \in L^\infty(\mathbb{R})$  then Assumption 4.6 about  $q_1^*(\cdot, 1)$  gives that  $q_1^*(\cdot, 1)Q \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$  and the Hausdorff-Young inequality (see, e.g., [7]) implies that  $k^4 \tilde{J}'_1(k) \in L^r([2k_0, \infty[)$  for some  $1 < r < \infty$ . Now Hölder's inequality gives that

$$k^3 \tilde{J}'_1(k) \in L^1([2k_0, \infty[). \quad (14)$$

Next, dividing the inner integral in  $J''_1$  into two parts and then integrating by parts with respect to  $z$  gives

$$J''_1(k) = -\frac{i}{(1-i)k^5} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) \overline{h_1(y)} dy \\ + \frac{i}{(1-i)k^5} \int_{-\infty}^{\infty} e^{i\frac{3k}{2}y} q_1^*(y, 1) e^{-\frac{k}{2}y} \int_{-\infty}^y e^{\frac{k}{2}y} e^{-i\frac{k}{2}z} \overline{h'_1(z)} dz dy \\ - \frac{i}{(1+i)k^5} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) \overline{h_1(y)} dy \\ - \frac{i}{(1+i)k^5} \int_{-\infty}^{\infty} e^{i\frac{3k}{2}y} q_1^*(y, 1) e^{\frac{k}{2}y} \int_y^{\infty} e^{-\frac{k}{2}y} e^{-i\frac{k}{2}z} \overline{h'_1(z)} dz dy.$$

2 The application of Lemma 4.7 in second and fourth integrals shows that all of the  
3 above integrals are  $O(k^{-5})$ .

The term  $J_2$  can be calculated similarly. We first split

$$J_2 = \frac{1}{2k^4} \int_{-\infty}^{\infty} e^{i\frac{k}{2}y} q_1^*(y, 1) \int_{-\infty}^{\infty} e^{i\frac{k}{2}|y-z|} h_1(z) e^{i\frac{k}{2}z} dz dy \\ + \frac{i}{2k^4} \int_{-\infty}^{\infty} e^{i\frac{k}{2}y} q_1^*(y, 1) \int_{-\infty}^{\infty} e^{-\frac{k}{2}|y-z|} h_1(z) e^{i\frac{k}{2}z} dz dy + O(k^{-5}) \\ =: J'_2(k) + J''_2(k) + O(k^{-5}).$$

As before, in  $J'_2$  we split the inner integral into two parts and integrate by parts with respect to  $z$  to get

$$J'_2 = \frac{1}{2k^4} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) Q(y) dy - \frac{1}{2ik^5} \int_{-\infty}^{\infty} e^{iky} q_1^*(y, 1) h_1(y) dy - \frac{1}{2ik^5} \int_{-\infty}^{\infty} q_1^*(y, 1) \int_y^{\infty} e^{ikz} h'_1(z) dz dy =: \tilde{J}'_2(k) + O(k^{-5}).$$

1 By the same argument as for  $\tilde{J}'_1$  we have that  $k^3 \tilde{J}'_2 \in L^1([2k_0, \infty[)$ . The term  $J''_2$   
 2 can be calculated similarly as  $J''_1$  to obtain another  $O(k^{-5})$  term.

3 To finish the proof we combine the results into the formula

$$b_1\left(\frac{k}{2}\right) = b_0\left(\frac{k}{2}\right) + \tilde{J}'_1(k) + \tilde{J}'_2(k) + O(k^{-5}), \quad k \geq 2k_0,$$

4 where  $k^3 \tilde{J}'_1(k)$  and  $k^3 \tilde{J}'_2(k)$  are in  $L^1([2k_0, \infty[)$ . Now using

$$h_{B,1}(x) = 2 \operatorname{Re} \left( \frac{i}{4\pi} \int_{2k_0}^{\infty} e^{-ikx} k^3 b_1\left(\frac{k}{2}\right) dk \right)$$

5 and Lemma 4.4 on the first term  $b_0$  gives the claimed explicit representation and an  
 6 additional  $\dot{C}$ -term. Since  $k^3 \tilde{J}'_1(k)$  and  $k^3 \tilde{J}'_2(k)$  are integrable outside the origin the  
 7 application of the Riemann-Lebesgue lemma shows that the corresponding parts of  
 8  $h_{B,1}$  are continuous and vanish at infinity. Finally, using (11) to the  $O(k^{-5})$ -part  
 9 of  $b_1$  completes the proof.  $\square$

10 **Theorem 4.9.** *The inverse Born approximation  $h_B$  of  $h$  is of the form*

$$h_B(x) = \operatorname{Re}(h)(x) + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\operatorname{Im}(h)(y)}{x-y} dy \quad (\text{mod } \dot{C}(\mathbb{R})).$$

11 *Proof.* Since  $b(k) = b_j(k) + O(1/k^{j+4})$ , then Definition 4.1 and (11) imply for  $j = 1$   
 12 that

$$h_B(x) = h_{B,1}(x) \quad (\text{mod } H^s(\mathbb{R}))$$

13 for all  $s < \frac{3}{2}$ . Then Lemma 4.8 gives the claimed representation.  $\square$

14 The preceding theorem already gives us some information about the unknown  
 15 coefficients  $q_0$  and  $q_1$ . The principal value integral appearing in Theorem 4.9 is  
 16 precisely the Hilbert transform of the imaginary part of  $h$ . By using the mapping  
 17 properties of the Hilbert transform (see, e.g., [3]) one can recover information about  
 18  $h$  even if the coefficients are complex-valued. If  $\operatorname{Im}(h)$  has some regularity in the  
 19 sense that  $\operatorname{Im}(h) \in H^r(\mathbb{R})$ ,  $r > \frac{1}{2}$ , we can use the Fourier transform and Sobolev  
 20 embedding theorems to prove the following result, which justifies our claim about  
 21 the solution to the inverse problem of recovering jumps and singularities of  $h$ .

1 **Corollary 4.10.** *Let  $q_0$  and  $q_1$  satisfy Assumptions 2.1 and 4.6. If  $\text{Im}(h) \in H^r(\mathbb{R})$*   
2 *for some  $r > \frac{1}{2}$  or if  $h$  is just real-valued, then the difference  $h_{\text{B}} - \text{Re}(h)$  is a*  
3 *continuous function. In particular, any jumps and singularities contained in  $\text{Re}(h)$*   
4 *can be recovered by calculating  $h_{\text{B}}$ .*

5 *Remark 4.11.* The non-linearity is only present in this recovery at point 1. Suppose  
6 that  $q_1(x, 1) = q_0(x, 1) \equiv 0$ . Then  $u_0(x, k) = e^{ikx}$  solves equation (4) uniquely and  
7 reflection coefficient  $b(k) \equiv 0$ , that is, there is no reflection. This shows that, e.g.  
8 non-linearities of type  $q(x, |u|) = f(x)(1 - |u|^p)$ ,  $p > 0$ , give trivial results regardless  
9 of  $f(x)$ . Also scaling  $u_0$  by a constant  $c > 0$  and doing appropriate changes to the  
10 assumptions through-out the text changes the reconstruction to correspond to the  
11 function  $h(x, c)$ .

## 12 Numerical example

13 Finally, to illustrate the method we give a numerical example. We consider the  
14 operator

$$Q_4 u = u^{(4)} + q_1 |u|^2 u' + q_0 \frac{|u|^2}{1 + |u|^2} u$$

15 with two non-linearities, first-order cubic non-linearity and zero-order saturation  
16 type non-linearity, both of which have applications in physics. Here we choose the  
17 scatterers

$$q_0(x) = \frac{i}{2} \chi_{[0, \frac{1}{2}]}(x), \quad q_1(x) = \begin{cases} x + \frac{5}{2}, & \text{when } -\frac{5}{2} < x < -2, \\ -x - \frac{3}{2}, & \text{when } -2 \leq x < -\frac{3}{2}, \\ 0, & \text{otherwise} \end{cases}$$

18 and note that  $q_1 \in W_1^1(\mathbb{R})$ ,  $q_0 \in L^1(\mathbb{R})$ , where  $q_0$  is pure imaginary when it is  
19 non-zero.

With these potentials we are able to calculate the required Hilbert transform in  
closed form, thus we expect to recover the (discontinuous and singular) combination

$$\begin{aligned} \text{Re}(h)(x) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im}(h)(y)}{x - y} dy \\ = -\frac{1}{2} \left( \chi_{[-\frac{5}{2}, -2]}(x) - \chi_{[-2, -\frac{3}{2}]}(x) \right) + \frac{1}{4\pi} \log \left( \frac{|x|}{|x - \frac{1}{2}|} \right). \end{aligned} \quad (15)$$

20 The numerical computation is done by following [10]. Figure 1 presents the un-  
21 known coefficients with the red line indicating the real coefficient  $q_1$  and the black  
22 dashed line indicating the imaginary part  $q_0$ . Figure 2 depicts the numerically  
23 reconstructed combination of the coefficients in the red line compared with the

1 actual unknown value (15) plotted in the black dashed line. This example shows  
2 that numerically we are able to recover the jumps and singularities in non-linear  
3 complex coefficients quite accurately.

## 4 **Conclusion**

5 The direct and inverse scattering problems for the first-order quasi-linear pertur-  
6 bation of the one-dimensional biharmonic operator in the frequency domain with  
7 singular coefficients (in the space coordinate) were considered. It is assumed that  
8 the non-linearities depend on the modulus of the wave and that they are complex-  
9 valued. The linear case and many well-known (in physics) types of non-linearities  
10 are included in the considerations. Under some additional regularity conditions  
11 (Lipschitz with certain Taylor-type expansions) for the non-linearities the classical  
12 inverse scattering Born approximation is justified for this non-linear operator of  
13 order 4.

14 Note that in principle one could consider many types of data, namely (at least)  
15 the transmission, reflection and connection coefficients in both directions  $\pm\infty$  on  
16 the line. For our purposes only part of this data alone is sufficient for recovering es-  
17 sential information about the unknown coefficients of our operator. More precisely,  
18 the Born approximation only requires the reflection coefficient (which corresponds  
19 to the asymptotic of the reflected wave at  $-\infty$ ) to be known for arbitrarily high  
20 frequencies. Under this limited data we proved the recovery of the jump discon-  
21 tinuities and infinite singularities of the coefficients in the space coordinate when  
22 the non-linearity is at point 1. These results generalize the well-known results for  
23 the linear and non-linear Schrödinger operator on the line. The considered method  
24 (Born approximation) also has natural generalization for the multi-dimensional  
25 case.

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## References

- 2 [1] Pausader B. Scattering for the beam equation in low dimensions. 2009.  
3 arXiv:0903.3777v2 [math.AP].
- 4 [2] Zeidler E. *Applied functional analysis*. Springer-Verlag, New York, 1995.
- 5 [3] Titchmarsh E.C. *Introduction to the Theory of Fourier Integrals*. Oxford  
6 University Press, 1948.
- 7 [4] Gazzola F., Grunau H.-C., and Sweers G. *Polyharmonic Boundary Value*  
8 *Problems*. Springer-Verlag, Berlin Heidelberg, 2010.
- 9 [5] Iwasaki K. Scattering theory for 4th order differential operators: I. *Japan. J.*  
10 *Math.*, 14:1–57, 1988.
- 11 [6] Iwasaki K. Scattering theory for 4th order differential operators: II. *Japan.*  
12 *J. Math.*, 14:59–96, 1988.
- 13 [7] Hörmander L. *The Analysis of Linear Partial Differential Operators: Dis-*  
14 *tribution Theory and Fourier Analysis*. Springer-Verlag, Berlin Heidelberg,  
15 2003.
- 16 [8] Kanwal R. *Generalized Functions: Theory and Technique*. Academic Press,  
17 New York, 1983.
- 18 [9] Aktosun T. and Papanicolaou V. G. Time evolution of the scattering data for  
19 a fourth-order linear differential operator. *Inverse Problems*, 24, 2008.
- 20 [10] Tyni T., Harju M., and Serov V. Recovery of singularities in fourth-order  
21 operator on the line from limited data. *Inverse Problems*, 32:075001, 2016.
- 22 [11] T. Tyni and M. Harju. Inverse backscattering problem for perturbations of  
23 biharmonic operator. *Inverse Problems*, 33:105002, 2017.
- 24 [12] T. Tyni and V. Serov. Scattering problems for perturbations of the multidimensional  
25 biharmonic operator. *Inverse Prob. Imag.*, 12(1):205–227, 2018.
- 26 [13] Serov V. An inverse Born approximation for the general nonlinear Schrödinger  
27 operator on the line. *Journal of Physics A: Mathematical and General*,  
28 42(33):332002, 2009.
- 29 [14] Serov V., Harju M., and Fotopoulos G. Direct and inverse scattering for  
30 nonlinear Schrödinger equation in 2d. *Journal of mathematical physics*, 53,  
31 2012.

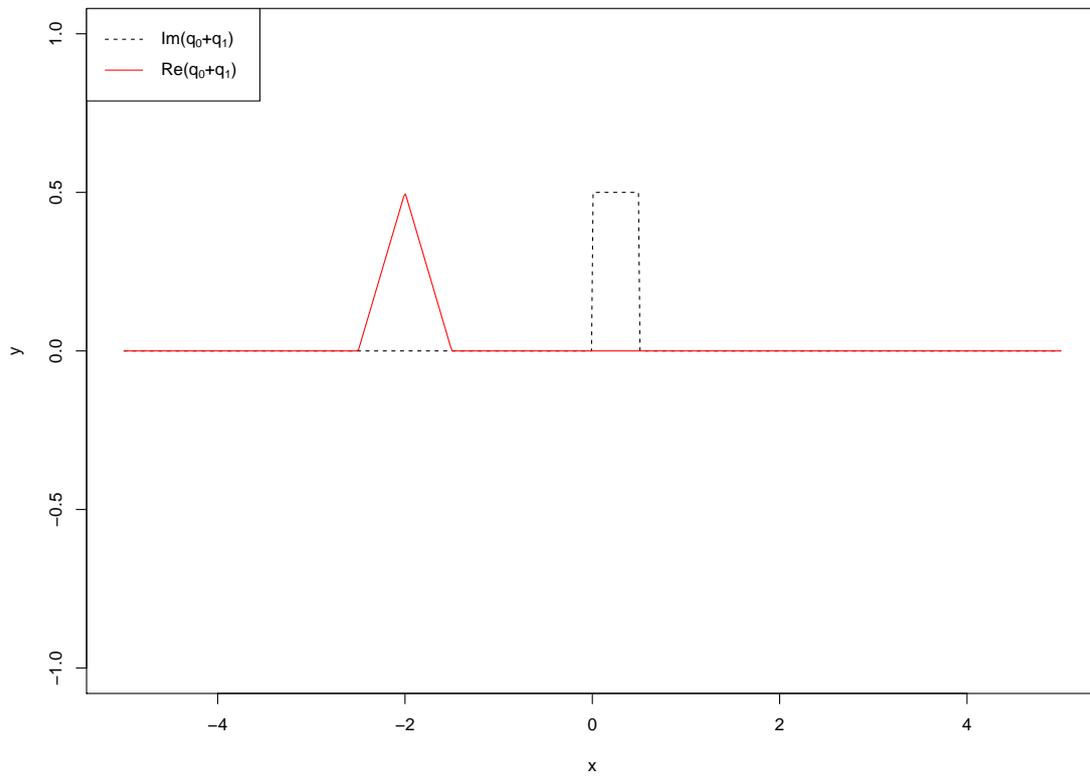


Figure 1: Coefficients  $q_1$  and  $q_0$

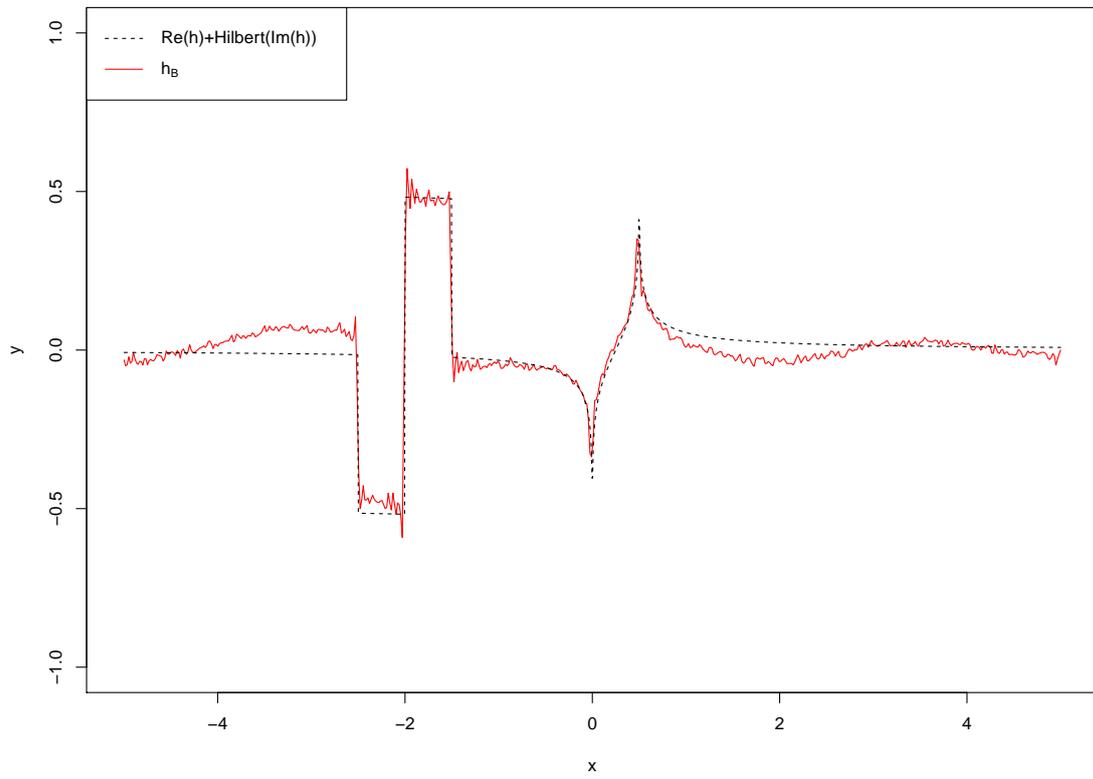


Figure 2: Numerical reconstruction (red) and the unknown combination (black)