Recovery of singularities in fourth order operator on the line from limited data

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Abstract

We consider a differential operator of order four with three coefficients which may be complex valued. We prove that the Born approximation can be used efficiently to recover essential information about a combination of the coefficients from the knowledge of the reflection coefficient. Numerical examples illustrate the feasibility of this method.

1 Introduction

Let us consider the one-dimensional operator of order four

$$L_4 u(x) := \frac{\mathrm{d}^4}{\mathrm{d}x^4} u(x) + (\alpha(x) u'(x))' + q(x) u'(x) + V(x) u(x) \tag{1}$$

with complex-valued coefficients α, q and V from function spaces that are specified later. We study the operator L_4 from the point of view of scattering problems. Operators of order four can be used for example in the theory of vibrations of beams and the study of elasticity, while for example operators of order two, such as the linear Schrödinger operator

$$L_2 u := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} u + qu$$

can be used to model scattering of particles.

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An operator similar to L_4 has been studied before with real-valued coefficients and we refer the reader to the studies by Aktosun and Papanicolaou [1] and Iwasaki [4, 5]. In those studies all the coefficients were assumed to be real-valued with the choice $q \equiv 0$. The analysis was carried out for complex k from the sector $\arg(k) \in \left[0, \frac{\pi}{4}\right]$ with $k \neq 0$. The coefficients α, α' and V were assumed to be integrable and to have special exponential type of decay [4] as $x \to \pm\infty$.

Iwasaki [5] defined the inverse problem for L_4 as a Riemann–Hilbert boundary value problem and was able to provide a uniqueness theorem for the problem. He assumed that the operator has no spectral or non-spectral [4] singularities and no negative eigenvalues. Under these assumptions he showed that given the so called reflection and connection coefficients R_+ and C_+ for all k in the rays $\arg(k) = 0$ and $\arg(k) = \frac{\pi}{4}$ respectively, then it is possible to uniquely recover the coefficients α and V of the operator L_4. Aktosun and Papanicolaou [1] on the other hand studied the direct problem of the relation of the time-evolving coefficients to the corresponding time-evolving scattering data, and our present work is motivated by this article.

The direct scattering problem for operator L_4 concerns finding the solution to the differential equation

$$L_4 u = k^4 u, \quad u = u_0 + u_{\mathrm{sc}}, \quad u_0(x,k) = \mathrm{e}^{\mathrm{i}kx}, \tag{2}$$

where u_0 is an in-coming plane-wave and u_{sc} the out-going (see below) scattered wave. By applying the definition of operator L_4 and rearranging the terms we obtain the new equation

$$L_0 u := u^{(4)} - k^4 u = -(\alpha u')' - q u' - V u, \tag{3}$$

where $-L_0$ is the bi-Laplacian Helmholtz operator. An out-going fundamental solution to operator L_0 is given by

$$G_k^+(|x|) = \frac{1}{4|k|^3}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}|k||x|} - \mathrm{e}^{-|k||x|}\right).$$

It is readily checked that for $k > 0$

$$\left(\frac{\partial}{\partial |x|} - \mathrm{i}k\right) G_k^+(|x|) = \frac{\mathrm{i}-1}{4k^2}\mathrm{e}^{-k|x|} = o(1), \quad |x| \to \infty,$$

which can be interpreted as a radiation condition for G_k^+. Further, when $k > 0$ we can write equation (3) as an integral equation in form

$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} - \int_{-\infty}^{\infty} G_k^+(|x-y|)((\alpha(y)u'(y,k))' + q(y)u'(y,k) + V(y)u(y,k))\mathrm{d}y. \tag{4}$$

We are looking for physical scattering solutions, so we assume that the solution is bounded i.e. $u, u' \in L^\infty(\mathbb{R})$. Furthermore, as it turns out in Section 3, the solution u has the asymptotic behaviour

$$u(x,k) = a(k)\mathrm{e}^{\mathrm{i}kx} + o(1), \quad x \to +\infty,$$
$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} + b(k)\mathrm{e}^{-\mathrm{i}kx} + o(1), \quad x \to -\infty,$$

where we call the functions $a(k)$ and $b(k)$ the transmission and reflection coefficients, respectively. We regard this asymptotic behaviour as the radiation condition for u_{sc}.

We formulate the inverse problem for equation (4) as follows: find the main singularities of β in (5), given the coefficient $b(k)$ for all $k \in \mathbb{R}$ large enough in absolute value.

Our objective is to explore the inverse Born approximation method for integral equation (4). We follow the procedure of Serov and Harju [7], where they applied the Born approximation to the linear Schrödinger operator on the line. It turns out that, given only the data $b(k)$, we can recover some essential information about the coefficients in form of

$$\beta := V - \frac{1}{2}q' - \frac{1}{4}\alpha''. \tag{5}$$

What is more, we do not need to know the reflection coefficient $b(k)$ for all $k \in \mathbb{R}$, just for k which is large enough in absolute value. Actually the Born approximation q_{B} is (up to some constants) just the Fourier transform of $k^3 b(k/2)$. Furthermore, we do not require as harsh decay at infinity from the coefficients as in [1, 4, 5], instead, for our purposes it is enough to assume that $V \in L^1(\mathbb{R})$ and that q and α both are L^1- and L^2-integrable up to the first and second order derivatives respectively. With this type of limited data, while we do not have uniqueness results, we avoid the handling of the problematic neighbourhood of $k = 0$ and are still able to recover essential information about β.

We use the following notations in this paper. The Fourier transform of a function (or distribution) f is defined by

$$\widehat{f}(\xi) := F(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi} f(x)\mathrm{d}x$$

and the inverse Fourier transform

$$F^{-1}(f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\xi} f(\xi)\mathrm{d}\xi,$$

and they satisfy $F(F^{-1}) = F^{-1}(F) = I$. The Sobolev space $W_p^s(\mathbb{R})$, $0 \leq s$, $1 \leq p < \infty$, is the space of those tempered distributions for which the norm

$$\|f\|_{W_p^s(\mathbb{R})} := \left(\int_{-\infty}^{\infty} \left| F^{-1}((1+|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi)) \right|^p \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Let $\|\cdot\|_p$ denote the usual L^p-norm for $1 \leq p \leq \infty$ and write $H^s(\mathbb{R}) := W_2^s(\mathbb{R})$ for the L^2-based Sobolev space. Finally we define the space of continuous functions vanishing at infinity by

$$\dot{C}(\mathbb{R}) := \{ f \in C(\mathbb{R}) \mid f(x) \to 0 \text{ as } x \to \pm\infty \}.$$

In Section 3 we show how the reflection coefficient $b(k)$ can be derived as part of the asymptotic of the solution u when $x \to -\infty$. More precisely, we can write $k_0 := \|\alpha\|_1 + \|q\|_1 + \|V\|_1$ and define a new function $\chi(k)$ by

$$\chi(k) = \begin{cases} 1, & \text{when } |k| \geq k_0, \\ 0, & \text{when } |k| < k_0. \end{cases}$$

With help of $\chi(k)$ we can define the reflection coefficient $b(k)$ for all $k \in \mathbb{R}$ by

$$b(k) = \chi(k) \begin{cases} -\frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky} \left[k\alpha u' + \mathrm{i}(qu' + Vu) \right] \mathrm{d}y, & \text{when } k > 0, \\ -\frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky} \left[k\overline{\alpha} u' + \mathrm{i}(\overline{q} u' + \overline{V} u) \right] \mathrm{d}y, & \text{when } k < 0. \end{cases}$$

Then the inverse Born approximation q_{B} of β is defined by

$$q_{\mathrm{B}}(\xi) = F\left(\frac{\mathrm{i}k^3}{2\sqrt{2\pi}} b\left(\frac{k}{2}\right) \right)(\xi)$$

in the sense of distributions.

Our main results can be summarized in the following theorem and its corollaries.

Theorem 1.1. *If $\alpha \in H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$, $q \in H^1(\mathbb{R}) \cap W_1^1(\mathbb{R})$ and $V \in L^1(\mathbb{R})$ then the inverse Born approximation q_{B} satisfies*

$$q_{\mathrm{B}}(x) = \mathrm{Re}(\beta)(x) + \frac{1}{\pi}\mathrm{p.v.} \int_{-\infty}^{\infty} \frac{\mathrm{Im}(\beta)(y)}{x-y} \mathrm{d}y + \widetilde{q}(x) + q_{\mathrm{rest}}(x),$$

where $\widetilde{q} \in \dot{C}(\mathbb{R})$ and $q_{\mathrm{rest}} \in H^s(\mathbb{R})$ for all $s < \frac{1}{2}$.

Corollary 1.2. *If α, q and V are as in Theorem 1.1 and in addition are real-valued, then the difference*

$$q_{\mathrm{B}} - \beta \in H^s(\mathbb{R}) + \dot{C}(\mathbb{R})$$

for all $s < \frac{1}{2}$.

Corollary 1.3. *If α, q and V are as in Theorem 1.1 and* $\mathrm{Im}(\beta) \in H^r(\mathbb{R})$ *for* $r > \frac{1}{2}$, *then the difference*

$$q_{\mathrm{B}} - \mathrm{Re}(\beta) \in H^s(\mathbb{R}) + \dot{C}(\mathbb{R}),$$

for all $s < \frac{1}{2}$.

According to Corollary 1.2, if α and q are smooth enough, then we can recover any local L^p-singularities of the coefficient V for $1 \leq p < \infty$, since globally we have the embedding $W_2^s(\mathbb{R}) \subset L^p(\mathbb{R})$, where $p = \frac{2}{1-2s}$ and $0 \leq s < \frac{1}{2}$. Similarly, Corollary 1.3 says that if the imaginary part of β is smooth enough, then we can recover the local L^p-singularities of the real part of β.

Remark 1.4. We can also consider the operator L_4 in the form

$$L_4 u(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4} u(x) + \alpha(x) u''(x) + q(x) u'(x) + V(x) u(x),$$

that is, not divergence form. Due to the equality $\alpha u'' + q u' = (\alpha u')' + (q - \alpha')u'$ the results for this operator are the same, except for β which in this case is given by the formula

$$\beta = V - \frac{1}{2} q' + \frac{1}{4} \alpha''.$$

In general, operator L_4 need not be self-adjoint.

This paper is organized as follows. In Section 2 we show some preliminary results regarding the existence and uniqueness of the solution u. We also give some estimates on the behaviour of the solution. In Section 3 we study the asymptotic behaviour of the solution and define the reflection coefficient $b(k)$ that is used as the data for the inverse problem. Section 4 is devoted to defining the inverse Born approximation and studying its properties in detail. More precisely, we prove Theorem 1.1 about the representation of q_{B}. Finally, in Section 5 we discuss the numerical computation of the Born approximation and give examples to illustrate this method.

2 Preliminaries

Let us investigate the first term of the integral in (4) under the assumption that $\alpha \in W_1^1(\mathbb{R})$. Dividing this integral into two parts with respect to x and integrating by parts once yields

$$\int_{-\infty}^{\infty} G_k^+(|x-y|)(\alpha(y)u'(y,k))'\mathrm{d}y = \int_{-\infty}^{x} \frac{1}{4k^3}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}k(x-y)} - \mathrm{e}^{-k(x-y)}\right)(\alpha u')'\mathrm{d}y$$
$$+\int_{x}^{\infty} \frac{1}{4k^3}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}k(y-x)} - \mathrm{e}^{-k(y-x)}\right)(\alpha u')'\mathrm{d}y$$
$$= -\int_{-\infty}^{x} \frac{1}{4k^2}\left(\mathrm{e}^{\mathrm{i}k(x-y)} - \mathrm{e}^{-k(x-y)}\right)\alpha u'\mathrm{d}y$$
$$+\int_{x}^{\infty} \frac{1}{4k^2}\left(\mathrm{e}^{-\mathrm{i}k(x-y)} - \mathrm{e}^{k(x-y)}\right)\alpha u'\mathrm{d}y$$
$$= -\int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(\mathrm{e}^{\mathrm{i}k|x-y|} - \mathrm{e}^{-k|x-y|}\right)\alpha u'\mathrm{d}y.$$

Then we obtain the integral equation

$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} + \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(\mathrm{e}^{\mathrm{i}k|x-y|} - \mathrm{e}^{-k|x-y|}\right)\alpha u'\mathrm{d}y$$
$$- \int_{-\infty}^{\infty} G_k^+(|x-y|)(q(y)u'(y,k) + V(y)u(y,k))\mathrm{d}y. \quad (6)$$

We will show that equation (6) can be solved by iterations defined as

$$u_j(x,k) = \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(\mathrm{e}^{\mathrm{i}k|x-y|} - \mathrm{e}^{-k|x-y|}\right)\alpha u_{j-1}'\mathrm{d}y$$
$$- \int_{-\infty}^{\infty} G_k^+(|x-y|)\left(qu_{j-1}' + Vu_{j-1}\right)\mathrm{d}y, \quad j = 1,2,\ldots \quad (7)$$

A straight-forward differentiation gives

$$u_j'(x,k) = \int_{-\infty}^{\infty} \frac{1}{4k}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}k|x-y|} + \mathrm{e}^{-k|x-y|}\right)\alpha u_{j-1}'\mathrm{d}y$$
$$+ \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(\mathrm{e}^{\mathrm{i}k|x-y|} - \mathrm{e}^{-k|x-y|}\right)(qu_{j-1}' + Vu_{j-1})\mathrm{d}y, \quad j = 1,2,\ldots \quad (8)$$

When $k < 0$ we define $u(x,k) := \overline{u(x,-k)}$ and $u'(x,k) := \overline{u'(x,-k)}$. In this case the integral equation becomes

$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} + \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(\mathrm{e}^{\mathrm{i}k|x-y|} - \mathrm{e}^{k|x-y|}\right)\overline{\alpha} u'\mathrm{d}y$$
$$- \int_{-\infty}^{\infty} \frac{1}{4k^3}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}k|x-y|} + \mathrm{e}^{k|x-y|}\right)\left(\overline{q}u' + \overline{V}u\right)\mathrm{d}y$$

and the iterations are

$$u_j(x,k) = \int_{-\infty}^{\infty} \frac{\mathrm{sgn}(x-y)}{4k^2}\left(\mathrm{e}^{\mathrm{i}k|x-y|} - \mathrm{e}^{k|x-y|}\right)\overline{\alpha} u'_{j-1}\mathrm{d}y$$
$$- \int_{-\infty}^{\infty} \frac{1}{4k^3}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}k|x-y|} + \mathrm{e}^{k|x-y|}\right)\left(\overline{q}u'_{j-1} + \overline{V}u_{j-1}\right)\mathrm{d}y, \quad j = 1,2,\dots$$

In both cases $u_0(x,k) = \mathrm{e}^{\mathrm{i}kx}$ as before. These iterations satisfy several properties that are listed in the following lemmata.

Lemma 2.1. *If $\alpha \in W_1^1(\mathbb{R})$ and $q, V \in L^1(\mathbb{R})$ then the iterations satisfy the estimates*

$$\|u_j\|_\infty \leq \left(\frac{C_0}{|k|}\right)^j, \quad j = 0,1,\dots$$

and

$$\|u'_j\|_\infty \leq k\left(\frac{C_0}{|k|}\right)^j, \quad j = 0,1,\dots,$$

where $C_0 := \frac{1}{2}(\|\alpha\|_1 + \|q\|_1 + \|V\|_1)$, when $|k| \geq 1$.

Proof. We prove the claim by induction with respect to both u_j and u'_j simultaneously. Clearly the claim holds for $j = 0$. Let us assume that the estimates

$$\|u_{j-1}\|_\infty \leq \left(\frac{C_0}{|k|}\right)^{j-1}$$

and

$$\|u'_{j-1}\|_\infty \leq |k|\left(\frac{C_0}{|k|}\right)^{j-1},$$

hold for some $j \geq 1$. Next we estimate

$$\begin{aligned}
|u_j(x,k)| &\leq \frac{1}{2k^2}\int_{-\infty}^{\infty} |\alpha||u'_{j-1}|\mathrm{d}y + \frac{1}{2|k|^3}\int_{-\infty}^{\infty}\left(|q||u'_{j-1}| + |V||u_{j-1}|\right)\mathrm{d}y \\
&\leq \frac{1}{2k^2}\|\alpha\|_1\|u'_{j-1}\|_\infty + \frac{1}{2|k|^3}\left(\|q\|_1\|u'_{j-1}\|_\infty + \|V\|_1\|u_{j-1}\|\right) \\
&\leq \left(\frac{\|\alpha\|_1}{2} + \frac{\|q\|_1}{2|k|} + \frac{\|V\|_1}{2k^2}\right)\frac{C_0^{j-1}}{|k|^j} \leq \left(\frac{C_0}{|k|}\right)^j.
\end{aligned}$$

Similarly

$$
\begin{aligned}
|u_j'(x,k)| &\leq \frac{1}{2|k|}\|\alpha\|_1\|u_{j-1}'\|_\infty + \frac{1}{2k^2}\left(\|q\|_1\|u_{j-1}'\|_\infty + \|V\|_1\|u_{j-1}\|_\infty\right) \\
&\leq \left(\frac{\|\alpha\|_1}{2} + \frac{\|q\|_1}{2|k|} + \frac{\|V\|_1}{2k^2}\right)\frac{C_0^{j-1}}{|k|^{j-1}} \leq |k|\left(\frac{C_0}{|k|}\right)^j .
\end{aligned}
$$

The claim follows. $\square$

Lemma 2.2. *Let $\alpha \in W_1^1(\mathbb{R})$ and $q, V \in L^1(\mathbb{R})$. The function $u(x,k)$ defined by the series*

$$
u(x,k) = \sum_{j=0}^{\infty} u_j(x,k) \tag{9}
$$

converges uniformly, when $|k| \geq 1$ and $|k| > C_0$ and is the unique solution to (6).

Proof. Lemma 2.1 shows us that the series (9) can be estimated by a geometric series

$$
\left|\sum_{j=0}^{\infty} u_j(x,k)\right| \leq \sum_{j=0}^{\infty}\left(\frac{C_0}{|k|}\right)^j ,
$$

which converges uniformly in x. Uniform convergence permits us to differentiate the series term by term and see that the series of u' also converges uniformly. Furthermore, we can change the order of integration and summation, so that substituting the series into the right-hand side of (6) shows that (9) is a solution to (6).

Let us assume that v is another solution to (6) that also satisfies the form $v = u_0 + v_{\text{sc}}$. By linearity of the integrals we can estimate the integrals corresponding to the scattering solutions in (6) as

$$
\begin{aligned}
|v - u| + |v' - u'| = |v_{\text{sc}} - u_{\text{sc}}| &+ |v_{\text{sc}}' - u_{\text{sc}}'| \\
&\leq \frac{1}{2k^2}\|\alpha\|_1\|v' - u'\|_\infty \\
&\quad + \frac{1}{2|k|^3}\left(\|q\|_1\|v' - u'\|_\infty + \|V\|_1\|v - u\|_\infty\right) \\
&\quad + \frac{1}{2|k|}\|\alpha\|_1\|v' - u'\|_\infty \\
&\quad + \frac{1}{2k^2}\left(\|q\|_1\|v' - u'\|_\infty + \|V\|_1\|v - u\|_\infty\right) \\
&\leq \frac{C_0}{|k|}\left(\|v - u\|_\infty + \|v' - u'\|_\infty\right).
\end{aligned}
$$

Then, by taking the supremum of the left-hand side with respect to x, we obtain

$$\|v - u\|_\infty + \|v' - u'\|_\infty \leq \frac{C_0}{|k|} \left(\|v - u\|_\infty + \|v' - u'\|_\infty\right).$$

Here, the constant $\frac{C_0}{|k|} < 1$, which means that $v = u$ concluding the proof. $\square$

Lemma 2.3. *Let $\alpha \in W_1^1(\mathbb{R})$ and $q, V \in L^1(\mathbb{R})$. The tails of the series representations of u and u' satisfy*

$$\sum_{j=m}^{\infty} |u_j(x,k)| \leq \frac{2C_0^m}{|k|^m},$$
$$\sum_{j=m}^{\infty} |u_j'(x,k)| \leq \frac{2C_0^m}{|k|^{m-1}},$$

when $m = 0, 1, \ldots$ and $|k| > 2C_0$.

Proof. We have

$$\sum_{j=m}^{\infty} |u_j(x,k)| \leq \sum_{j=m}^{\infty} \left(\frac{C_0}{|k|}\right)^j = \left(\frac{C_0}{|k|}\right)^m \sum_{j=0}^{\infty} \left(\frac{C_0}{|k|}\right)^j$$
$$= \left(\frac{C_0}{|k|}\right)^m \frac{1}{1 - \frac{C_0}{|k|}} = \left(\frac{C_0}{|k|}\right)^m \frac{|k|}{|k| - C_0} \leq \frac{2C_0^m}{|k|^m}.$$

The only difference for u' is an additional k in the numerator. $\square$

3 Asymptotics for the solution u

In what follows we assume, that $\alpha \in W_1^2(\mathbb{R})$, $q \in W_1^1(\mathbb{R})$ and $V \in L^1(\mathbb{R})$. These assumptions are sufficient for the moment, but later we need to require some stronger conditions.

First we check how the solution behaves for large values of $|x|$. When $k > 0$ we have

$$
\begin{aligned}
u(x,k) &= \mathrm{e}^{\mathrm{i}kx} + \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(\mathrm{e}^{\mathrm{i}k|x-y|} - \mathrm{e}^{-k|x-y|}\right)\alpha u' \mathrm{d}y \\
&\quad - \int_{-\infty}^{\infty} \frac{1}{4k^3}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}k|x-y|} - \mathrm{e}^{-k|x-y|}\right)(qu' + Vu)\,\mathrm{d}y \\
&= \mathrm{e}^{\mathrm{i}kx} + \int_{-\infty}^{x} \frac{1}{4k^2}\mathrm{e}^{\mathrm{i}k(x-y)}\alpha u' \mathrm{d}y - \int_{x}^{\infty} \frac{1}{4k^2}\mathrm{e}^{\mathrm{i}k(y-x)}\alpha u' \mathrm{d}y \\
&\quad - \int_{-\infty}^{x} \frac{1}{4k^2}\mathrm{e}^{-k(x-y)}\alpha u' \mathrm{d}y + \int_{x}^{\infty} \frac{1}{4k^2}\mathrm{e}^{-k(y-x)}\alpha u' \mathrm{d}y \\
&\quad - \int_{-\infty}^{x} \frac{\mathrm{i}}{4k^3}\mathrm{e}^{\mathrm{i}k(x-y)}(qu' + Vu)\mathrm{d}y - \int_{x}^{\infty} \frac{\mathrm{i}}{4k^3}\mathrm{e}^{\mathrm{i}k(y-x)}(qu' + Vu)\mathrm{d}y \\
&\quad + \int_{-\infty}^{x} \frac{1}{4k^3}\mathrm{e}^{-k(x-y)}(qu' + Vu)\mathrm{d}y + \int_{x}^{\infty} \frac{1}{4k^3}\mathrm{e}^{-k(y-x)}(qu' + Vu)\mathrm{d}y.
\end{aligned}
$$

Next we split the first and fifth integral as $\int_{-\infty}^{x} = \int_{-\infty}^{\infty} - \int_{x}^{\infty}$ to obtain

$$
\begin{aligned}
u(x,k) = \mathrm{e}^{\mathrm{i}kx} &+ \frac{\mathrm{e}^{\mathrm{i}kx}}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ky} k\alpha u' \mathrm{d}y - \frac{\mathrm{e}^{\mathrm{i}kx}}{4k^3}\int_{x}^{\infty} \mathrm{e}^{-\mathrm{i}ky} k\alpha u' \mathrm{d}y \\
&- \frac{\mathrm{e}^{-\mathrm{i}kx}}{4k^3}\int_{x}^{\infty} \mathrm{e}^{\mathrm{i}ky} k\alpha u' \mathrm{d}y \\
&- \frac{\mathrm{e}^{-kx}}{4k^3}\int_{-\infty}^{x} \mathrm{e}^{ky} k\alpha u' \mathrm{d}y + \frac{\mathrm{e}^{kx}}{4k^3}\int_{x}^{\infty} \mathrm{e}^{-ky} k\alpha u' \mathrm{d}y \\
&- \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}kx}}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ky}(qu' + Vu)\mathrm{d}y + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}kx}}{4k^3}\int_{x}^{\infty} \mathrm{e}^{-\mathrm{i}ky}(qu' + Vu)\mathrm{d}y \\
&- \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}kx}}{4k^3}\int_{x}^{\infty} \mathrm{e}^{\mathrm{i}ky}(qu' + Vu)\mathrm{d}y \\
&+ \frac{\mathrm{e}^{-kx}}{4k^3}\int_{-\infty}^{x} \mathrm{e}^{ky}(qu' + Vu)\mathrm{d}y + \frac{\mathrm{e}^{kx}}{4k^3}\int_{x}^{\infty} \mathrm{e}^{-ky}(qu' + Vu)\mathrm{d}y.
\end{aligned}
$$

Since u and u' are bounded and α, q and V are integrable we are left with only the first and sixth integrals, while the other integrals tend to 0 as $x \to +\infty$. Similarly, if $x \to -\infty$ then we are left with just the third and eighth integrals, the other integrals tend to 0. Thus, u behaves asymptotically like

$$
u(x,k) = \mathrm{e}^{\mathrm{i}kx} + \frac{\mathrm{e}^{\mathrm{i}kx}}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ky}\left[k\alpha u' - \mathrm{i}(qu' + Vu)\right]\mathrm{d}y + o(1),
$$

when $x \to +\infty$ and

$$
u(x,k) = \mathrm{e}^{\mathrm{i}kx} - \frac{\mathrm{e}^{-\mathrm{i}kx}}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\left[k\alpha u' + \mathrm{i}(qu' + Vu)\right]\mathrm{d}y + o(1),
$$

when $x \to -\infty$. Similar calculation shows that if $k < 0$ then u admits asymptotically the representations

$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} + \frac{\mathrm{e}^{\mathrm{i}kx}}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ky} \left[k\overline{\alpha} u' - \mathrm{i}(\overline{q}u' + \overline{V}u) \right] \mathrm{d}y + o(1),$$

when $x \to +\infty$ and

$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} - \frac{\mathrm{e}^{-\mathrm{i}kx}}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky} \left[k\overline{\alpha} u' + \mathrm{i}(\overline{q}u' + \overline{V}u) \right] \mathrm{d}y + o(1),$$

when $x \to -\infty$. These asymptotic relations allow us to define the transmission and reflection coefficients $a(k)$ and $b(k)$ as follows

$$a(k) = \begin{cases} 1 + \frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ky} \left[k\alpha u' - \mathrm{i}(qu' + Vu) \right] \mathrm{d}y, & \text{when } k > 0, \\ 1 + \frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ky} \left[k\overline{\alpha} u' - \mathrm{i}(\overline{q}u' + \overline{V}u) \right] \mathrm{d}y, & \text{when } k < 0 \end{cases}$$

and

$$b(k) = \begin{cases} -\frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky} \left[k\alpha u' + \mathrm{i}(qu' + Vu) \right] \mathrm{d}y, & \text{when } k > 0, \\ -\frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky} \left[k\overline{\alpha} u' + \mathrm{i}(\overline{q}u' + \overline{V}u) \right] \mathrm{d}y, & \text{when } k < 0. \end{cases}$$

This fact is often written in form

$$\begin{aligned} u(x,k) &= a(k)\mathrm{e}^{\mathrm{i}kx} + o(1), \quad x \to +\infty, \\ u(x,k) &= \mathrm{e}^{\mathrm{i}kx} + b(k)\mathrm{e}^{-\mathrm{i}kx} + o(1), \quad x \to -\infty, \end{aligned}$$

which, in view of the coefficients $a(k)$ and $b(k)$, can be understood as a radiation condition for u_{sc}, as stated in Section 1.

Due to Lemma 2.3 we have $u \approx u_0$ for large $|k|$, which allows us to approximate

$$\begin{aligned} a(k) &= 1 + \frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ky} \left[k\alpha u' - \mathrm{i}(qu' + Vu) \right] \mathrm{d}y \\ &\approx 1 + \frac{1}{4k^3} \int_{-\infty}^{\infty} \left[\mathrm{i}k^2\alpha + kq - \mathrm{i}V \right] \mathrm{d}y \approx 1, \qquad k \to \infty \end{aligned}$$

and

$$\begin{aligned} b(k) &= -\frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky} \left[k\alpha u' + \mathrm{i}(qu' + Vu) \right] \mathrm{d}y \\ &\approx -\frac{1}{4k^3} \int_{-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}ky} \left[\mathrm{i}k^2\alpha - kq + \mathrm{i}V \right] \mathrm{d}y, \qquad k \to \infty. \end{aligned}$$

We choose to investigate the reflection coefficient $b(k)$. By integrating by parts formally we see that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{i}kf(x)\mathrm{d}x = \frac{1}{\sqrt{2\pi}}f(x)\mathrm{e}^{\mathrm{i}kx}\Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}f'(x)\mathrm{d}x.$$

Since $\alpha \in W_1^2(\mathbb{R})$ and $q \in W_1^1(\mathbb{R})$ then α, α' and q belong to $\dot{C}(\mathbb{R})$. The good behaviour of α, α' and q at infinity allows us to write

$$b\left(\frac{k}{2}\right) \approx \frac{2\sqrt{2\pi}}{\mathrm{i}k^3}F^{-1}\left(V - \frac{1}{2}q' - \frac{1}{4}\alpha''\right)(k).$$

Same calculation for $k < 0$ yields similar formula, but with complex conjugates of V, q' and α'', i.e. when $k < 0$

$$b\left(\frac{k}{2}\right) \approx \frac{2\sqrt{2\pi}}{\mathrm{i}k^3}F^{-1}\left(\overline{V} - \frac{1}{2}\overline{q}' - \frac{1}{4}\overline{\alpha}''\right)(k).$$

Recall that in (5) we defined $\beta = V - \frac{1}{2}q' - \frac{1}{4}\alpha''$.

4 The inverse Born approximation of β

This section is devoted to proving Theorem 1.1. First we reiterate and justify the claims of Section 1.

Let us denote $k_0 := 2C_0 = \|\alpha\|_1 + \|q\|_1 + \|V\|_1$ and define a new function $\chi(k)$ by

$$\chi(k) = \begin{cases} 1, & \text{when } |k| \geq k_0, \\ 0, & \text{when } |k| < k_0. \end{cases}$$

With help of $\chi(k)$ we can write the reflection coefficient $b(k)$ for all $k \in \mathbb{R}$. We define

$$b(k) = \chi(k)\begin{cases} -\frac{1}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\left[k\alpha u' + \mathrm{i}(qu' + Vu)\right]\mathrm{d}y, & \text{when } k > 0, \\ -\frac{1}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\left[k\overline{\alpha}u' + \mathrm{i}(\overline{q}u' + \overline{V}u)\right]\mathrm{d}y, & \text{when } k < 0. \end{cases}$$

Now we can define the inverse Born approximation of β in the sense of distributions.

Definition 4.1. The inverse Born approximation q_{B} of β is defined by

$$q_{\mathrm{B}}(\xi) = F\left(\frac{\mathrm{i}k^3}{2\sqrt{2\pi}}b\left(\frac{k}{2}\right)\right)(\xi).$$

Remark 4.2. Since $b(k/2) = \overline{b(-k/2)}$, when $k < 0$, we find that

$$
\begin{aligned}
q_{\mathrm{B}}(\xi) &= \frac{\mathrm{i}}{4\pi}\int_{-\infty}^{-2k_0} \mathrm{e}^{-\mathrm{i}k\xi} k^3 b\left(\frac{k}{2}\right)\mathrm{d}k + \frac{\mathrm{i}}{4\pi}\int_{2k_0}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi} k^3 b\left(\frac{k}{2}\right)\mathrm{d}k \\
&= \overline{\frac{\mathrm{i}}{4\pi}\int_{2k_0}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi} k^3 b\left(\frac{k}{2}\right)\mathrm{d}k} + \frac{\mathrm{i}}{4\pi}\int_{2k_0}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi} k^3 b\left(\frac{k}{2}\right)\mathrm{d}k,
\end{aligned}
$$

where we changed k to $-k$ in the first term. This means that q_{B} is real-valued and we can focus on recovering information about the real part of β.

Writing $b(k) = \chi(k)(b_0(k) + b_1(k) + b_2(k) + b_{\mathrm{rest}}(k))$, in view of the series representation $u = u_0 + u_1 + u_2 + \sum_{j=3}^{\infty} u_j$, allows us to write the Born approximation as

$$
\begin{aligned}
q_{\mathrm{B}}(\xi) = {}& F\left(\frac{\mathrm{i}k^3}{2\sqrt{2\pi}} b_0\left(\frac{k}{2}\right)\right)(\xi) + F\left(\frac{\mathrm{i}k^3(\chi(k/2)-1)}{2\sqrt{2\pi}} b_0\left(\frac{k}{2}\right)\right)(\xi) \\
&+ F\left(\frac{\mathrm{i}k^3\chi(k/2)}{2\sqrt{2\pi}} b_1\left(\frac{k}{2}\right)\right)(\xi) + F\left(\frac{\mathrm{i}k^3\chi(k/2)}{2\sqrt{2\pi}} b_2\left(\frac{k}{2}\right)\right)(\xi) \\
&+ F\left(\frac{\mathrm{i}k^3\chi(k/2)}{2\sqrt{2\pi}} b_{\mathrm{rest}}\left(\frac{k}{2}\right)\right)(\xi) \\
=:{}& q_0(\xi) + \widetilde{q}(\xi) + q_1(\xi) + q_2(\xi) + q_{\mathrm{rest}}(\xi).
\end{aligned}
$$

Next we start investigating the terms appearing on the right-hand side of this representation.

Lemma 4.3. *Let $\alpha \in W_1^2(\mathbb{R})$, $q \in W_1^1(\mathbb{R})$ and $V \in L^1(\mathbb{R})$. Then we have the formula*

$$
q_0(\xi) = \mathrm{Re}(\beta)(\xi) + \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(\beta)(y)}{\xi - y}\mathrm{d}y. \tag{10}
$$

Proof. Recall that $u_0(x,k) = \mathrm{e}^{\mathrm{i}kx}$ and $u_0'(x,k) = \mathrm{i}k\mathrm{e}^{\mathrm{i}kx}$. First we integrate by parts the inner integrals in the formula

$$
\begin{aligned}
q_0(\xi) = \frac{1}{2\pi}\int_0^{\infty} \mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\left(\frac{k^2}{4}\alpha + \mathrm{i}\frac{k}{2}q + V\right)\mathrm{d}y\mathrm{d}k \\
+ \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\left(\frac{k^2}{4}\overline{\alpha} + \mathrm{i}\frac{k}{2}\overline{q} + \overline{V}\right)\mathrm{d}y\mathrm{d}k
\end{aligned}
$$

to obtain

$$
q_0(\xi) = \frac{1}{2\pi}\int_0^{\infty} \mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\beta\mathrm{d}y\mathrm{d}k + \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\overline{\beta}\mathrm{d}y\mathrm{d}k.
$$

Then we can combine the integrals into the real and imaginary parts of the integrands as follows

$$q_0(\xi) = F(F^{-1}(\mathrm{Re}(\beta))(\xi) + \mathrm{i}F(\mathrm{sgn}(\cdot)F^{-1}(\mathrm{Im}(\beta))(\xi).$$

From [6] we find that in the sense of distributions $\widehat{\mathrm{sgn}}(\xi) = -\mathrm{i}\sqrt{\frac{2}{\pi}}\mathrm{p.v.}\frac{1}{\xi}$, where $\mathrm{p.v.}\frac{1}{\xi}$ denotes the principal value distribution of $\frac{1}{\xi}$. By applying this identity we obtain

$$q_0(\xi) = \mathrm{Re}(\beta)(\xi) + \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(\beta)(y)}{\xi - y}\mathrm{d}y,$$

as claimed. □

In Remark 4.2 we noted that the Born approximation is real-valued, and equation (10) shows nicely, how the real-part of β appears in the formulas.

Next, being the Fourier transform of a compactly supported distribution $\widetilde{q} \in C^{\infty}(\mathbb{R})$. Moreover, we can prove the following.

Lemma 4.4. *If $\alpha \in W_1^2(\mathbb{R})$, $q \in W_1^1(\mathbb{R})$ and $V \in L^1(\mathbb{R})$ then $\widetilde{q} \in \dot{C}(\mathbb{R})$.*

Proof. While $b_0(k)$ is not defined when $k = 0$, for $k^3 b_0(k/2)$ it is possible to write

$$\begin{aligned}\widetilde{q}(\xi) = &-\frac{\mathrm{i}}{2\pi}\int_{-2k_0}^{0} \mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}y}\left[\frac{k}{2}\overline{\alpha}u_0' + \mathrm{i}(\overline{q}u_0' + \overline{V}u_0)\right]\mathrm{d}y\mathrm{d}k \\ &- \frac{\mathrm{i}}{2\pi}\int_{0}^{2k_0} \mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}y}\left[\frac{k}{2}\alpha u_0' + \mathrm{i}(qu_0' + Vu_0)\right]\mathrm{d}y\mathrm{d}k.\end{aligned}$$

We then substitute $u_0(x,k) = \mathrm{e}^{\mathrm{i}kx}$ and change the variables k to $-k$ in the first integral. Then integration by parts in the inner integral yields

$$\widetilde{q}(\xi) = \mathrm{Re}\left(\frac{1}{\pi}\int_{0}^{2k_0} \mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\beta(y)\mathrm{d}y\mathrm{d}k\right).$$

By assumption $\beta \in L^1(\mathbb{R})$, which means that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\beta(y)\mathrm{d}y \in L^{\infty}(\mathbb{R}).$$

Because bounded functions are locally integrable, the Riemann–Lebesgue lemma implies that $\widetilde{q}$ vanishes at infinity. □

The next result concerns the behaviour of q_{rest}, that is, the tail of the Born series. In what follows we let $C > 0$ denote a generic constant that can have different values from step to step.

Lemma 4.5. *If $\alpha \in W_1^2(\mathbb{R})$, $q \in W_1^1(\mathbb{R})$, $V \in L^1(\mathbb{R})$ and q_{rest} is as above, then $q_{\mathrm{rest}} \in H^s(\mathbb{R})$ for all $s < \frac{1}{2}$.*

Proof. Let us estimate b_{rest}. Since $u_{\mathrm{rest}}(x,k) = \sum_{j=3}^{\infty} u_j(x,k)$, when $k > 2k_0$ we can apply Lemma 2.3 to obtain

$$\begin{aligned}
\left| b_{\mathrm{rest}}\left(\frac{k}{2}\right)\right| &\leq \frac{2}{k^3}\int_{-\infty}^{\infty}\left[\frac{k}{2}|\alpha|\left|\sum_{j=3}^{\infty} u_j'\right| + |q|\left|\sum_{j=3}^{\infty} u_j'\right| + |V|\left|\sum_{j=3}^{\infty} u_j\right|\right]\mathrm{d}y \\
&\leq \frac{C}{k^3}\int_{-\infty}^{\infty}\left[\frac{1}{k}|\alpha| + \frac{1}{k^2}|q| + \frac{1}{k^3}|V|\right]\mathrm{d}y \leq \frac{C}{k^4}.
\end{aligned}$$

Similarly, if $k < -2k_0$ then

$$\left| b_{\mathrm{rest}}\left(\frac{k}{2}\right)\right| \leq \frac{C}{k^4}.$$

Then the norm of q_{rest} in the Sobolev space $H^s(\mathbb{R}) = W_2^s(\mathbb{R})$ can be estimated as follows

$$\begin{aligned}
\|q_{\mathrm{rest}}\|^2_{W_2^s(\mathbb{R})} &= \int_{-\infty}^{\infty}(1+|k|^2)^s|\widehat{q_{\mathrm{rest}}}(k)|^2\mathrm{d}k \\
&\leq \int_{|k|>k_0}(1+|k|^2)^s\frac{C}{k^2}\mathrm{d}k \\
&\leq \int_{k>k_0}\frac{C}{k^{2-2s}}\mathrm{d}k.
\end{aligned}$$

The last integral converges if and only if $s < \frac{1}{2}$, so $q_{\mathrm{rest}} \in H^s(\mathbb{R})$ when $s < \frac{1}{2}$. $\square$

Let us define a suitable expression for b_1. First we simply list the iterations u_1 and u_1' for $k > 0$. We have

$$\begin{aligned}
u_1\left(x, \frac{k}{2}\right) &= \int_{-\infty}^{\infty}\frac{\mathrm{sgn}(x-y)}{k^2}\left(\mathrm{e}^{\mathrm{i}\frac{k}{2}|x-y|} - \mathrm{e}^{-\frac{k}{2}|x-y|}\right)\alpha(y)\frac{\mathrm{i}k}{2}\mathrm{e}^{\mathrm{i}\frac{k}{2}y}\mathrm{d}y \\
&\quad - \frac{2}{k^3}\int_{-\infty}^{\infty}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{k}{2}|x-y|} - \mathrm{e}^{-\frac{k}{2}|x-y|}\right)\left(\frac{\mathrm{i}k}{2}q(y) + V(y)\right)\mathrm{e}^{\mathrm{i}\frac{k}{2}y}\mathrm{d}y
\end{aligned}$$

and

$$\begin{aligned}
u_1'\left(x, \frac{k}{2}\right) &= \int_{-\infty}^{\infty}\frac{1}{2k}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{k}{2}|x-y|} + \mathrm{e}^{-\frac{k}{2}|x-y|}\right)\alpha(y)\frac{\mathrm{i}k}{2}\mathrm{e}^{\mathrm{i}\frac{k}{2}y}\mathrm{d}y \\
&\quad + \int_{-\infty}^{\infty}\frac{\mathrm{sgn}(x-y)}{k^2}\left(\mathrm{e}^{\mathrm{i}\frac{k}{2}|x-y|} - \mathrm{e}^{-\frac{k}{2}|x-y|}\right)\left(\frac{\mathrm{i}k}{2}q(y) + V(y)\right)\mathrm{e}^{\mathrm{i}\frac{k}{2}y}\mathrm{d}y.
\end{aligned}$$

The corresponding part of reflection coefficient is

$$b_1(k) = \chi(k)\left\{\begin{array}{ll} -\frac{1}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\left[k\alpha u_1' + \mathrm{i}(qu_1' + Vu_1)\right]\mathrm{d}y, & \text{when } k > 0,\\ -\frac{1}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\left[k\overline{\alpha}u_1' + \mathrm{i}(\overline{q}u_1' + \overline{V}u_1)\right]\mathrm{d}y, & \text{when } k < 0.\end{array}\right.$$

Since $b(k) = \overline{b(-k)}$ for $k < 0$, we can conclude that

$$\begin{aligned} q_1(\xi) &= \frac{\mathrm{i}}{4\pi}\int_{-\infty}^{-2k_0} \mathrm{e}^{-\mathrm{i}k\xi}k^3 b_1\left(\frac{k}{2}\right)\mathrm{d}k + \frac{\mathrm{i}}{4\pi}\int_{2k_0}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi}k^3 b_1\left(\frac{k}{2}\right)\mathrm{d}k\\ &= -\frac{\mathrm{i}}{4\pi}\int_{2k_0}^{\infty} \mathrm{e}^{\mathrm{i}k\xi}k^3 b_1\left(-\frac{k}{2}\right)\mathrm{d}k + \frac{\mathrm{i}}{4\pi}\int_{2k_0}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi}k^3 b_1\left(\frac{k}{2}\right)\mathrm{d}k\\ &= \frac{1}{2\pi}\,\mathrm{Im}\left(\int_{2k_0}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi}k^3 b_1\left(\frac{k}{2}\right)\mathrm{d}k\right). \end{aligned} \tag{11}$$

Next we decompose b_1 as $b_1 = \widetilde{b_1} + b_{1,\mathrm{rest}}$, where $|b_{1,\mathrm{rest}}| \le \frac{C}{k^4}$. By (11) it is enough to check b_1 for $k \ge 2k_0$. Plugging the definitions of u_1 and u_1' into b_1 yields

$$\begin{aligned} b_1\left(\frac{k}{2}\right) = -\frac{2}{k^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}y}\Bigg[& \\ & \left(\frac{k}{2}\alpha + \mathrm{i}q\right)\left\{\frac{1}{2k}\int_{-\infty}^{\infty}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{k}{2}|y-z|} + \mathrm{e}^{-\frac{k}{2}|y-z|}\right)\frac{\mathrm{i}k}{2}\alpha \mathrm{e}^{\mathrm{i}\frac{k}{2}z}\mathrm{d}z\right.\\ & \left. + \int_{-\infty}^{\infty}\frac{\mathrm{sgn}(y-z)}{k^2}\left(\mathrm{e}^{\mathrm{i}\frac{k}{2}|y-z|} - \mathrm{e}^{-\frac{k}{2}|y-z|}\right)\left(\mathrm{i}\frac{k}{2}q + V\right)\mathrm{e}^{\mathrm{i}\frac{k}{2}z}\mathrm{d}z\right\}\\ & + \mathrm{i}V\left\{\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(y-z)}{k^2}\left(\mathrm{e}^{\mathrm{i}\frac{k}{2}|y-z|} - \mathrm{e}^{-\frac{k}{2}|y-z|}\right)\frac{\mathrm{i}k}{2}\alpha \mathrm{e}^{\mathrm{i}\frac{k}{2}z}\mathrm{d}z\right.\\ & \left. - \frac{2}{k^3}\int_{-\infty}^{\infty}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{k}{2}|y-z|} - \mathrm{e}^{-\frac{k}{2}|y-z|}\right)\left(\frac{\mathrm{i}k}{2}q + V\right)\mathrm{e}^{\mathrm{i}\frac{k}{2}z}\mathrm{d}z\right\}\Bigg]\mathrm{d}y. \end{aligned}$$

The first six terms of this expression must be investigated separately, while the remaining terms can be directly estimated by $\frac{C}{k^4}$ for some $C > 0$. Let us denote the first six terms by $\widetilde{b_1}$ and the remaining terms by $b_{1,\mathrm{rest}}$.

Now we are ready to study the behaviour of the first non-linear term in the Born approximation. To conclude the appropriate continuity, we will need to assume little more from the coefficients α and q.

Lemma 4.6. *Let $\alpha \in H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$, $q \in H^1(\mathbb{R}) \cap W_1^1(\mathbb{R})$ and $V \in L^1(\mathbb{R})$. Then the first non-linear term q_1 of the inverse Born approximation can be written as a sum $q_1 = \widetilde{q_1} + q_{1,\mathrm{rest}}$, where*

$$\widetilde{q_1}(\xi) = F\left(\frac{\mathrm{i}k^3\chi(k/2)}{2\sqrt{2\pi}}\widetilde{b_1}\left(\frac{k}{2}\right)\right)(\xi)$$

and

$$q_{1,\mathrm{rest}}(\xi) = F\left(\frac{\mathrm{i}k^3\chi(k/2)}{2\sqrt{2\pi}} b_{1,\mathrm{rest}}\left(\frac{k}{2}\right)\right)(\xi)$$

satisfy $\widetilde{q}_1 \in \dot{C}(\mathbb{R})$ *and* $q_{1,\mathrm{rest}} \in H^s(\mathbb{R})$ *for all* $s < \frac{1}{2}$.

Proof. Note that under our assumptions certain products of α and q and their first integer derivatives belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. This is because $f \in W_1^1(\mathbb{R})$ implies $f \in \dot{C}(\mathbb{R})$, which in turn means that α, α' and q are bounded.

Since $|b_{1,\mathrm{rest}}(k/2)| \le \frac{C}{k^4}$ for some $C > 0$, then $q_{1,\mathrm{rest}} \in H^s(\mathbb{R})$ for all $s < \frac{1}{2}$ as in Lemma 4.5. It remains to verify that $\widetilde{q}_1$ is a continuous function. To this end we need to check six terms. Since the calculation is very technical, we write down the details of the three first terms (that are also the most ill-behaved ones) and note that the remaining terms can be investigated in similar fashion.

The constant factors do not affect continuity of $\widetilde{q}_1$, so to simplify the notation we drop the constants in the sequel. According to (11) the first term of $\widetilde{q}_1$ is determined by the imaginary part of

$$\begin{aligned}&\int_{2k_0}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}y}\alpha(y) \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}|y-z|} k\alpha(z)\mathrm{e}^{\mathrm{i}\frac{k}{2}z}\mathrm{d}z\mathrm{d}y\mathrm{d}k \\ &= \int_{2k_0}^{\infty} k\mathrm{e}^{-\mathrm{i}k\xi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}(|y-z|+y+z)}\alpha(y)\alpha(z)\mathrm{d}y\mathrm{d}z\mathrm{d}k. \end{aligned} \tag{12}$$

Due to symmetry the two innermost integrals assume the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}(|y-z|+y+z)}\alpha(y)\alpha(z)\mathrm{d}y\mathrm{d}z = 2\int_{-\infty}^{\infty}\int_{-\infty}^{y} \mathrm{e}^{\mathrm{i}ky}\alpha(z)\alpha(y)\mathrm{d}z\mathrm{d}y.$$

Denote

$$A(y) := \int_{-\infty}^{y} \alpha(z)\mathrm{d}z$$

and note that A is a bounded function with the property that $A' = \alpha$. Then formula (12) becomes

$$2\int_{2k_0}^{\infty} k\mathrm{e}^{-\mathrm{i}k\xi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\alpha(y)A(y)\mathrm{d}y\mathrm{d}k.$$

Since $A\alpha \in H^2(\mathbb{R})$, we find that $kF^{-1}(A\alpha)(k)$ is integrable over $[2k_0, \infty[$. By the Riemann–Lebesgue lemma (12) defines a continuous function.

The second term of $\widetilde{q_1}$ is little more complicated, because we can not directly resort to properties of the Fourier transform. Analogously to the first term, the second one can be written in the form

$$\int_{2k_0}^{\infty} k\mathrm{e}^{-\mathrm{i}k\xi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}(y+z)} \mathrm{e}^{-\frac{k}{2}|y-z|} \alpha(y)\alpha(z)\mathrm{d}z\mathrm{d}y\mathrm{d}k. \tag{13}$$

This term does not have as convenient symmetry as (12), but we nevertheless find that the two inner integrals of (13) equal

$$\begin{aligned} I := &\int_{-\infty}^{\infty} \int_{-\infty}^{y} \mathrm{e}^{\frac{k}{2}(\mathrm{i}-1)y} \mathrm{e}^{\frac{k}{2}(\mathrm{i}+1)z} \alpha(y)\alpha(z)\mathrm{d}z\mathrm{d}y \\ &+ \int_{-\infty}^{\infty} \int_{y}^{\infty} \mathrm{e}^{\frac{k}{2}(\mathrm{i}+1)y} \mathrm{e}^{\frac{k}{2}(\mathrm{i}-1)z} \alpha(y)\alpha(z)\mathrm{d}z\mathrm{d}y. \end{aligned}$$

To proceed from this expression we integrate both terms by parts with respect to z to obtain

$$\begin{aligned} I = &\int_{-\infty}^{\infty} \frac{2}{(\mathrm{i}+1)k} \mathrm{e}^{\mathrm{i}ky} \left(\alpha(y)\right)^2 \mathrm{d}y - \int_{-\infty}^{\infty} \frac{2}{(\mathrm{i}-1)k} \mathrm{e}^{\mathrm{i}ky} \left(\alpha(y)\right)^2 \mathrm{d}y \\ &- \int_{-\infty}^{\infty} \int_{-\infty}^{y} \frac{2}{(\mathrm{i}+1)k} \mathrm{e}^{\frac{k}{2}(\mathrm{i}+1)z} \mathrm{e}^{\frac{k}{2}(\mathrm{i}-1)y} \alpha(y)\alpha'(z)\mathrm{d}z\mathrm{d}y \\ &- \int_{-\infty}^{\infty} \int_{y}^{\infty} \frac{2}{(\mathrm{i}-1)k} \mathrm{e}^{\frac{k}{2}(\mathrm{i}-1)z} \mathrm{e}^{\frac{k}{2}(\mathrm{i}+1)y} \alpha(y)\alpha'(z)\mathrm{d}z\mathrm{d}y =: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Here $\alpha^2 \in H^2(\mathbb{R})$, so that terms I_1 and I_2 in combination with (13) define continuous functions.

Then we integrate I_3 and I_4 by parts twice more. The first integration gives

$$\begin{aligned} I_3 + I_4 = &\int_{-\infty}^{\infty} \frac{2\mathrm{i}}{k^2} \mathrm{e}^{\mathrm{i}ky} \alpha(y)\alpha'(y)\mathrm{d}y \\ &- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2\mathrm{i}\ \mathrm{sgn}(y-z)}{k^2} \mathrm{e}^{\mathrm{i}\frac{k}{2}(y+z)} \mathrm{e}^{-\frac{k}{2}|y-z|} \alpha(y)\alpha''(z)\mathrm{d}z\mathrm{d}y. \end{aligned}$$

Again, the first term in combination with (13) defines a continuous function. The last integral is integrated by parts with respect to y. To do this, we change the order of integration, which can be done by Fubini's theorem because all the functions are integrable. Then separating the integral back into two parts gives (without constants)

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\operatorname{sgn}(y-z)}{k^2}\mathrm{e}^{\mathrm{i}\frac{k}{2}(y+z)}\mathrm{e}^{-\frac{k}{2}|y-z|}\alpha(y)\alpha''(z)\mathrm{d}y\mathrm{d}z\\
&=\frac{1}{k^2}\int_{-\infty}^{\infty}\int_{-\infty}^{z}\mathrm{e}^{\frac{k}{2}(\mathrm{i}-1)z}\mathrm{e}^{\frac{k}{2}(\mathrm{i}+1)y}\alpha(y)\alpha''(z)\mathrm{d}y\mathrm{d}z\\
&\quad-\frac{1}{k^2}\int_{-\infty}^{\infty}\int_{z}^{\infty}\mathrm{e}^{\frac{k}{2}(\mathrm{i}-1)z}\mathrm{e}^{\frac{k}{2}(\mathrm{i}+1)y}\alpha(y)\alpha''(z)\mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

One final integration by parts yields

$$
\begin{aligned}
&-\int_{-\infty}^{\infty}\frac{2}{(\mathrm{i}+1)k^3}\mathrm{e}^{\mathrm{i}kz}\alpha(z)\alpha''(z)\mathrm{d}z-\int_{-\infty}^{\infty}\frac{2}{(\mathrm{i}-1)k^3}\mathrm{e}^{\mathrm{i}kz}\alpha(z)\alpha''(z)\mathrm{d}z\\
&+\frac{1}{k^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{k}{2}(y+z)}\mathrm{e}^{-\frac{k}{2}|y-z|}\alpha'(y)\alpha''(z)\mathrm{d}y\mathrm{d}z\\
&+\frac{\mathrm{i}}{k^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{sgn}(y-z)\mathrm{e}^{\mathrm{i}\frac{k}{2}(y+z)}\mathrm{e}^{-\frac{k}{2}|y-z|}\alpha'(y)\alpha''(z)\mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

All the integrals can be simply estimated by $\frac{C}{k^3}$, because $\alpha \in W_1^2(\mathbb{R})$, and in combination with (13) they define continuous functions.

The third term of $\widetilde{q_1}$ is

$$
\int_{2k_0}^{\infty}\mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{k}{2}(|y-z|+y+z)}q(y)\alpha(z)\mathrm{d}z\mathrm{d}y\mathrm{d}k. \tag{14}
$$

The two inner integrals in (14) can be rewritten as the sum

$$
\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kz}\alpha(z)\int_{-\infty}^{z}q(y)\mathrm{d}y\mathrm{d}z+\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}ky}q(y)\int_{-\infty}^{y}\alpha(z)\mathrm{d}z\mathrm{d}y.
$$

Following the procedure of the first term (12), we define

$$
Q(z):=\int_{-\infty}^{z}q(y)\mathrm{d}y \quad \text{and} \quad A(y):=\int_{-\infty}^{y}\alpha(z)\mathrm{d}z,
$$

and note that A and Q are bounded functions with $A'=\alpha$ and $Q'=q$. Then (14) becomes

$$
\int_{2k_0}^{\infty}\mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kz}\alpha(z)Q(z)\mathrm{d}z\mathrm{d}k+\int_{2k_0}^{\infty}\mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}ky}A(y)q(y)\mathrm{d}y\mathrm{d}k. \tag{15}
$$

Here, the functions αQ and Aq are in $H^1(\mathbb{R})$, which in turn implies that $F^{-1}(\alpha Q)(k)$ and $F^{-1}(Aq)(k)$ are integrable over $[2k_0,\infty[$. Thus (15) defines a continuous function.

The remaining three terms of $\widetilde{q_1}$ contain different combinations of α and q, but the technique to obtain the continuity is similar, so we omit the details here. $\square$

Remark 4.7. It may be possible to weaken the assumption of Lemma 4.6 to $\alpha \in H^s(\mathbb{R}) \cap W_1^2(\mathbb{R})$ for $s > \frac{3}{2}$ and $q \in H^t(\mathbb{R}) \cap W_1^1(\mathbb{R})$ for $t > \frac{1}{2}$ with minor modifications in proof, but from our point of view it is more natural to assume integer derivatives.

Remark 4.8. By embedding theorems [2] we have

$$W_1^2(\mathbb{R}) \subset H^{3/2}(\mathbb{R}), \quad \text{and} \quad W_1^1(\mathbb{R}) \subset H^{1/2}(\mathbb{R})$$

so the assumptions in Remark 4.7 are quite strict.

To solve the inverse problem of finding the main singularities of β we still need to conclude that q_2 behaves well enough. It turns out, that we can write $q_2 = \widetilde{q_2} + q_{2,\mathrm{rest}}$ similarly as for q_1 and see that $\widetilde{q_2}$ is a continuous function and $q_{2,\mathrm{rest}}$ belongs to $H^s(\mathbb{R})$ for all $s < \frac{1}{2}$. It is evident from previous results that the formulas regarding q_2 are very long and it is impractical to write them down. Instead, we first estimate b_2 and divide it into two parts $b_2 = \widetilde{b_2} + b_{2,\mathrm{rest}}$.

It is possible to write down the iterations u_2 and u_2' for $k > 0$ and plug them in

$$b_2\left(\frac{k}{2}\right) = -\frac{2}{k^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}y}\left[\frac{k}{2}\alpha u_2' + \mathrm{i}(qu_2' + Vu_2)\right]\mathrm{d}y.$$

Then it becomes clear, that most of the terms of b_2 can be directly estimated by $\frac{C}{k^4}$. Let us denote those terms by $b_{2,\mathrm{rest}}$. The only term of b_2 that contains parts that are only of type $\frac{C}{k^3}$ or worse is

$$\frac{2}{k^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}z} k\alpha(z) u_2'\left(z, \frac{k}{2}\right)\mathrm{d}z\mathrm{d}k,$$

and more specifically within this term (after plugging in the definition of u_2' and dropping out the unnecessary constants)

$$\begin{aligned}\frac{2}{k^3}\int_{-\infty}^{\infty}\alpha(z)\mathrm{e}^{\mathrm{i}\frac{k}{2}z}\int_{-\infty}^{\infty}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{k}{2}|z-y|} + \mathrm{e}^{-\frac{k}{2}|z-y|}\right)\alpha(y)\mathrm{e}^{\mathrm{i}\frac{k}{2}y}\\ \times\int_{-\infty}^{\infty}\left(\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{k}{2}|y-w|} + \mathrm{e}^{-\frac{k}{2}|y-w|}\right)\alpha(w)\mathrm{e}^{\mathrm{i}\frac{k}{2}w}\mathrm{d}w\mathrm{d}y\mathrm{d}z.\end{aligned}$$

Then we can rearrange the integrals and multiply the exponentials, to obtain

$$\begin{aligned}\widetilde{b_2}\left(\frac{k}{2}\right) := \frac{2}{k^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(-\mathrm{e}^{\mathrm{i}\frac{k}{2}(|z-y|+|y-w|)} + \mathrm{i}\mathrm{e}^{-\frac{k}{2}|z-y|}\mathrm{e}^{\mathrm{i}\frac{k}{2}|y-w|} + \mathrm{i}\mathrm{e}^{-\frac{k}{2}|y-w|}\mathrm{e}^{\mathrm{i}\frac{k}{2}|y-z|}\\ + \mathrm{e}^{-\frac{k}{2}(|z-y|+|y-w|)}\Big)\mathrm{e}^{\mathrm{i}\frac{k}{2}(z+y+w)}\alpha(y)\alpha(z)\alpha(w)\mathrm{d}w\mathrm{d}y\mathrm{d}z. \qquad (16)\end{aligned}$$

Lemma 4.9. *Let $\alpha \in H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$, $q \in H^1(\mathbb{R}) \cap W_1^1(\mathbb{R})$ and $V \in L^1(\mathbb{R})$. Then the second non-linear term q_2 of the inverse Born approximation can be written as the sum $q_2 = \widetilde{q_2} + q_{2,\mathrm{rest}}$, where*

$$\widetilde{q_2}(\xi) = F\left(\frac{\mathrm{i}k^3\chi(k/2)}{2\sqrt{2\pi}}\widetilde{b_2}\left(\frac{k}{2}\right)\right)(\xi)$$

and

$$q_{2,\mathrm{rest}}(\xi) = F\left(\frac{\mathrm{i}k^3\chi(k/2)}{2\sqrt{2\pi}}b_{2,\mathrm{rest}}\left(\frac{k}{2}\right)\right)(\xi)$$

satisfy $\widetilde{q_2} \in \dot{C}(\mathbb{R})$ and $q_{2,\mathrm{rest}} \in H^s(\mathbb{R})$ for all $s < \frac{1}{2}$.

Proof. Since $b_{2,\mathrm{rest}}$ can be estimated by $\frac{C}{k^4}$, we find that $q_{2,\mathrm{rest}} \in H^s(\mathbb{R})$ for $s < \frac{1}{2}$. Then we note that

$$\widetilde{q_2}(\xi) = \frac{1}{2\pi}\operatorname{Im}\left(\int_{2k_0}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi}k^3\widetilde{b_2}\left(\frac{k}{2}\right)\mathrm{d}k\right).$$

Now it is enough to check that $k^3\widetilde{b_2}$ is integrable over $[2k_0, \infty[$ and apply the Riemann-Lebesgue lemma to conclude the continuity of $\widetilde{q_2}$. According to (16), the first term of $k^3\widetilde{b_2}$ is

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{k}{2}(|z-y|+|y-w|)}\mathrm{e}^{\mathrm{i}\frac{k}{2}(z+y+w)}\alpha(y)\alpha(z)\alpha(w)\mathrm{d}z\mathrm{d}y\mathrm{d}w \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{y} \mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}\alpha(y)\alpha(z)\alpha(w)\mathrm{d}z\mathrm{d}y\mathrm{d}w \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{y}^{\infty} \mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}\alpha(y)\alpha(z)\alpha(w)\mathrm{d}z\mathrm{d}y\mathrm{d}w =: I_1 + I_2.
\end{aligned}$$

By denoting

$$A(y) := \int_{-\infty}^{y} \alpha(z)\mathrm{d}z$$

we can write

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}A(y)\alpha(y)\alpha(w)\mathrm{d}y\mathrm{d}w \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{w} \mathrm{e}^{\mathrm{i}kw}\mathrm{e}^{\mathrm{i}\frac{k}{2}y}A(y)\alpha(y)\alpha(w)\mathrm{d}y\mathrm{d}w + \int_{-\infty}^{\infty}\int_{w}^{\infty} \mathrm{e}^{\mathrm{i}\frac{3k}{2}y}A(y)\alpha(y)\alpha(w)\mathrm{d}y\mathrm{d}w.
\end{aligned}$$

Integrating this expression by parts with respect to y yields

$$\begin{aligned}
I_1 = \frac{4}{3\mathrm{i}k} & \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{3k}{2}w} A(w) \left(\alpha(w)\right)^2 \mathrm{d}w \\
& + \int_{-\infty}^{\infty} \frac{32}{9k^2} \mathrm{e}^{\mathrm{i}\frac{3k}{2}w} \left(A\cdot\alpha\right)'(w)\alpha(w)\mathrm{d}w \\
& - \int_{-\infty}^{\infty}\int_{-\infty}^{w} \frac{4}{k^2}\mathrm{e}^{\mathrm{i}\frac{k}{2}y}\mathrm{e}^{\mathrm{i}kw}\left(A\cdot\alpha\right)''(y)\alpha(w)\mathrm{d}y\mathrm{d}w \\
& - \int_{-\infty}^{\infty}\int_{w}^{\infty} \frac{4}{9k^2}\mathrm{e}^{\mathrm{i}\frac{3k}{2}y}\left(A\cdot\alpha\right)''(y)\alpha(w)\mathrm{d}y\mathrm{d}w.
\end{aligned}$$

Since $A\alpha^2 \in L^2(\mathbb{R})$ then $F^{-1}(A\alpha) \in L^2(\mathbb{R})$. Then by Hölder's inequality the first term in the above expression belongs to $L^1([2k_0,\infty[)$. All other terms can be estimated by $\frac{C}{k^2}$, which belongs to $L^1([2k_0,\infty[)$.

To conclude the proof, we integrate by parts

$$\begin{aligned}
I_2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{y}^{\infty} \mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}\alpha(y)\alpha(z)\alpha(w)\mathrm{d}z\mathrm{d}y\mathrm{d}w \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\mathrm{i}k}\mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}(\alpha(y))^2\alpha(w)\mathrm{d}y\mathrm{d}w \\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{y}^{\infty} \frac{1}{\mathrm{i}k}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}\alpha(y)\alpha'(z)\alpha(w)\mathrm{d}z\mathrm{d}y\mathrm{d}w.
\end{aligned} \tag{17}$$

The first integral in (17) is the same as in I_1, the only difference being the fact that in I_2 we have α^2 instead of $A\alpha$. Therefore it is an integrable function of k.

Finally the latter integral in (17) is integrated by parts again to get

$$\begin{aligned}
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{y}^{\infty} \frac{1}{\mathrm{i}k}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}\alpha(y)\alpha'(z)\alpha(w)\mathrm{d}z\mathrm{d}y\mathrm{d}w \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(\mathrm{i}k)^2}\mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}\alpha'(y)\alpha(y)\alpha(w)\mathrm{d}y\mathrm{d}w \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{y}^{\infty} \frac{1}{(\mathrm{i}k)^2}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\frac{k}{2}(w+|y-w|)}\alpha''(z)\alpha(y)\alpha(w)\mathrm{d}y\mathrm{d}w.
\end{aligned}$$

This expression is estimated by $\frac{C}{k^2}$.

The remaining three terms of $k^3\widetilde{b_2}$ are investigated in similar manner, and the details are omitted here. □

At this point we have the tools to prove Theorem 1.1.

Proof of Theorem 1.1. Recall that $q_{\mathrm{B}} = q_0 + q_1 + q_2 + \widetilde{q} + q_{\mathrm{rest}}$, where $\widetilde{q} \in \dot{C}(\mathbb{R})$ and $q_{\mathrm{rest}} \in H^s(\mathbb{R})$ by Lemmata 4.4 and 4.5. Then, by Lemmata 4.6 and 4.9 we have that $q_1 = \widetilde{q_1} + q_{1,\mathrm{rest}}$ and $q_2 = \widetilde{q_2} + q_{2,\mathrm{rest}}$. Here $\widetilde{q_1}$ and $\widetilde{q_2}$ belong to $H^s(\mathbb{R})$ for $s < \frac{1}{2}$ and can be included in $\widetilde{q}$. Similarly, $q_{1,\mathrm{rest}}$ and $q_{2,\mathrm{rest}}$ are in $\dot{C}(\mathbb{R})$ and can be included in q_{rest}. Finally Lemma 4.3 gives the representation of q_{B}. $\square$

5 Numerical examples

In this section we compute q_{B} numerically to illustrate the recovery of singularities of β. For simplicity, all our examples are real-valued and we assume that α'', q' are continuous and concentrate our attention on singularities of V. What is more, we will explore numerically if also jumps can be recovered from q_{B}.

The computation of q_{B} is carried out as in [3]. By Remark 4.2 it suffices to compute

$$f(\xi) = \int_0^\infty \mathrm{e}^{-\mathrm{i}k\xi} k^3 b(k/2)\mathrm{d}k.$$

By Fourier inversion f satisfies

$$\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}kx} f(x)\mathrm{d}x = 2\pi k^3 b(k/2), \quad k > 0.$$

In order to discretize this equation we interpolate f in piecewise linear form

$$f(x) = \sum_{j=1}^N f_j \phi_j(x),$$

where $f_j = f(x_j)$ are the (unknown) values at the grid points

$$x_j = a_1 + (j-1)(b_1 - a_1)/N, \quad j = 1, 2, \dots, N$$

over some interval of interest $[a_1, b_1]$. Here the hat functions ϕ_j are given by

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j+1}}{x_j - x_{j+1}}, & x_j \le x \le x_{j+1} \\ \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x_{j-1} \le x \le x_j \\ 0, & \text{otherwise.} \end{cases}$$

This yields

$$\sum_{j=1}^N f_j \int_{a_1}^{b_1} \mathrm{e}^{\mathrm{i}kx} \phi_j(x)\mathrm{d}x = 2\pi k^3 b(k/2) =: g(k), \quad k > 0, \tag{18}$$

where the integrals in the left-hand side are easily computed in closed form. The reflection coefficient is computed from

$$b(k) \approx \mathrm{e}^{\mathrm{i}kx_0}(u(x_0,k) - u_0(x_0,k)).$$

We use $x_0 = -10^6$ and $u \approx \sum_{j=0}^{5} u_j$. Here u_j is computed recursively from its definition (7) by splitting the integrals in three parts over supports of q, α and V and using $2n+1$ point Simpson quadratute rule to evaluate the integrals. Similarly for u_j', see (8).

Writing (18) at $k = k_s, s = 1, 2, \ldots, M$ gives us a linear system for the determination of f_j. We use the following parameter values:

$$n = 128, N = 400, k_s = 1 + \frac{(s-1)(k_{\max}-1)}{M-1}, k_{\max} = 140, M = 300, a_1 = 5 = -b_1.$$

The ill-posed linear system $Af = g$ from (18) is solved with the truncated singular value decomposition (TSVD) method. More precisely, let $A = USV^*$ be the singular value decomposition of A with singular values $s_j, j = 1, \ldots, M$ in descending order. Then the regularized solution is

$$f = VLU^*g,$$

where

$$L = \mathrm{diag}(s_1^{-1}, \ldots, s_r^{-1}, 0, \ldots, 0)$$

with s_r being the last singular value satisfying $s_r \geq s_{\mathrm{tol}}$. We use $s_{\mathrm{tol}} = 0.003$. The data is corrupted with Gaussian noise with standard deviation $\sigma = 0.01 \max |g|$.

Next we describe the sample problems. Let us denote

$$\chi_{a,b}(x) = \begin{cases} 1, & x \in]a,b[\\ 0, & x \notin]a,b[\end{cases}, \quad \varphi_{a,d,h}(x) = \begin{cases} h \exp\left(1 + \frac{d^2}{(x-a)^2-d^2}\right), & |x-a| < d \\ 0, & \text{otherwise.} \end{cases}$$

Then $\varphi_{a,d,h}(x) \in C_0^\infty(\mathbb{R})$ has height h, centered at a with support of length $2d$. We consider the following sample coefficients:

1. $V(x) = 0.5\chi_{0,0.5}(x), q(x) = \varphi_{-3,0.5,0.75}(x), \alpha(x) = 0$
2. $V(x) = 0.5\chi_{0,0.5}(x), q(x) = 0, \alpha(x) = \varphi_{3,0.4,0.75}(x)$
3. $V(x) = 0.5\chi_{0,0.5}(x), q(x) = \varphi_{-3,0.5,0.75}(x), \alpha(x) = \varphi_{3,0.4,0.75}(x)$
4. $V(x) = 0.25\chi_{0,1}(x)(1/\sqrt{x} - 1), q(x) = \varphi_{-3,0.5,0.75}(x), \alpha(x) = \varphi_{3,0.4,0.75}(x),$

see Figure 1. The TSVD reconstructions are depicted in Figure 2. For each example they indicate rather good recovery of both jumps and singularities of V even though our theory is limited to singularities.

Conclusions

We have investigated an inverse scattering problem for a fourth order operator that is a second order perturbation of the one-dimensional bi-Laplacian. The operator we have considered is not necessarily self-adjoint. Even if the coefficients are real-valued and there is no first order perturbation, our operator is more general than e.g. the squared Schrödinger operator. We have introduced inverse scattering Born approximation for this operator. Using this Born approximation and under quite limited data (in absence of uniqueness result), we have established the recovery of singularities of the coefficients. Numerical examples demonstrate the feasibility of the method. Moreover, numerically we were able to recover also the jumps of zero order perturbation, even though our theoretical results provide only the recovery of infinite singularities.

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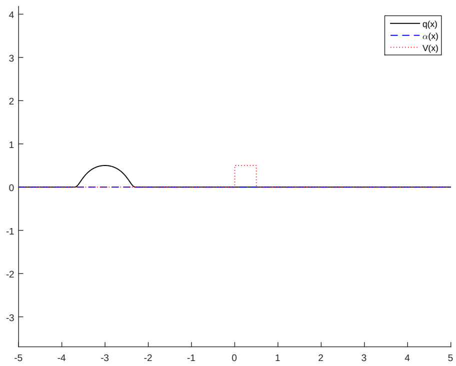
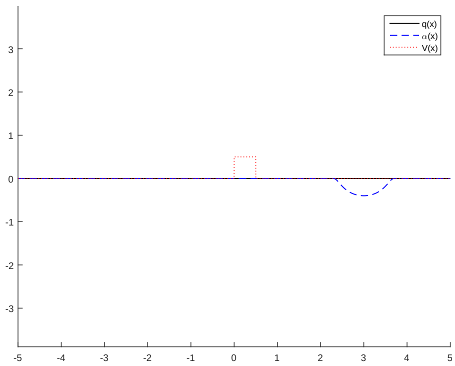
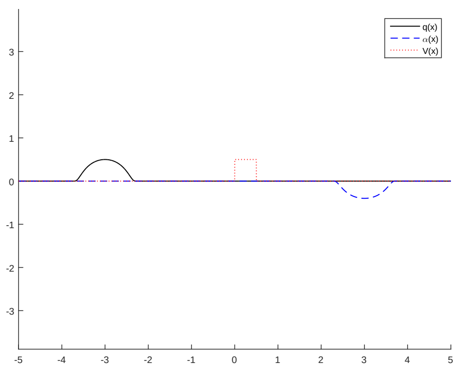


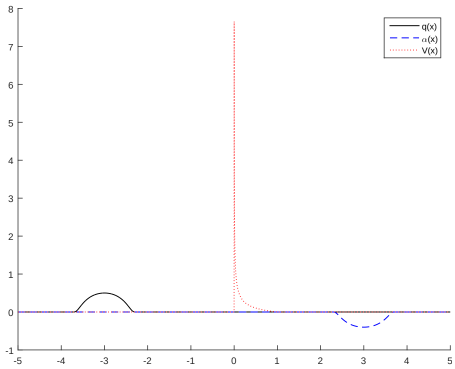
(a) Example 1



(b) Example 2

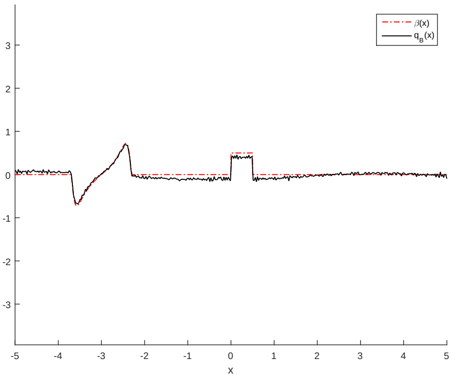


(c) Example 3

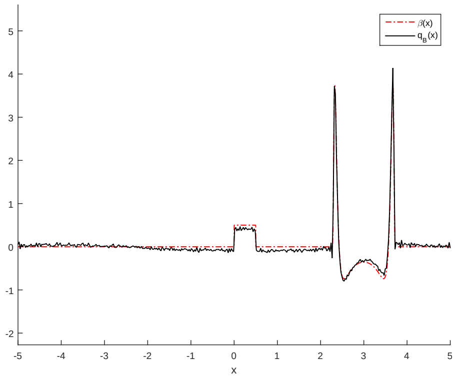


(d) Example 4

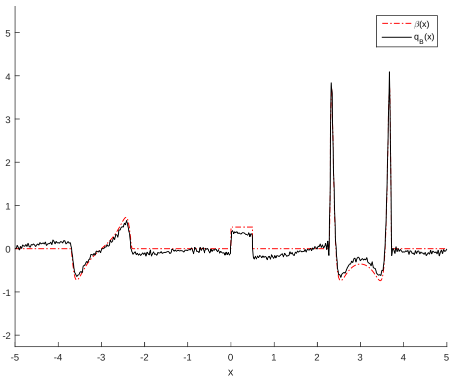
Figure 1: Sample coefficients



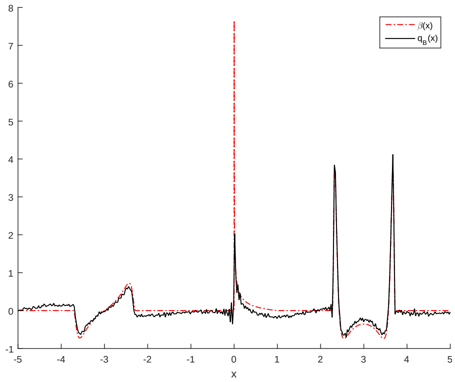
(a) Example 1



(b) Example 2



(c) Example 3



(d) Example 4

Figure 2: TSVD reconstructions, noisy data, $s_{\mathrm{tol}} = 0.003$.