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Inverse Scattering Problems for Perturbed Bi-harmonic Operator

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Abstract. Some inverse scattering problems for operator of order 4 which is the perturbation (in smaller terms) of the biharmonic operator in one and three dimensions are considered. The coefficients of this perturbation are assumed to be from some Sobolev spaces (they might be singular). The classical (as for the Schrödinger operator) scattering theory is developed for this operator of order 4. The classical inverse scattering problems are considered and their uniqueness is proved. The method of inverse scattering Born approximation and an analogue of Saito's formula are justified for this operator of order 4. Using this approximate method the reconstruction of the singularities of the unknown coefficients is proved in the scale of Sobolev spaces. The results have natural generalization for any dimensions.

INTRODUCTION

The linear Euler-Lagrange equation that arises from vibrating of the beam contains both 4th and 2nd terms (see [1])

$$u^{(4)}(x) - cu''(x) = p(x),$$

where $u(x)$ denotes the deviation from the equilibrium of the beam at point x and $p(x)$ is the density of the lateral load at x. Another example concerns to a suspension bridge which can be considered as a beam of length L, with hinged ends and whose downward deflection is measured by $u(x,t)$ subject to three forces (see [1] and [2])

$$\gamma u_{xxxx}(x,t) + u_{tt}(x,t) = -ku^{+}(x,t) + W + f(x,t), \quad u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0.$$

where γ, W, k are constants, and $f(x,t)$ is the external forcing term.

Three-dimensional transmission eigenvalue problem, i.e. the problem of finding a non-trivial pair (u,v) solving with some k^2 the problem

$$\Delta u(x) + k^2(1+m(x))u(x) = 0, \quad x \in D \subset R^3,$$

$$\Delta v(x) + k^2 v(x) = 0, \quad x \in D, \quad u(x) = v(x), \quad \frac{\partial u}{\partial \nu}(x) = \frac{\partial v}{\partial \nu}(x), \quad x \in \partial D$$

is reduced to the following operator of order 4 with singular coefficients in smaller terms (see, for example, [3])

$$(\Delta + k^2)\left(\frac{1}{m(x)}(\Delta + k^2)\right) + k^2(\Delta + k^2).$$

where $m(x)$ is the perturbation of the index of refraction.

ONE-DIMENSIONAL CASE

We consider one-dimensional 4th order equation of the form

$$u^{(4)}(x) + \alpha(x)u''(x) + q(x)u'(x) + V(x)u(x) = k^4 u(x), \quad x \in R,$$

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where $u(x)$ denotes, for example, the deflection (displacement) at the point x of the ideal beam, $k \neq 0$ is real number and the potentials $\alpha(x)$, $q(x)$ and $V(x)$ are complex-valued and integrable (for the first step).

In the problems we consider the main role is played by the special solutions, i.e., the solutions of the form

$$u(x,k) = e^{ikx} + u_{sc}(x,k),$$

where the scattered part u_{sc} satisfy the Sommerfeld radiation condition at the infinity in the one-dimensional case. In that case u_{sc} is the unique solution of the so-called Lippmann-Schwinger integral equation

$$u_{sc} = -\int_{-\infty}^{\infty} G_k^+(|x-y|)(\alpha(y)u''(y) + q(y)u'(y) + V(y)u(y))\,dy,$$

where G_k^+ is the outgoing fundamental solution of the one-dimensional Helmholtz operator $\frac{d^4}{dx^4} - k^4$.

We prove that for $|k|$ large there is a unique solution of the Lippmann-Schwinger equation

$$u_{sc} = \sum_{j=1}^{\infty} u_j, \quad u_j = -\int_{-\infty}^{\infty} G_k^+(|x-y|)(\alpha(y)u''_{j-1} + q(y)u'_{j-1} + V(y)u_{j-1})\,dy,$$

where $j = 1, 2, ...$ and $u_0 = e^{ikx}$. This solution satisfies the estimates

$$\|u - u_0\|_{L^\infty(R)} \leq \sum_{j=1}^{\infty}\left(\frac{c_0}{|k|}\right)^j \leq \frac{2c_0}{|k|}, \quad \|u' - iku_0\|_{L^\infty(R)} \leq c_0 \sum_{j=1}^{\infty}\left(\frac{c_0}{|k|}\right)^{j-1} \leq 2c_0$$

uniformly in $|k| \geq 2c_0$ with $c_0 = \|\alpha\|_{W_1^1(R)} + \|q\|_{L^1(R)} + \|V\|_{L^1(R)}$ and admits the following asymptotical representations:

$$u(x,k) = a(k)e^{ikx} + o(1), \quad u(x,k) = e^{ikx} + b(k)e^{-ikx} + o(1), \quad x \to \pm\infty,$$

respectively, where the coefficients $a(k)$ and $b(k)$ are defined as

$$a(k) = 1 - \frac{i}{4k^3}\int_{-\infty}^{\infty} e^{-iky}(\alpha(y)u'' + q(y)u' + V(y)u)\,dy, \quad b(k) = -\frac{i}{4k^3}\int_{-\infty}^{\infty} e^{iky}(\alpha(y)u'' + q(y)u' + V(y)u)\,dy$$

and they are called the ”transmission” and the ”reflection” coefficients, respectively.

The properties of $u(x,k)$ allow us to conclude that for $k \to +\infty$

$$b(k) \approx -\frac{i\sqrt{2\pi}}{4k^3}F(\beta)(2k),$$

where $\beta(y) = \frac{\alpha''(y)}{4} - \frac{q'(y)}{2} + V(y)$. This asymptotic leads to the direct scattering Born approximation u_B and to the inverse scattering Born approximation V_B, respectively

$$u_B(x,k) = e^{ikx} - \frac{i\sqrt{2\pi}}{4k^3}F(\beta)(2k)e^{-ikx}, \quad V_B(x) := F^{-1}\left(\frac{i}{2\sqrt{2\pi}}k^3 b\left(\frac{k}{2}\right)\right),$$

where F^{-1} denotes the inverse Fourier transform on the line.

The following result is valid (see [4]):

Theorem 1 *If $\alpha \in W_1^2(R) \cap H^s(R)$ for $s > \frac{3}{2}$, $q \in W_1^1(R) \cap H^r(R)$ for $r > \frac{1}{2}$ and $V \in L^1(R)$ then the inverse scattering Born approximation V_B admits the representation*

$$V_B(x) = \Re(\beta)(x) + \frac{1}{\pi}p.v.\int_{-\infty}^{\infty}\frac{\Im(\beta)(y)}{x-y}\,dy + \tilde{V}(x) + V_{rest}(x),$$

where $\tilde{V} \in C_0(R)$ and $V_{rest} \in H^t(R)$ for any $t < \frac{1}{2}$.

According to this result, if α and q are smooth enough, then we can recover any local L^p-singularities of the unknown potential $V(x)$ for any $1 \leq p < \infty$ using the Born approximation.

The next result can be considered as one of the main results of the present work. Let us consider now one-dimensional 4th order equation in the form

$$u^{(4)}(x) + 2q(x)u'(x) + q'(x)u(x) + V(x)u(x) = k^4 u(x), \quad x \in R,$$

where q is a complex-valued function and V is a real-valued function. In that case we can prove the following result.

Theorem 2 *Assume that the potentials $q(x)$ and $V(x)$ belong to the Sobolev space $W_1^1(R)$ and the Lebesgue space $L^1(R)$, respectively. Then*

$$V_B(x) - V(x) \in C(R),$$

i.e. the difference is continuous everywhere.

This result shows that all singularities and all jumps of the unknown potential $V(x)$ can be obtained exactly by the inverse scattering Born approximation with limited data. In particular, we can prove that for the function $V(x)$ being the characteristic function of an interval on the line, this interval is uniquely determined by the scattering data. Moreover, in the class of piece-constant functions V with compact support this theorem provides the uniqueness result of the inverse scattering problem.

THREE-DIMENSIONAL CASE

Concerning the scattering theory in three dimensions. We consider the operator of order 4 in the form

$$L_4 u(x) := \Delta^2 u(x) + \vec{G}(x)\nabla u(x) + V(x)u(x), \quad x \in R^3,$$

with complex-valued functions $\vec{G}$ and V. The bi-Laplacian is perturbed by first and zero order perturbations $\vec{G}$ and V, respectively. We are looking for the scattering solutions of this equation in the form (the same as in one-dimensional case)

$$u(x, k, \theta) = e^{ik(x,\theta)} + u_{sc}(x, k, \theta),$$

where $\theta \in S^2$ is the angle of the incident wave, and u_{sc} satisfies the Sommerfeld radiation condition at the infinity

$$\left(\frac{\partial}{\partial|x|} - ik\right) u_{sc}(x, k, \theta) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty.$$

The outgoing fundamental solution of three-dimensional operator $\Delta^2 - k^4$ (which corresponds to the radiation condition) is equal to

$$G_k^+(|x|) = \frac{e^{i|k||x|} - e^{-|k||x|}}{8\pi|x|k^2}.$$

These scattering solutions are the unique solutions of the analogue of Lippmann-Schwinger equation for 4th order operator

$$u(x, k, \theta) = e^{ik(x,\theta)} - \int_{R^3} G_k^+(|x - y|)\left(\vec{G}(y)\nabla u(y) + V(y)u(y)\right) dy.$$

Our basic assumptions for the coefficients $\vec{G}$ and V are: they belong to $L^1(R^3) \cap L^p(R^3)$ with $3 < p \leq \infty$. Under these conditions there exists a unique solution u of this integral equation as a series of corresponding iterations

$$u(x, k, \theta) = \sum_{j=0}^{\infty} u_j(x, k, \theta), \quad ||u - u_0||_{L^\infty(R^3)} \leq \frac{8c_0}{|k|}, \quad ||\nabla u - ik\theta u_0||_{L^\infty(R^3)} \leq 8c_0, \quad |k| \geq 8c_0,$$

where $c_0 = \sqrt{3}\max(c_1, c_2)$ and the constant $c_m, m = 1, 2$ are defined as

$$c_m = \sup_{x \in R^3} \int_{R^3} \frac{|\vec{G}(y)| + |V(y)|}{|x - y|^m} dy.$$

If we assume now that $\vec{G}$ and V belong to the weighted Lebesgue space $L^p_\delta(R^3)$ with $3 < p \leq \infty$ and $\delta > 3 - \frac{3}{p}$. Then the following asymptotical representation holds: $k \geq k_0$ with big enough $k_0 > 0$ and fixed,

$$u(x,k,\theta) = e^{ik(x,\theta)} - \frac{e^{ik|x|}}{8\pi|x|} A(k,\theta,\theta') + o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

where $\theta' = \frac{x}{|x|}$ is the angle of observation. The function $A(k,\theta,\theta')$ is called the scattering amplitude and is defined by

$$A(k,\theta,\theta') = \frac{1}{k^2} \int\limits_{R^3} e^{-ik(y,\theta')} \left(\vec{G}(y)\nabla u(y,k,\theta) + V(y)u(y,k,\theta)\right) dy.$$

This function $A(k,\theta,\theta')$ gives us the data for inverse scattering problems. The first result is the analogue of Saito's formula for 4th order operator (see, for example, [5]).

Theorem 3 *(Saito's formula). If $\vec{G} \in W^1_{p,\delta}(R^3)$ and $V \in L^p_\delta(R^3)$ where $3 < p \leq \infty$ and $\delta > 3 - \frac{3}{p}$, then the limit*

$$\lim_{k\to+\infty} k^4 \int\limits_{S^2\times S^2} e^{-ik(\theta-\theta',x)} A(k,\theta,\theta')\, d\theta\, d\theta' = 8\pi^2 \int\limits_{R^3} \frac{\beta(y)}{|x-y|^2}\, dy$$

holds uniformly in $x \in R^3$ and with $\beta := -\frac{1}{2}\nabla\vec{G} + V$.

An important consequence of Saito's formula is the following uniqueness result.

Theorem 4 *(Uniqueness). Let $\vec{G}_1, V_1$ and $\vec{G}_2, V_2$ be as before. If the corresponding scattering amplitudes $A_1(k,\theta,\theta')$ and $A_2(k,\theta,\theta')$ coincide for some sequence $k_j \to +\infty$ and for all angles $\theta,\theta' \in S^2$ then the coefficients β_1 and β_2 are equal a.e.*

Another inverse scattering problem is the inverse backscattering problem. The data for this problem is given by the scattering amplitude for all $k > 0$ arbitrary large and for all angles $\theta \in S^2$ with $\theta' = -\theta$. Then

$$A(k,\theta,-\theta) = \frac{1}{k^2} F(\beta)(2k\theta) + o(1), \quad k \to \infty,$$

where F is the Fourier transform in R^3. This asymptotic first leads to the direct backscattering Born approximation

$$A^b_B(k,\theta,-\theta) = \frac{1}{k^2} F(\beta)(2k\theta), \quad u_B(x,k,\theta) = e^{ik(x,\theta)} - \frac{e^{ik|x|}}{8\pi|x|} A^b_B(k,\theta,-\theta).$$

It justifies the definition of the inverse backscattering Born approximation $V^b_B(x)$ of the potential function $\beta(x)$

$$V^b_B(x) = F^{-1}_{(k,\theta)\to x}\left(A\left(\frac{k}{2},\theta,-\theta\right)\right)(x) = \frac{1}{32\pi^3} \int_0^\infty k^4\, dk \int\limits_{S^2} e^{ik(x,\theta)} A\left(\frac{k}{2},\theta,-\theta\right) d\theta.$$

In the backscattering problem we may recover the singularities of $\beta(x)$ via the Born approximation $V^b_B(x)$.

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