Two-dimensional inverse scattering for quasi-linear biharmonic operator

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February 8, 2021

Abstract

The subject of this work concerns the classical direct and inverse scattering problems for quasi-linear perturbations of the two-dimensional biharmonic operator. The quasi-linear perturbations of the first and zero order might be complex-valued and singular. We show the existence of the scattering solutions to the direct scattering problem in the Sobolev space $W^1_\infty(\mathbb{R}^2)$. Then the inverse scattering problem can be formulated as follows: does the knowledge of the far field pattern uniquely determine the unknown coefficients for given differential operator? It turns out that the answer to this classical question is affirmative for quasi-linear perturbations of the biharmonic operator. Moreover, we present a numerical method for the reconstruction of unknown coefficients, which from the practical point of view can be thought of as recovery of the coefficients from fixed energy measurements.

1 Introduction

We consider the following two-dimensional quasi-linear differential operator

$$H_4 u(x) := \Delta^2 u(x) + \vec{W}(x,|u|)\cdot\nabla u(x) + V(x,|u|)u(x),$$

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where $\vec{W}$ is a vector-valued complex (in general) function and V is a scalar complex-valued potential. Our basic assumptions for the coefficients of H_4 are:

Assumption 1.1. *We assume that functions $\overrightarrow{W}$ and V satisfy the following conditions:*

- $|\overrightarrow{W}(x,s)| \leq C_\rho\ \alpha_W(x)$ and $|V(x,s)| \leq C_\rho\ \alpha_V(x)$
- $|\overrightarrow{W}(x,s_1) - \overrightarrow{W}(x,s_2)| \leq \widetilde{C}_\rho |s_1 - s_2|\ \beta_W(x)$
- $|V(x,s_1) - V(x,s_2)| \leq \widetilde{C}_\rho |s_1 - s_2| \beta_V(x)$,

where $0 < s, s_1, s_2 \leq \rho$. Precise conditions for functions α_W, α_V, β_W and β_V will be given in each theorem, but they are always assumed to satisfy the following decay property

$$|f(x)| \leq \frac{C}{|x|^\mu}, \quad |x| \geq R, \tag{1}$$

where R is big enough and $\mu > 2$.

The motivation and the interest to study multi-dimensional operator of order four appear for example in the study of elasticity and in the theory of vibration of beams. As a concrete example, the non-linear beam equation (see [7])

$$\partial_t^2 U(x,t) + \Delta_x^2 U(x,t) + m(x)|U(x,t)|^p U(x,t) = 0,$$

where $p \geq 0$, under time-harmonic assumptions $U(x,t) = u(x)\mathrm{e}^{-\mathrm{i}\omega t}$ leads to the equation

$$\Delta^2 u(x) + m(x)|u(x)|^p u(x) = \omega^2 u(x).$$

The wave parameter ω is fixed (in general), but nevertheless we can consider it fixed but big enough in order to apply limiting process and appropriate numerical methods. This allows to consider some scattering problems with high frequency for this potential equation. For the scattering problems (including non-linear equations), see for example [13] and references therein. Concerning inverse problems for biharmonic and polyharmonic operators we mention some solutions to inverse boundary value problems (see [10, 11]).

Assumption 1.1 includes the power-type nonlinearities of the nonlinear beam equation described above, and most other physically relevant nonlinearities, such as the saturation and sinc nonlinearities

$$q(x)\frac{|u|^2}{1+|u|^2}, \quad \text{and} \quad q(x)\frac{\sin(|u|)}{|u|}.$$

The present work follows the footsteps of [5, 6, 9, 15, 17]. In [5], [9] and [15] the inverse scattering problems for multi-dimensional nonlinear Schrödinger operators were considered. In [17] similar study was carried out for a multi-dimensional biharmonic operator with linear perturbations of first and zero order (see also [18]). In [6] fixed energy problem (the inverse scattering problem with fixed wave number) for nonlinear Schrödinger operator is studied. In [19] these problems were considered for biharmonic operator with first and zero order nonlinear perturbations on the line while a general nonlinear Schrödinger operator on the line was investigated in [14]. The purpose of this work is to initiate similar studies in the two-dimensional case.

The present work is concerned with the following scattering problem for the operator H_4 given by

$$H_4 u(x) = k^4 u(x), \quad u(x) = u_0(x) + u_{\text{sc}}(x), \quad u_0(x) = \mathrm{e}^{\mathrm{i}k(x,\theta)}, \quad \theta \in \mathbb{S}^1, \tag{2}$$

where u_0 is a plane wave travelling in direction θ with wavenumber k and the scattered wave u_{sc} and its Laplacian Δu_{sc} are required to satisfy Sommerfeld radiation condition at the infinity

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial f(x)}{\partial r} - \mathrm{i}k f(x) \right) = 0, \quad r = |x|, \text{ for both } f = u_{\text{sc}} \text{ and } f = \Delta u_{\text{sc}}. \tag{3}$$

We are looking for the scattering solutions u_{sc} to the equation (2) in the Sobolev spaces $H^4_{\text{loc}}(\mathbb{R}^2) \cap W^2_\infty(\mathbb{R}^2)$. Literally repeating the proof of Theorem 3.3 in [17] (see also [3]) and using the radiation conditions (3) we obtain that if $u = u_0 + u_{sc}, u_{sc} \in H^4_{\text{loc}}(\mathbb{R}^2) \cap W^2_\infty(\mathbb{R}^2)$ solves (2) then it also solves the integral Lippmann-Schwinger equation (see [17] for details)

$$u(x) = u_0(x) - \int_{\mathbb{R}^2} G_k^+(|x-y|)(\vec{W}(y,|u|) \cdot \nabla u(y) + V(y,|u|)u(y))\mathrm{d}y,$$

where G_k^+ is the outgoing fundamental solution of the operator $(\Delta^2 - k^4)$ in $\mathbb{R}^2$, i.e., the kernel of the integral operator $(\Delta^2 - k^4 - \mathrm{i}0)^{-1}$. This function G_k^+ in $\mathbb{R}^2$ has the following form

$$G_k^+(|x|) = \frac{\mathrm{i}}{8k^2} \left(H_0^{(1)}(k|x|) + \frac{2\mathrm{i}}{\pi} K_0(k|x|) \right), \quad k > 0,$$

where $H_0^{(1)}$ is the Hankel function of the first kind and order zero and K_0 is the Macdonald function of order zero.

Since u_0 is just a bounded function with the norm $\|u_0\|_{L^\infty(\mathbb{R}^2)} = 1$ it is more convenient to study (instead of (2)) the equivalent integral equation for the scattered wave, namely

$$u_{\mathrm{sc}}(x) = -\int_{\mathbb{R}^2} G_k^+(|x-y|)\Big(\vec{W}(y,|u_0+u_{\mathrm{sc}}|)\cdot\nabla\left(u_0+u_{\mathrm{sc}}\right)(y)$$
$$+V(y,|u_0+u_{\mathrm{sc}}|)\left(u_0+u_{\mathrm{sc}}\right)(y)\Big)\mathrm{d}y. \quad (4)$$

Once we have shown that the unique solution exists, by repeating the calculation that was done in [9, 17] we obtain for fixed $k>0$ the asymptotic behaviour for $|x|\to\infty$ of the function u_{sc}

$$u_{\mathrm{sc}}(x) = -\frac{\mathrm{i}+1}{8\sqrt{\pi}}\frac{\mathrm{e}^{\mathrm{i}k|x|}}{k^{\frac{5}{2}}|x|^{\frac{1}{2}}}A(k,\theta',\theta)+o\left(\frac{1}{|x|^{\frac{1}{2}}}\right),$$

where $\theta' = x/|x|$ is the angle of observation and function A is called the scattering amplitude and it is given via the formula

$$A(k,\theta',\theta) = \int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}k(\theta',y)}\left[\overrightarrow{W}(y,|u|)\cdot\nabla u+V(y,|u|)u\right]\mathrm{d}y.$$

From the point of view of inverse problems one regards this scattering amplitude as one possible scattering data. For these purposes one requires the scattering amplitude to be known for all possible angles θ and θ' and all arbitrarily high frequencies $k>0$.

The main result of this work is the following Saito's formula. Similarly to other scattering problems it allows us to obtain a uniqueness result for the inverse problem and a representation formula for the unknown combination $\beta(x) := V(x,1)-\frac{1}{2}\nabla\cdot\overrightarrow{W}(x,1)$ which appears in Theorem 1.2. What is more, it was shown in [16] that Saito's formula can be inverted numerically by considering large value for $k>0$ and solving the convolution type equation for β.

Theorem 1.2 (Saito's formula)**.** *Let functions $\overrightarrow{W}$ and V satisfy Assumption 1.1 with functions α_W, α_V, β_W and β_V belonging to space $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ where $2<p\le\infty$ and they all satisfy condition* (1)*. If in addition we assume that $\nabla\cdot\overrightarrow{W}(\cdot,1)\in L^p_{\mathrm{loc}}(\mathbb{R}^2)$, with $2<p\le\infty$, and has the behavior at the infinity* (1) *then*

$$\lim_{k\to\infty} k\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}A(k,\theta',\theta)\mathrm{d}\theta'\mathrm{d}\theta = 4\pi\int_{\mathbb{R}^2}\frac{V(y,1)-\frac{1}{2}\nabla\cdot\overrightarrow{W}(y,1)}{|x-y|}\mathrm{d}y, \quad (5)$$

uniformly in $x\in\mathbb{R}^2$.

The most significant consequences of Saito's formula are contained in the following corollaries.

Corollary 1.3. *Let* $\beta_1(x) = -\frac{1}{2}\nabla\cdot\overrightarrow{W}_1(x,1) + V_1(x,1)$ *and* $\beta_2(x) = -\frac{1}{2}\nabla\cdot\overrightarrow{W}_2(x,1) + V_2(x,1)$ *be as in Theorem 1.2 and* $A_1(k,\theta',\theta)$ *and* $A_2(k,\theta',\theta)$ *be the corresponding scattering amplitudes arising from these two scattering problems. If these scattering amplitudes coincide for all angles* θ,θ' *and for some sequence* $k_j\to\infty$ *as* $j\to\infty$, *then* $\beta_1=\beta_2$ *almost everywhere in* $\mathbb{R}^2$.

Corollary 1.4. *If all conditions of Theorem 1.2 are satisfied, then*

$$\beta(x) = \frac{1}{8\pi^2}\lim_{k\to\infty} k^2 \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} A(k,\theta',\theta)|\theta-\theta'|\mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}\mathrm{d}\theta\mathrm{d}\theta',$$

in the sense of tempered distributions.

Proofs for these corollaries can be found for example in [9].

Remark 1.5. Heuristically, if the non-linearity can be expanded by the Taylor formula, we recover the principal part of the expansion. If the non-linearity is of some known type, for example $V(x,|u|) = q(x)|u|^r$ for some function q and $r\geq 1$, then we can recover the unknown potential q uniquely.

The following notations are used throughout the text. The symbol $L^p_\delta(\mathbb{R}^2)$, $1\leq p\leq\infty, \delta\in\mathbb{R}$ denotes the p-based Lebesgue space over $\mathbb{R}^2$ with norm

$$\|f\|_{L^p_\delta} = \left(\int_{\mathbb{R}^2}(1+|x|)^{\delta p}|f(x)|^p\mathrm{d}x\right)^{1/p}.$$

The weighted Sobolev spaces $W^m_{p,\delta}(\mathbb{R}^2)$ are defined as the spaces of functions whose weak derivatives up to order $m\geq 0$ belong to $L^p_\delta(\mathbb{R}^3)$ and the norm is defined as follows,

$$\|f\|_{W^m_{p,\delta}} = \sum_{|\alpha|\leq m}\|D^\alpha f\|_{L^p_\delta}.$$

For L^2-based space we use the special notation $H^m_\delta(\mathbb{R}^2) = W^m_{2,\delta}(\mathbb{R}^2)$. Throughout the text the symbol C (compare with the constants C with some special index and special meaning) is used to denoted generic positive constant whose value may change from line to line.

The paper is organized as follows. In Section 2 we study the direct scattering problem and establish its unique solvability under some suitable assumptions. Section 3 is devoted to proving the main result of the paper, i.e., Saito's formula. In Section 4 we demonstrate a new numerical method for reconstruction of function β.

2 Existence of the scattering solutions

The goal of this section is to show that the scattering problem (2) has a unique solution. To this end the following theorem holds.

Theorem 2.1. *Let functions $\vec{W}$ and V be as in Assumption 1.1, with $\alpha_W, \beta_W \in L^p_{\text{loc}}(\mathbb{R}^2)$, where $1 < p \leq \infty$ and $\alpha_V, \beta_V \in L^1_{\text{loc}}(\mathbb{R}^2)$. Then for any $\rho > 0$ there exists $k_0 > 0$ such that the equation (4) has a unique solution in $B_\rho(0) := \{f \in W^1_\infty(\mathbb{R}^2) : \|f\|_{W^1_\infty} \leq \rho\}$, for all $k \geq k_0$. Moreover, depending on $1 < p \leq \infty$, the solution will agree to the following norm estimates*

$$\|u_{\text{sc}}\|_{W^1_\infty} \leq C \begin{cases} k^{-1/2}, & 4/3 < p \leq \infty \\ k^{-\varepsilon}, & 1 < p \leq 4/3, \end{cases}$$

for some constant C and any $0 < \varepsilon < \frac{2p-2}{p}$.

Proof. Let us start by defining an operator F by setting

$$F(\varphi)(x) = -\int_{\mathbb{R}^2} G_k^+(|x-y|)\Big[\overrightarrow{W}(y, |u_0+\varphi|) \cdot \nabla(u_0+\varphi) + V(y, |u_0+\varphi|)(u_0+\varphi)\Big]\mathrm{d}y,$$

where $\varphi \in W^1_\infty(\mathbb{R}^2)$. We will show that operator F is a contraction from $B_\rho(0)$ to itself. The asymptotic behaviour of functions $H_j^{(1)}$ and K_j (see [12]) leads us to the following estimates for the fundamental solution

$$\left|G_k^+(|x-y|)\right| \leq \frac{C_0}{k^2}$$

and

$$\left|\nabla_x G_k^+(|x-y|)\right| \leq C_0 \begin{cases} \dfrac{1}{k}, & k|x-y| < 1 \\ \dfrac{1}{k^{3/2}|x-y|^{1/2}}, & k|x-y| > 1. \end{cases} \tag{6}$$

The latter estimate gives us that for any $0 \leq \varepsilon \leq \frac{1}{2}$ there exists constant C_ε, such that we have

$$|\nabla_x G_k^+(|x-y|)| \leq \frac{C_\varepsilon}{k^{1+\varepsilon}|x-y|^\varepsilon}, \tag{7}$$

for all $x, y \in \mathbb{R}^2$ and $k > 0$. From (6) it follows that the value of ε can not be bigger than $\frac{1}{2}$ in (7). Using this behaviour of function G_k^+, the following inequality holds for all $\varphi \in B_\rho(0)$, when $k \geq 1$,

$$
\begin{aligned}
|F(\varphi)(x)| &\leq \frac{C_0}{k^2}\int_{\mathbb{R}^2}\Big(|\vec{W}(y,|u_0+\varphi|)|(k+\rho)+|V(y,|u_0+\varphi|)|(1+\rho)\Big)\,\mathrm{d}y \\
&\leq \frac{C'}{k},
\end{aligned}
\tag{8}
$$

where

$$
C' = C_0 C_{\rho+1}(1+\rho)\left(\|\alpha_W\|_{L^1}+\|\alpha_V\|_{L^1}\right),
$$

uniformly in $x \in \mathbb{R}^2$. Conditions for functions α_W and α_V guarantee that this value is always finite. For the gradient of $F(\varphi)$, we use the estimate (7). If $\frac{4}{3} < p \leq \infty$, we choose $\varepsilon = \frac{1}{2}$ and otherwise we pick any $0 < \varepsilon < \frac{2p-2}{p}$ and we have

$$
\begin{aligned}
|\nabla_x F(\varphi)(x)| \leq &\int_{\mathbb{R}^2}\frac{C_\varepsilon}{k^{1+\varepsilon}|x-y|^\varepsilon}C_{\rho+1}\,|\alpha_W(y)||\,(k+\rho)\mathrm{d}y \\
&+\int_{\mathbb{R}^2}\frac{C_0}{k}C_{\rho+1}\,|\alpha_V(y)|\,(1+\rho)\mathrm{d}y \leq \frac{C^*}{k^\varepsilon},
\end{aligned}
\tag{9}
$$

where

$$
C^* = C_\varepsilon C_{\rho+1}(1+\rho)\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\alpha_W(y)|}{|x-y|^\varepsilon}\mathrm{d}y + C_0C_{\rho+1}(1+\rho)\|\alpha_V\|_{L^1},
$$

which is finite due to $\alpha_W \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$, with $1 < p \leq \infty$. Indeed,

$$
\int_{\mathbb{R}^2}\frac{|\alpha_W(y)|}{|x-y|^\varepsilon}\mathrm{d}y \leq \int_{|y|\leq R}\frac{|\alpha_W(y)|}{|x-y|^\varepsilon}\mathrm{d}y + C\int_{|y|>R}\frac{1}{|x-y|^\varepsilon|y|^\mu}\mathrm{d}y = I_1 + I_2.
$$

For I_1 we may use Hölder inequality and we have

$$
I_1 \leq \|\alpha_W\|_{L^p}\left(\int_{|y|\leq R}|x-y|^{\frac{-\varepsilon p}{p-1}}\mathrm{d}y\right)^{\frac{p-1}{p}} \leq C_R\|\alpha_W\|_{L^p}
$$

uniformly in $x \in \mathbb{R}^2$, since $\varepsilon < \frac{2p-2}{p}$.

In order to show that I_2 is bounded uniformly in $x \in \mathbb{R}^2$, we consider two cases. Let us first assume that $|x| \leq R/2$. Then, inside the area of integration in I_2, we have $|x-y| \geq R/2$ and therefore (for these values of x)

$$
I_2 \leq C\left(\frac{2}{R}\right)^\varepsilon \int_{|y|>R}|y|^{-\mu}\mathrm{d}y \leq C'_R
$$

due to $\mu > 2$.

In case when $|x| > R/2$, for any $\mu - 2 < \alpha < \mu + \varepsilon - 2$ we have

$$\begin{aligned} I_2 &= C \int_{|y|>R} \frac{1}{|x-y|^{\varepsilon}|y|^{\mu}} \mathrm{d}y \leq C R^{-\alpha} \int_{\mathbb{R}^2} \frac{1}{|x-y|^{\varepsilon}|y|^{\mu-\alpha}} \mathrm{d}y \\ &\leq R^{-\alpha} \frac{C}{|x|^{\varepsilon+\mu-\alpha-2}} \leq C_R''. \end{aligned}$$

Therefore, we may conclude that uniformly in $x \in \mathbb{R}^2$

$$\sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\alpha_W(y)|}{|x-y|^{\varepsilon}} \mathrm{d}y \leq C.$$

Combining estimates (8) and (9), gives us that

$$\|F(\varphi)\|_{W^1_\infty} \leq \rho,$$

whenever $k \geq k_1 := \max\left\{1, \left(\frac{C'+C^*}{\rho}\right)^{1/\varepsilon}\right\}$. Next, we will show that F is contraction, i.e., there exists $0 < \tau < 1$ such that for all $\psi, \varphi \in B_\rho(0)$,

$$\|F(\psi) - F(\varphi)\|_{W^1_\infty} \leq \tau \|\psi - \varphi\|_{W^1_\infty}.$$

Let $\psi, \varphi \in B_\rho(0)$. We start by splitting the difference into two parts

$$\begin{aligned} &F(\varphi)(x) - F(\psi)(x) \\ &= -\int_{\mathbb{R}^2} G_k^+(|x-y|) \Big[\overrightarrow{W}(y, |u_0+\varphi|) \cdot \nabla(u_0+\varphi) \\ &\qquad\qquad - \overrightarrow{W}(y, |u_0+\psi|) \cdot \nabla(u_0+\psi)\Big] \mathrm{d}y \\ &- \int_{\mathbb{R}^2} G_k^+(|x-y|) \left[V(y, |u_0+\varphi|)(u_0+\varphi) - V(y,|u_0+\psi|)(u_0+\psi)\right] \mathrm{d}y \\ &= I' + I''. \end{aligned}$$

The first part can be estimated as

$$\begin{aligned} |I'| \leq &\int_{\mathbb{R}^2} |G_k^+(|x-y|)| \left|\vec{W}(y,|u_0+\psi|) - \vec{W}(y,|u_0+\varphi|)\right| |\nabla u_0(y)| \mathrm{d}y \\ &+ \int_{\mathbb{R}^2} |G_k^+(|x-y|)| \left|\vec{W}(y,|u_0+\psi|) - \vec{W}(y,|u_0+\varphi|)\right| |\nabla \psi| \mathrm{d}y \\ &+ \int_{\mathbb{R}^2} |G_k^+(|x-y|)| \left|\vec{W}(y,|u_0+\psi|)\right| |\nabla\varphi - \nabla\psi| \mathrm{d}y \\ \leq &\frac{\|\beta_W\|_{L^1}}{k} \widetilde{C}_{\rho+1}(1+\rho)\|\varphi-\psi\|_{L^\infty} + \frac{\|\alpha_W\|_{L^1}}{k^2} C_{\rho+1} \|\nabla\varphi - \nabla\psi\|_{L^\infty}. \end{aligned} \tag{10}$$

Similar calculation shows that

$$|I''| \leq \frac{\|\beta_V\|_{L^1}}{k^2}\widetilde{C}_{\rho+1}(1+\rho)\|\varphi-\psi\|_{L^\infty} + \frac{\|\alpha_V\|_{L^1}}{k^2}C_{\rho+1}\|\varphi-\psi\|_{L^\infty}. \tag{11}$$

Finally, for the difference of gradients of functions $F(\varphi)$ and $F(\psi)$, we may again pick $0 < \varepsilon < 2 - \frac{2}{p}$ and similarly as above, we have

$$|\nabla F(\varphi)(x) - \nabla F(\psi)(x)| \leq \left[\frac{A_1}{k^\varepsilon} + \frac{A_2}{k}\right]\|\varphi-\psi\|_{W^1_\infty}, \tag{12}$$

where the constants A_1 and A_2 are given as

$$A_1 = C_\varepsilon \widetilde{C}_{\rho+1} \sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\beta_W(y)|}{|x-y|^\varepsilon}\mathrm{d}y \quad \text{and}$$

$$A_2 = C_0\left[\widetilde{C}_{\rho+1}\rho\|\beta_W\|_{L^1} + C_{\rho+1}\|\alpha_W\|_{L^1} + \widetilde{C}_{\rho+1}(1+\rho)\|\beta_V\|_{L^1} + C_{\rho+1}\|\alpha_V\|_{L^1}\right].$$

Our conditions for functions α_W, β_W, α_V and β_W guarantee that these constants are finite. Now, by combining estimates (10), (11) and (12) we have that

$$\|F(\varphi) - F(\psi)\|_{W^1_\infty} \leq \frac{\widetilde{\tau}}{k^\varepsilon}\|\varphi-\psi\|_{W^1_\infty}, \tag{13}$$

where $\varepsilon > 0$ and

$$\widetilde{\tau} = C_\varepsilon\widetilde{C}_{\rho+1}\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\beta_W(y)|}{|x-y|^\varepsilon}\mathrm{d}y + \widetilde{C}_{\rho+1}(1+\rho(1+C_0))\|\beta_W\|_{L^1}$$
$$+ C_{\rho+1}(1+C_0)\left(\|\alpha_W\|_{L^1} + \|\alpha_V\|_{L^1}\right) + \widetilde{C}_{\rho+1}(1+\rho)(1+C_0)\|\beta_V\|_{L^1}.$$

From this it follows that operator F is a contraction for all $k \geq k_2 := \max\{1, \widetilde{\tau}^{1/\varepsilon}\}$. The estimate (13) also yields an estimate for the scattered field $u_{\rm sc}$ in terms of iterations defined by $\widetilde{u}_j := F(\widetilde{u}_{j-1}), j = 1, 2, \ldots$ and $\widetilde{u}_0 = 0$. This follows from Banach fixed point theorem's a priori estimates (see e.g, [20, Theorem 1.A(iii)]). Indeed, using $\widetilde{u}_1 = F(0)$, we have

$$\|u_{\rm sc} - \widetilde{u}_j\|_{W^1_\infty(\mathbb{R}^2)} \leq \frac{(\widetilde{\tau}/k^\varepsilon)^j}{1-\widetilde{\tau}/k}\|F(0)\|_{W^1_\infty(\mathbb{R}^2)} \leq Ck^{-(j+1)\varepsilon}, \tag{14}$$

when $k \geq k_0$. In particular, when we consider functions $\vec{W}$ and V that satisfy Assumption 1.1 with $2 < p \leq \infty$, we have

$$\|u_{\rm sc} - \widetilde{u}_j\|_{W^1_\infty(\mathbb{R}^2)} \leq Ck^{-\frac{j+1}{2}}. \tag{15}$$

The norm estimate for $u_{\rm sc}$ given in Theorem 2.1 follows immediately from the error estimate (14) and estimates (8) and (9) for $\varphi = 0$. This finishes the proof. □

Lemma 2.2. *Let $\vec{W}$ and V satisfy Assumption 1.1. Then the following norm estimates hold*

$$\|u_{\mathrm{sc}}\|_{H^j_{-\delta}(\mathbb{R}^2)} \leq \frac{C}{k^{2-j}}, \quad j = 0, 1, 2$$

with some constant $C > 0$, where $\delta > \frac{1}{2}$.

Proof. This lemma follows from the mapping properties of integral operator with kernel G_k^+ and from the Agmon's estimate [1, Appendix A, Remark 2], which states that for $\delta > \frac{1}{2}$ and $k \geq 1$,

$$\sum_{|\alpha| \leq 4} k^{3-|\alpha|} \|D^\alpha f\|_{L^2_{-\delta}(\mathbb{R}^n)} \leq C \left\|(\Delta^2 - k^4) f\right\|_{L^2_\delta(\mathbb{R}^n)},$$

where the constant $C > 0$ depends only on n and δ. For details, see [17]. □

Theorem 2.3. *Let u_{sc} be the solution of (4) obtained in Theorem 2.1. Then for fixed $k > k_0$ it has the following asymptotic representation as $|x| \to \infty$*

$$u_{\mathrm{sc}}(x) = -\frac{\mathrm{i}+1}{8\sqrt{\pi}} \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{1/2} k^{5/2}} A(k, \theta', \theta) + o\left(\frac{1}{|x|^{\frac{1}{2}}}\right), \tag{16}$$

where, $\theta' \in \mathbb{S}^2$ is the angle of observation, i.e., $\theta' = \frac{x}{|x|}$ and function A is called a scattering amplitude and it is given as

$$A(k, \theta', \theta) = \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta', y)} \left[\overrightarrow{W}(y, |u|) \cdot \nabla u + V(y, |u|) u\right] \mathrm{d}y.$$

Proof. See [17, Theorem 5.2]. □

3 Proof of Saito's formula

Proof. We start by splitting the left hand side of (5) into four parts

$$\begin{aligned}
&k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta - \theta', x)} A(k, \theta', \theta) \mathrm{d}\theta' \mathrm{d}\theta \\
&\quad = k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta - \theta', x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta' - \theta, y)} \mathrm{i}k\theta \cdot \vec{W}(y, |u|) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\
&\qquad + k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta - \theta', x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta', y)} \vec{W}(y, |u|) \cdot \nabla u_{\mathrm{sc}}(y) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\
&\qquad + k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta - \theta', x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta' - \theta, y)} V(y, |u|) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\
&\qquad + k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta - \theta', x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta', y)} V(y, |u|) u_{\mathrm{sc}}(y) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

In what follows, we will make a frequent use of the following result,

$$\int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta, x-y)} \mathrm{d}\theta = 2\pi J_0(k|x-y|), \tag{17}$$

where J_0 is the Bessel function of first kind and order 0. Some justification for this can be found in [8]. Function J_0 satisfies the following asymptotic behaviour

$$J_0(z) = \begin{cases} O(1), & z \to 0 \\ \sqrt{\dfrac{2}{\pi z}} \cos(z - \dfrac{1}{4}\pi) + O(z^{-\frac{3}{2}}), & z \to +\infty. \end{cases}$$

Let us first consider the term I_1. By substituting $\vec{W}(y, |u|) = \vec{W}(y, 1) + \vec{W}(y, |u|) - \vec{W}(y, 1)$ we obtain

$$\begin{aligned} I_1 &= k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta', x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta, y)} \mathrm{i}k\theta \cdot \vec{W}(y, 1) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\ &\quad + k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta', x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta, y)} \mathrm{i}k\theta \cdot \left(\vec{W}(y, |u|) - \vec{W}(y, 1) \right) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\ &= I' + I''. \end{aligned}$$

When we substitute $\theta = \theta - \theta' + \theta'$ into I', we have

$$\begin{aligned} I' &= k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta', x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta, y)} \mathrm{i}k(\theta - \theta') \cdot \vec{W}(y, 1) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\ &\quad + k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta', x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta, y)} \mathrm{i}k\theta' \cdot \vec{W}(y, 1) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\ &= k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta', x)} \int_{\mathbb{R}^2} \nabla_y \mathrm{e}^{-\mathrm{i}k(\theta'-\theta, y)} \cdot \vec{W}(y, 1) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta - I', \end{aligned}$$

where $-I'$ appears on the last row due to asymmetricity with respect to θ and θ'.

If we now rearrange the equation above, we may integrate by parts and use the result (17) to obtain

$$
\begin{aligned}
2I' &= -k \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)} \nabla \cdot \vec{W}(y,1) \mathrm{d}y \mathrm{d}\theta' \mathrm{d}\theta \\
&= -4\pi^2 k \int_{\mathbb{R}^2} J_0^2(k|x-y|) \nabla \cdot \vec{W}(y,1) \mathrm{d}y \\
&= -4\pi^2 k \int_{k|x-y|<1} \nabla \cdot \vec{W}(y,1) \mathrm{d}y \\
&\quad - 4\pi^2 k \int_{k|x-y|>1} \left(\frac{\sqrt{2}\cos(k|x-y| - \frac{1}{4}\pi)}{\sqrt{k\pi|x-y|}} + O\left(\frac{1}{k^{3/2}|x-y|^{3/2}}\right)\right)^2 \nabla \cdot \vec{W}(y,1) \mathrm{d}y \\
&= -4\pi^2 k \int_{k|x-y|<1} \nabla \cdot \vec{W}(y,1) \mathrm{d}y \\
&\quad - 4\pi \int_{k|x-y|>1} \frac{2}{|x-y|} \cos^2(k|x-y| - \frac{1}{4}\pi) \nabla \cdot \vec{W}(y,1) \mathrm{d}y \\
&\quad + O(1) \int_{k|x-y|>1} \frac{\nabla \cdot \vec{W}(y,1)}{k|x-y|^2} \mathrm{d}y. \qquad (18)
\end{aligned}
$$

Here the first term goes to zero as $k \to \infty$ uniformly in x due to Assumption 1.1 and

$$
\begin{aligned}
&4\pi^2 k \int_{k|x-y|<1} |\nabla \cdot \vec{W}(y,1)| \mathrm{d}y \\
&\leq 4\pi^2 k \left(\int_{k|x-y|<1} 1 \mathrm{d}y \right)^{1/p'} \left(\int_{k|x-y|<1} |\nabla \cdot \vec{W}(y,1)|^p \mathrm{d}y \right)^{1/p} \\
&\leq C k^{1-2/p'} \left(\int_{k|x-y|<1} |\nabla \cdot \vec{W}(y,1)|^p \mathrm{d}y \right)^{1/p},
\end{aligned}
$$

where p' is the Hölder conjugate of $2 < p \leq \infty$ and thus $1 - 2/p' < 0$.

The second term in (18) can be further split into two parts and we obtain

$$
\begin{aligned}
&4\pi \int_{k|x-y|>1} \frac{\nabla \cdot \vec{W}(y,1)}{|x-y|} 2\cos^2(k|x-y| - \frac{1}{4}\pi) \mathrm{d}y \\
&= 4\pi \int_{k|x-y|>1} \frac{\nabla \cdot \vec{W}(y,1)}{|x-y|} \mathrm{d}y + 4\pi \int_{k|x-y|>1} \frac{\nabla \cdot \vec{W}(y,1)}{|x-y|} \sin(2k|x-y|) \mathrm{d}y,
\end{aligned}
$$

where at the limit when $k \to \infty$, the first term gives us what we wanted. In order to show that the rest goes to zero as $k \to \infty$ we will use Riemann-Lebesgue lemma. It suffices to show that $\frac{\nabla \cdot \vec{W}(\cdot,1)}{|x-\cdot|} \in L^1(\mathbb{R}^2)$, uniformly in

$x \in \mathbb{R}^2$. Now

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|\nabla \cdot \vec{W}(y,1)|}{|x-y|}\mathrm{d}y &\leq \int_{|y|\leq R} \frac{|\nabla \cdot \vec{W}(y,1)|}{|x-y|}\mathrm{d}y + C\int_{|y|>R} \frac{\mathrm{d}y}{|y|^\mu |x-y|} \\ &\leq \left(\int_{|y|\leq R} |\nabla \cdot \vec{W}(y,1)|^p \mathrm{d}y\right)^{1/p} \left(\int_{|y|\leq R} \frac{1}{|x-y|^{p'}}\mathrm{d}y\right)^{1/p'} \\ &\quad + C\int_{|y|>R} \frac{\mathrm{d}y}{|y|^\mu |x-y|},\end{aligned}$$

where the first term is finite uniformly in $x \in \mathbb{R}^2$ due to $p' < 2$. In the second term we consider two cases. First, let $|x| \leq \frac{R}{2}$. Then for all $|y| > R$, we have $|x-y| > \frac{R}{2}$ and therefore

$$C\int_{|y|>R} \frac{\mathrm{d}y}{|y|^\mu |x-y|} \leq \frac{2C}{R}\int_{\mathbb{S}^1} \mathrm{d}\omega \int_R^\infty r^{1-\mu}\mathrm{d}r \leq C.$$

In the case when $|x| > \frac{R}{2}$, we may choose $\varepsilon > 0$ such that $1 < \mu - \varepsilon < 2$. Now for the convolution it holds that

$$C\int_{|y|>R} \frac{\mathrm{d}y}{|y|^\mu |x-y|} \leq \frac{C}{R^\varepsilon}\int_{\mathbb{R}^2} \frac{\mathrm{d}y}{|y|^{\mu-\varepsilon}|x-y|} \leq \frac{C}{R^\varepsilon}\frac{1}{|x|^{\mu-\varepsilon-1}} \leq \frac{C2^{\mu-\varepsilon-1}}{R^{\mu-1}}$$

and therefore $\frac{\nabla\cdot\vec{W}(\cdot,1)}{|x-\cdot|} \in L^1(\mathbb{R}^2)$, uniformly in $x \in \mathbb{R}^2$. Thus Riemann-Lebesgue lemma gives us

$$\lim_{k\to\infty} 4\pi \int_{k|x-y|>1} \frac{\nabla \cdot \vec{W}(y,1)}{|x-y|}\sin(2k|x-y|)\mathrm{d}y = 0,$$

uniformly in $x \in \mathbb{R}^2$.

The third term of (18) can be estimated as

$$\left|\int_{k|x-y|>1} \frac{\nabla \cdot \vec{W}(y,1)}{k|x-y|^2}\mathrm{d}y\right| \leq \frac{1}{k^\sigma}\int_{\mathbb{R}^2} \frac{|\nabla \cdot \vec{W}(y,1)|}{|x-y|^{1+\sigma}}\mathrm{d}y,$$

where the last integral is finite when $0 < \sigma < 1 - \frac{2}{p}$.

In order to estimate term I'', we prove the following inequality. For a function $g \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$, $2 < p \leq \infty$ satisfying the decay-property (1) and $\delta < \mu - \frac{1}{2}$, we have

$$\int_{\mathbb{R}^2} (1+|y|)^{2\delta}|J_0(k|x-y|)|^2|g(y)|^2\mathrm{d}y \leq \frac{C}{k^\eta}, \tag{19}$$

for some $\eta > 0$ small enough. Indeed,

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (1+|y|)^{2\delta} |J_0(k|x-y|)|^2 |g(y)|^2 \mathrm{d}y \\
&\quad \le \int_{k|x-y|<1} (1+|y|)^{2\delta} |g(y)|^2 \mathrm{d}y + C \int_{k|x-y|>1} (1+|y|)^{2\delta} \frac{|g(y)|^2}{k|x-y|} \mathrm{d}y \\
&\quad \le \int_{k|x-y|<1,\ |y|\le R} (1+|y|)^{2\delta} |g(y)|^2 \mathrm{d}y + C \int_{k|x-y|<1,\ |y|>R} (1+|y|)^{2\delta} \frac{1}{|y|^{2\mu}} \mathrm{d}y \\
&\qquad + \frac{C}{k^\eta} \int_{k|x-y|>1, |y|\le R} (1+|y|)^{2\delta} \frac{|g(y)|^2}{|x-y|^\eta} \mathrm{d}y \\
&\qquad + \frac{C}{k} \int_{k|x-y|>1,\ |y|>R} (1+|y|)^{2\delta} \frac{1}{|x-y||y|^{2\mu}} \mathrm{d}y \\
&\quad = K_1 + K_2 + K_3 + K_4.
\end{aligned}
$$

Using Hölder's inequality, we may estimate the first term as follows

$$
K_1 \le (1+R)^{2\delta} \left(\int_{k|x-y|<1} |g(y)|^{2q} \mathrm{d}y \right)^{1/q} \left(\int_{k|x-y|<1} 1 \mathrm{d}y \right)^{1/q'} \le \frac{C}{k^{2/q'}},
$$

where $q = \frac{p}{2}$ and $q' = \frac{p}{p-2}$, and the second term as

$$
K_2 \le C R^{2\delta - 2\mu} \int_{|x-y|<1/k} 1 \mathrm{d}y \le \frac{C}{k^2}.
$$

Again, by Hölder's inequality we have

$$
K_3 \le \frac{C}{k^\eta} (1+R)^{2\delta} \left(\int_{|y|\le R} |g(y)|^{2q} \mathrm{d}y \right)^{1/q} \left(\int_{|y|\le R} \frac{1}{|x-y|^{\eta q'}} \mathrm{d}y \right)^{1/q'} \le \frac{C}{k^\eta},
$$

where both integrals are finite when $q = \frac{p}{2}$ and $0 < \eta < 2 - \frac{4}{p}$. And finally the fourth term can be estimated as

$$
K_4 \le \frac{C}{k} \int_{|y|>R} \frac{1}{|x-y||y|^{2\mu-2\delta}} \mathrm{d}y \le \frac{C}{k},
$$

since $\delta < \mu - \frac{1}{2}$. This proves (19).

Next we will show that the term I'' goes to zero as $k \to \infty$. Using Hölder inequality and Lemma 2.2 together with estimate (19) we have

$$
\begin{aligned}
|I''| &\le C\, k^2 \int_{\mathbb{S}^1} \mathrm{d}\theta \int_{\mathbb{R}^2} |J_0(k|x-y|)|\, |\beta_W(y)||u_{\mathrm{sc}}(y)| \mathrm{d}y \\
&\le C\, k^2 \int_{\mathbb{S}^1} \mathrm{d}\theta \left(\int_{\mathbb{R}^2} (1+|y|)^{2\delta} |J_0(k|x-y|)|^2 |\beta_W(y)|^2 \mathrm{d}y \right)^{1/2} \\
&\qquad \times \left(\int_{\mathbb{R}^2} (1+|y|)^{-2\delta} |u_{\mathrm{sc}}(y)|^2 \mathrm{d}y \right)^{1/2} \le \frac{C}{k^{\eta/2}},
\end{aligned}
$$

for some $\eta > 0$ small enough.

Let us now consider I_3. We start again by substituting $V(y,|u|) = V(y,1) + V(y,|u|) - V(y,1)$ and we have

$$
\begin{aligned}
I_3 &= k\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}\int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}V(y,1)\mathrm{d}y\mathrm{d}\theta'\mathrm{d}\theta \\
&\quad + k\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}\int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}\left(V(y,|u|) - V(y,1)\right)\mathrm{d}y\mathrm{d}\theta'\mathrm{d}\theta \\
&= I^* + I^{**}.
\end{aligned}
$$

The term I^{**} can be considered in the same manner as we did earlier with I''. In I^* we may calculate integrals with respect to both θ and θ' using (17) and we obtain

$$
\begin{aligned}
I^* &= 4\pi^2 k\int_{\mathbb{R}^2} J_0^2(k|x-y|)V(y,1)\mathrm{d}y \\
&= 4\pi^2 k\int_{k|x-y|<1} V(y,1)\mathrm{d}y \\
&+ 4\pi^2 k\int_{k|x-y|>1}\left(\sqrt{\frac{2}{\pi k|x-y|}}\cos(k|x-y| - \frac{1}{4}\pi) + O\left(\frac{1}{k^{3/2}|x-y|^{1/2}}\right)\right)^2 V(y,1)\mathrm{d}y \\
&= 4\pi^2 k\int_{k|x-y|<1} V(y,1)\mathrm{d}y \\
&+ 4\pi\int_{k|x-y|>1}\frac{2}{|x-y|}\cos^2(k|x-y| - \frac{1}{4}\pi)V(y,1)\mathrm{d}y \\
&+ O\left(1\right)\int_{k|x-y|>1}\frac{V(y,1)}{k|x-y|^2}\mathrm{d}y = J_1 + J_2 + J_3.
\end{aligned}
$$

We may estimate term J_1 straightforwardly by using Hölder's inequality and we have

$$
|J_1| \le 4\pi^2 k\left(\int_{k|x-y|<1} 1\mathrm{d}y\right)^{1/p'}\left(\int_{k|x-y|<1}|V(y,1)|^p\mathrm{d}y\right)^{1/p} \le Ck^{1-2p'},
$$

where $1 \le p' < 2$.

Let us now split J_2 into two parts,

$$
J_2 = 4\pi\int_{k|x-y|>1}\frac{V(y,1)}{|x-y|}\mathrm{d}y + 4\pi\int_{k|x-y|>1}\frac{V(y,1)}{|x-y|}\sin(2k|x-y|)\mathrm{d}y,
$$

where the second term goes to zero due to Riemann-Lebesgue lemma and the first term is as we wanted. For J_3 there exists some constant C such that the following estimate holds

$$|J_3| \leq \frac{C}{k^{\varepsilon}} \int_{\mathbb{R}^2} \frac{|V(y,1)|}{|x-y|^{1+\varepsilon}} \mathrm{d}y,$$

where the integral is finite when $\varepsilon > 0$ is small enough. To combine, we have now shown that

$$\lim_{k\to\infty} (I_1 + I_3) = 4\pi \int_{\mathbb{R}^2} \frac{V(y,1) - \frac{1}{2}\nabla \cdot \vec{W}(y,1)}{|x-y|} \mathrm{d}y,$$

uniformly in $x \in \mathbb{R}^2$.

Finally, in order to finish the proof of Theorem 1.2 we need to show that both I_2 and I_4 go to zero as $k \to \infty$. When we first calculate the integral with respect to $\theta' \in \mathbb{S}^1$ and then use Hölder's inequality, we obtain

$$\begin{aligned}
|I_2| &\leq 2\pi k \int_{\mathbb{S}^1} \mathrm{d}\theta \int_{\mathbb{R}^2} |J_0(k|x-y|)||\alpha_W(y)||\nabla u_{\mathrm{sc}}(y)|\mathrm{d}y \\
&\leq 2\pi k \int_{\mathbb{S}^1} \mathrm{d}\theta \left(\int_{\mathbb{R}^2} (1+|y|)^{2\delta} |J_0(k|x-y|)|^2 |\alpha_W(y)|^2 \mathrm{d}y \right)^{1/2} \\
&\qquad \times \left(\int_{\mathbb{R}^2} (1+|y|)^{-2\delta} |\nabla u_{\mathrm{sc}}(y)|^2 \mathrm{d}y \right)^{1/2} \\
&\leq 2\pi k \int_{\mathbb{S}^1} \|u_{\mathrm{sc}}\|_{H^1_{-\delta}} \mathrm{d}\theta \left(\int_{\mathbb{R}^2} (1+|y|)^{2\delta} |J_0(k|x-y|)|^2 |\alpha_W(y)|^2 \mathrm{d}y \right)^{1/2}.
\end{aligned}$$

The term inside brackets tends to zero as $k \to \infty$ due to estimate (19) and therefore by Lemma 2.2 the whole term tends to zero when $k \to \infty$. Similar calculations can be done for I_4 (even simpler than for I_2) and we may conclude that

$$\lim_{k\to\infty} (I_2 + I_4) = 0.$$

Thus Theorem 1.2 is completely proved. □

4 Numerical examples

In this section we demonstrate numerically how we can recover β from the knowledge of the scattering amplitude. We take two different approaches. Firstly, we shall compute numerically the left hand side of Saito's formula and invert the convolution type integral equation for β. Our second approach makes use of the representation formula which gives us a direct access to β. In both cases we fix large enough k to simulate the high frequency limit.

To simulate the scattering data A we need to solve the direct scattering problem (2). Due to the non-linearity of the problem we need to approximate the scattered field u_{sc} iteratively. To this end, we follow [4, 19]. Let us briefly describe this approach for the convenience of the reader. Since the solution to the direct scattering problem can be obtained from the iterations $\widetilde{u}_j = F(\widetilde{u}_{j-1})$ (see Theorem 2.1), we need to calculate the integrals

$$F\left(\widetilde{u}_{j-1}\right)(x) = -\int_{\mathbb{R}^2} G_k^+(|x-y|)\Big[\overrightarrow{W}(y, |u_0+\widetilde{u}_{j-1}|)\cdot\nabla(u_0+\widetilde{u}_{j-1}) \\ + V(y, |u_0+\widetilde{u}_{j-1}|)(u_0+\widetilde{u}_{j-1})\Big]\mathrm{d}y,$$

where $u_0(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(x,\theta)}$ and $\widetilde{u}_0 = 0$. We note that the kernel G_k^+ is essentially bounded, so we can use for example Gauss-Legendre n-quadrature rule for the integrals. These integrals are evaluated iteratively. Then the scattered field is obtained as the limit $u_{\mathrm{sc}} = \lim_{j\to\infty}\widetilde{u}_j$, where we take $j = 3$, choice justified by the estimate (15).

In order to obtain the scattering amplitude we use the asymptotic behaviour of the solution u, see (16). By discarding the error term $o(|x|^{-1/2})$, we have

$$u(x,k,\theta) \approx \mathrm{e}^{\mathrm{i}k(x,\theta)} - \frac{\mathrm{i}+1}{8\sqrt{\pi R k^5}}\mathrm{e}^{\mathrm{i}kR}A(k,\theta,\theta'),$$

where we put $x = R\theta'$ with $R = 10^4$. This will be our synthetic data. To simulate noisy measurements, we add Gaussian white noise to the data with standard deviation 5% of the maximum of the measurement.

We use the abbreviation

$$\beta(y) := V(y,1) - \frac{1}{2}\nabla\cdot\overrightarrow{W}(y,1).$$

For the inverse problem of recovering β from A we will follow [16]. Saito's formula says that

$$\lim_{k\to\infty} k\int_{\mathbb{S}^1\times\mathbb{S}^1}\mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}A(k,\theta,\theta')\mathrm{d}\theta\mathrm{d}\theta' = 4\pi\int_{\mathbb{R}^2}\frac{\beta(y)}{|x-y|}\mathrm{d}y \\ = 4\pi\left(\beta * \frac{1}{|\cdot|}\right)(x) =: S(x),$$

where $*$ denotes the convolution. The convolution-type integral on the right-hand side can be inverted by using a Calderón operator as follows. Note first that the Fourier transform of $g(x) = 1/|x|$ is

$$\mathcal{F}(g)(\xi) = \frac{1}{|\xi|}.$$

Therefore, in the sense of distributions, we have that

$$\frac{1}{4\pi}\mathcal{F}^{-1}\big(|\xi|\mathcal{F}(S)\big)(x) = \mathcal{F}\left[|\xi|\mathcal{F}\left(\beta * \frac{1}{|x|}\right)(\xi)\right](x) = \mathcal{F}^{-1}\left[\mathcal{F}(\beta)(\xi)\right] = \beta(x).$$

This high-pass filtering is implemented numerically with the fast-Fourier transform.

Remark 4.1. This approach of inverting Saito's formula has the practical benefit that one can choose to use a different filter instead of the simple high-pass filter $|\xi|$. Depending on the features that one wants to emphasize one can choose filters such as Shepp&Logan, Hamming or Blackman filters. The effects of different filters in the context of computed tomography are described for instance in [2].

The integral over incident and measurement angles in Saito's formula is computed in polar coordinates

$$\theta(t) := (\cos(t), \sin(t)), \quad \theta'(s) := (\cos(s), \sin(s))$$

as

$$\int_{\mathbb{S}^1\times\mathbb{S}^1} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} A(k,\theta,\theta')\mathrm{d}\theta\mathrm{d}\theta' \\ = \int_0^{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k(\theta(t)-\theta'(s),x)} A(k,\theta(t),\theta'(s))\mathrm{d}t\mathrm{d}s. \quad (20)$$

This double integral can be evaluated by using standard quadrature rules, and we resort to Gauss-Legendre rule. We used 64 incident and measurement angles, totalling 4096 measurements evaluated at the quadrature nodes.

Similar integral appears in the representation formula (see Corollary 1.4). We evaluated this integral in the same manner as (20) to test the applicability of this approach.

While Saito's formula requires a limit $k \to \infty$, for numerical purposes large $k > 0$ will suffice. In our experiments $k > 15$ yields reasonable reconstructions. The examples below are done with $k = 25$.

Let us now describe our sample potentials. We denote by

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A \end{cases}$$

the characteristic function of the set A.

For the component functions of $\vec{W}$ we will use C_0^∞-functions on an ellipse $\{(x,y) \in \mathbb{R}^2 \mid (x/a)^2 + (y/b)^2 < 1,\ a, b > 0\}$, given by the formula

$$\varphi_{\text{ellipse}}(x,y) = \begin{cases} \exp\left(\frac{1}{(x/a)^2+(y/b)^2-1}\right), & (x/a)^2 + (y/b)^2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The ellipses will also be shifted and rotated in different examples.

We consider the following nonlinear examples

1. $\vec{W} \equiv 0$ and $V(x,|u|) = \chi_{\text{ellipse}}(x)|u|^2$.
2. $\vec{W}(x,|u|) \equiv (0, \varphi_2)\sin(|u|)$ and $V(x,|u|) = \chi_{\text{L-shape}}(x)\dfrac{|u|^2}{1+|u|^2}$.
3. $\vec{W}(x,|u|) = (\varphi_1 \frac{|u|^2}{1+|u|^2}, \varphi_2|u|^2)$ and $V(x,|u|) = \varphi_3(x)|u|^2$.
4. $\vec{W}(x,|u|) = (\varphi_1, \varphi_2)|u|^2$ and $V(x,|u|) = \varphi_3(x)|u|^2$.

These examples contain both smooth functions and functions with jump discontinuities. In Example 4 we will choose the smooth bump functions φ_1, φ_2 and φ_3 so that their supports intersect. Figure 1 depicts the potential combinations β for each example. Figure 2 displays the scattered fields u_{sc}. The corresponding reconstructions by using inversion of Saito's formula by FFT are shown in Figures 3-6. We conclude that the shape and location of the scatterers are reconstructed reasonably well. It must be mentioned, that with this approach we can not distinguish the different functions V, W_1 and W_2 from each other.

Without details we report that the reconstructions obtained with inversion of Saito's formula and the representation theorem are indistinguishable to the eye, since the absolute difference between the reconstructions is within 10^{-6}. To us this signals that the end user may choose which one of the approaches best suits application. On one hand, representation formula Corollary 1.4 is very simple to implement. On the other hand, inversion of Saito's formula allows one to choose different filtering functions and possibly different regularisation methods. This approach is also considerably faster numerically due to effectiveness of fast-Fourier transform.

Acknowledgments

This work was supported by the Academy of Finland (grant number 312123, the Centre of Excellence of Inverse Modelling and Imaging 2018-2025). T. T. was supported by the Academy of Finland (grant number 312119, Centre of Excellence in Inverse Modelling and Imaging, 2018-2025).

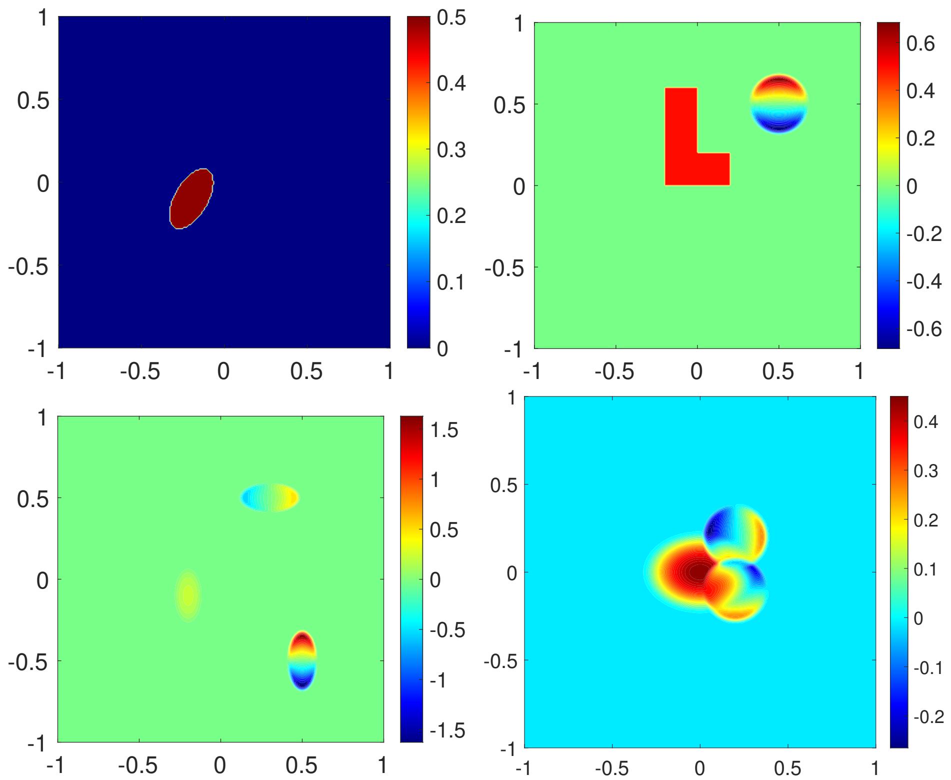

Figure 1: The potentials β for Examples 1 (top left), 2 (top right), 3 (bottom left) and 4 (bottom right). In Example 1 we have only potential V which is a characteristic function of an ellipse. In Example 2 V is the characteristic function of an L-shaped domain and $\vec{W} = (0, \varphi_2)\sin(|u|)$ has one component, where φ_2 is a smooth bump function in a circular domain. In Example 3 both components of $\vec{W} = (\varphi_1 \frac{|u|^2}{1+|u|^2}, \varphi_2|u|^2)$ are multiplied by smooth bump functions φ_1 and φ_2 supported in ellipses located at the top and bottom right in the figure, respectively. The coefficient φ_3 of potential V is also a smooth bump function supported in an ellipse, located in the middle-left side of the figure. In example 4 all coefficients are smooth bump functions (see also Figure 6), but their supports are intersecting.

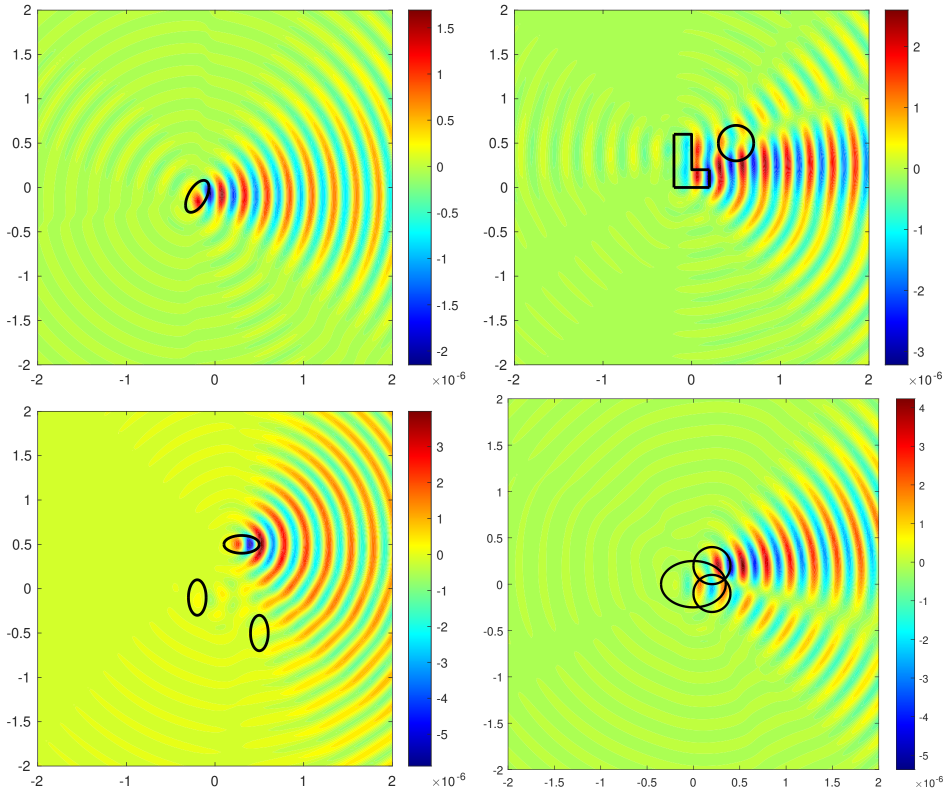

Figure 2: The scattered fields for Examples 1 (top left), 2 (top right), 3 (bottom left) and 4 (bottom right) with $k = 25$. The locations of the supports of the potentials are presented in black. Here the incident field is travelling from the left to the right.

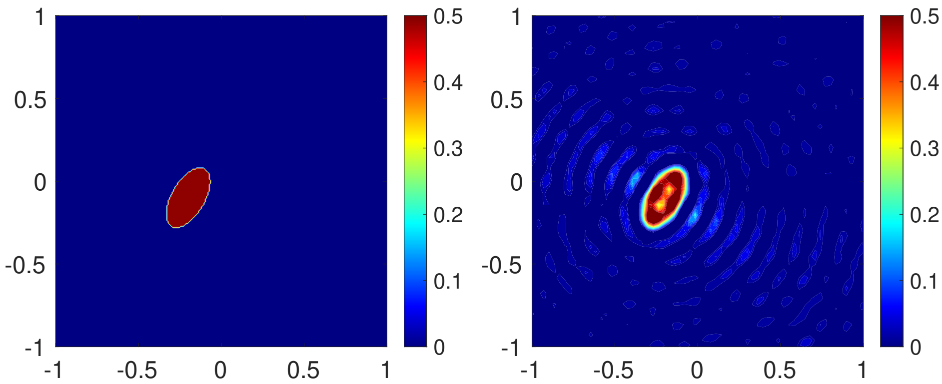

Figure 3: Example 1. Left: The unknown target β. Right: The numerical reconstruction β_{num}.

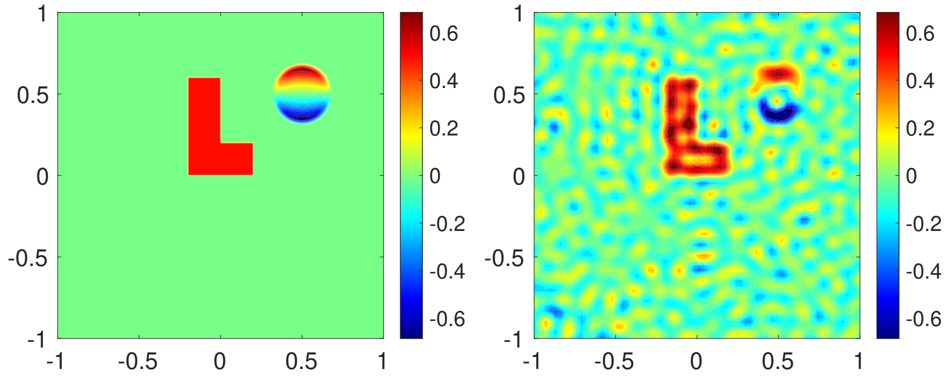

Figure 4: Example 2. Left: The unknown target β. Right: The numerical reconstruction β_{num}. This example shows recovery of corners and recovery of a shape with piece-wise smooth boundary.

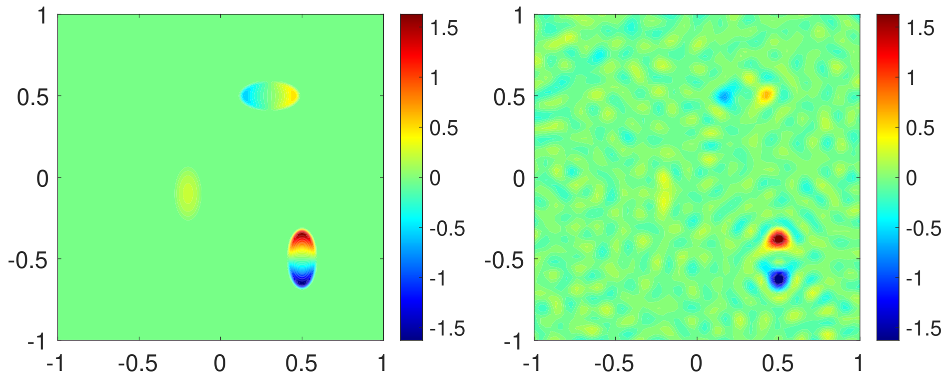

Figure 5: Example 3. Left: The unknown target β. Right: The numerical reconstruction β_{num}. We see that weak potentials are quite difficult to detect while stronger potentials are clearly visible in comparison, as is expected.

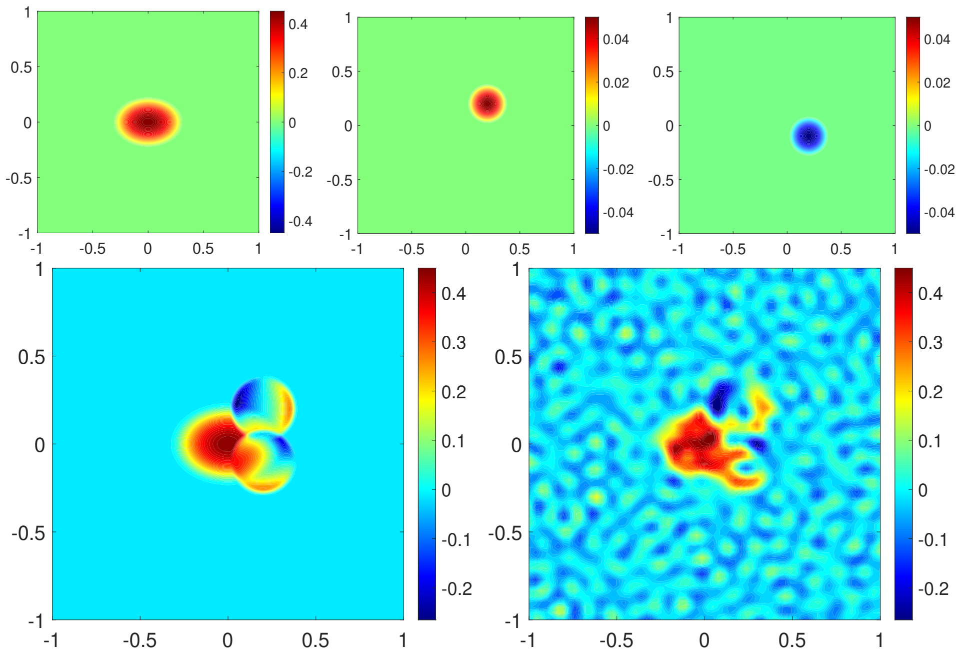

Figure 6: Example 4. Top left $V(x, 1)$, top middle $W_1(x, 1)$ and top right $W_2(x, 1)$. Bottom left: The unknown target β. Bottom right: The numerical reconstruction β_{num}. In this example the supports of potentials V, W_1 and W_2 overlap. We can not distinguish these functions from each other.

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