Inverse problem for the minimal surface equation and nonlinear CGO calculus in dimension 2

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Based on a work with C. Carstea, M. Lassas and L. Tzou,
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Introduction

The minimal surface equation (1)

Several ways to define minimal surfaces. A minimal surface, which is given as a graph $\subset \mathbb{R}^{n+1}$ of a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ satisfies the minimal surface equation

$$\begin{cases} \nabla \cdot \left(\frac{\nabla u}{(1+|\nabla u|)^{1/2}} \right) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

- Quasilinear elliptic. If you linearize at $u = 0$, the principal term is Laplacian.
- A minimal surface has vanishing mean curvature. That is, trace of the tensor $(X, Y) \mapsto \langle \nabla_X N, Y \rangle$ vanishes, X, Y tangential and N the normal of the surface.

More generally, a minimal surface embedded in an $(n+1)$-dimensional Riemannian manifold $(M, \overline{g})$ can be defined to be an n-dimensional submanifold whose mean curvature vanishes.

The minimal surface equation (2)

The class of minimal surfaces even in Euclidean space $\mathbb{R}^3$ is quite rich.

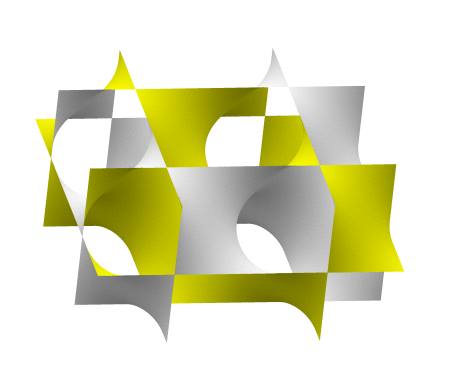


Figure 1: Schwarz D Surface

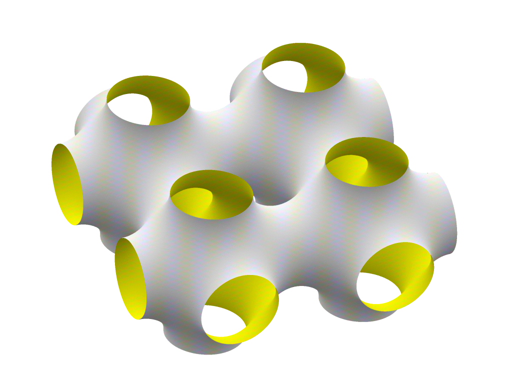


Figure 2: Schwarz P Surface

The minimal surface equation (3)

We set up notation. We consider dimension of the surface $n = 2$. We work in Fermi coordinates relative to a 2D surface Ω embedded in 3D manifold M. In Fermi-coordinates, the metric of $(M, \overline{g})$ reads

$$\overline{g}(s, x) = ds^2 + \sum_{k,l=1}^{2} g_{kl}(s, x) dx^k dx^l.$$

- Fermi coordinates always exist. Not a restriction of generality.
- Metric of the same form as in boundary normal coordinates.
- The 2D metric g on Ω is given by $g_{kl}(s, x)|_{s=0} = g_{kl}(0, x)$.
- If u is a function over Ω, we write $g_u(x) = g(u(x), x)$.
- The minimal surface equation can also be written in other coordinate system.

As mean curvature depends not only on the metric on the minimal surface, but also on the surrounding "ambient" 3D metric $\overline{g}$, the minimal surface equation will depend on $\overline{g}$.

The minimal surface equation (4)

If a minimal surface embedded in $(M, \overline{g})$ is given as a graph of u over general Ω, then u satisfies

$$-\frac{1}{\text{Det}(g_u)^{1/2}} \nabla \cdot \left(g_u^{-1} \frac{\text{Det}(g_u)^{1/2}}{\sqrt{1 + |\nabla u|_{g_u}^2}} \right) \nabla u + f(u, \nabla u) = 0 \text{ in } \Omega,$$

where

$$f(u, \nabla u) = \frac{1}{2} \frac{1}{(1 + |\nabla u|_{g_u}^2)^{1/2}} (\partial_s g_u^{-1})(\nabla u, \nabla u) + \frac{1}{2} (1 + |\nabla u|_{g_u}^2)^{1/2} \text{Tr}(g_u^{-1} \partial_s g_u).$$

Here ∇ and $\cdot$ are the Euclidean ones.

- Quite complicated, which will result in quite long computations.
- The equation is invariant and thus invariant under isometries of the ambient space $(M, \overline{g})$.
- Not conformally invariant $\Rightarrow$ Hope for recovery up to an isometry.
- Note that $u = 0$ is a solution only if $\text{Tr}(g_u^{-1} \partial_s g_u)|_{u=0} = 0$, which is equivalent to Ω itself being a minimal surface, equivalent to Ω having vanishing mean curvature.

Areas of minimal surfaces determine the DN map

From now on we consider minimal surfaces given as graphs over Ω, which itself is a minimal surface (Σ, g). DN map Λ of the minimal surface equation on Σ given as usual

$$\Lambda(f) = \partial_\nu u^f|_{\partial\Sigma},$$

where u^f solves the minimal surface eq. with boundary value $f \in C^\infty(\partial\Sigma)$ small.

- Local solvability near a given minimal surface follows from the usual implicit function theorem/contraction method.

DN map of the minimal surface equation is determined by areas of minimal surfaces.

- An exercise in calculus of variations. Minimal surfaces are extremals of the energy functional

$$u \mapsto \text{Area}(\text{graph}(u)) = \int_\Sigma \sqrt{1 + |\nabla u|^2_{g(x,u)}}\, \det\left(g(x,u)\right)^{1/2} dx^1 \wedge dx^2.$$

The inverse problem and main results (1)

The first inverse problem we consider is finding a 2D minimal surface (Σ, g) from the DN map.

Theorem (C. Carstea, M. Lassas, T. L, L. Tzou 2023)
Let $(\Sigma_1, g_1) \subset (M_1, \overline{g}_1)$ and $(\Sigma_2, g_2) \subset (M_2, \overline{g}_2)$ be $2D$ minimal surfaces embedded in 3D Riemannian manifold, with a mutual boundary $\partial\Sigma$. Assume that Σ_1, Σ_2 are diffeomorphic to a fixed domain in $\mathbb{R}^2$. (Assume also boundary determination.)

If the DN maps of the associated minimal surface equations satisfy $\Lambda_{g_1} f = \Lambda_{g_2} f$, for $f \in C^\infty(\partial\Sigma)$ sufficiently small, then there is an isometry $F : \Sigma_1 \to \Sigma_2$,

$$F^* g_2 = g_1, \quad F|_{\partial\Sigma} = Id.$$

Also $F^* \eta_2 = \eta_1$, where η_β are the second fundamental forms of (Σ_β, g_β), $\beta = 1, 2$.

- The recovery is up to diffeomorphism, not just conformal mapping as often in 2D cases.
- Also the the second fundamental form recovered, which depends on the 3D metric.
- We work in 2D because we need to solve anisotropic Calderón problem at one stage. Many of the arguments work in all dimensions however.

The inverse problem and main results (2)

The only real assumption in the theorem was the topological assumption that the domains are topologically (not necessarily isometrically) a domain in $\mathbb{R}^2$.

- The assumption due to the best result, by Imanuvilov-Uhlmann-Yamamoto 2012, for 2D Calderón where both metric and potential for the Shrödinder are unknown.

If we only consider recovering a conformal class by assuming a priori $g_2 = cg_1$, then the assumption that Σ_1 and Σ_2 are topologically a fixed domain in $\mathbb{R}^2$ can be dropped.

Theorem (C. Carstea, M. Lassas, T. L, L. Tzou 2023)

Let (Σ, g_1) and (Σ, g_2) are conformally equivalent Riemannian surfaces, $g_2 = cg_1$. Assume corresponding DN maps agree (and boundary determination). Then

$$c \equiv 1$$

and $\eta_1 = \eta_2$.

Generalized boundary rigidity

In the generalized boundary rigidity problem the aim is to construct a manifold from the areas of minimal surfaces.

- The usual boundary rigidity asks if a manifold is determined by lengths of minimal geodesics between pairs of points on its boundary. Minimal geodesics are 1D minimal surfaces.

The first mathematical paper on generalized boundary rigidity is by S. Alexakis, T. Balehowsky & A. Nachman (2020) "How to determine a 3 dimensional manifold from the areas of their minimal surfaces".

- In generalized boundary rigidity the task is two-fold: By knowing areas of minimal surfaces (1) recover minimal surfaces and (2) find a way to glue them together.
- The above paper considered both (1) and (2).
- In this talk we only consider (1). We introduce the higher order linearization method for the problem.

Motivation from AdS/CFT duality in physics (1)

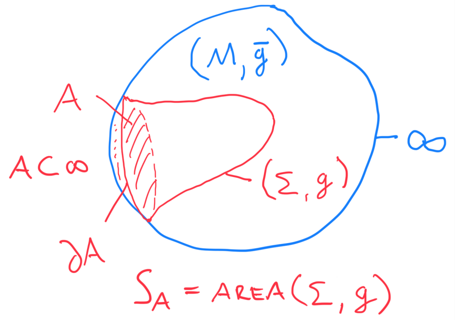
An AdS/CFT duality conjecture in physics by Ryu and Takayanagi states that "entanglement entropies" of a quantum field theory living on the boundary are equivalent to areas of related minimal surfaces, (2006, thousands of citations).

- Entanglement entropy S_A is the experienced entropy (state of disorder) of a physical system for an observer who has only access to a subregion A of a larger space.
- Physcists ask: "Is a (static) spacetime determined by entanglement entropies, that is areas of minimal surfaces by duality, of a quantum field theory (QFT) living on the (asymptotic) boundary?"
  - Conformal field theories (CFTs) are special type of QFTs.
- Physicists give examples where the answer to the above question is yes. Especially these are examples where the generalized boundary rigidity problem is solvable.

Motivation from AdS/CFT duality in physics (2)

The real physical situation is in the noncompact setting.

- The typical physical setting is in asymptotically hyperbolic Riemannian manifolds, such as time slices of Anti de Sitter (AdS) space.
  - Areas of minimal surfaces become infinite $\leftrightarrow$ QFTs have infinite degrees of freedom.
- Considered by conformal compactification, in which case the metric blows up on the boundary:



In this talk everything is compact and nothing blows up on the boundary.

Other earlier results

Inverse problems for the minimal surface equation:

- C. Carstea, Lassas, T. L, L. Oksanen (2022), determination of a minimal surface (Σ, g) embedded in $\Sigma \times \mathbb{R}$, "toy model".
- J. Nurminen (2022, 2023), results for conformally Euclidean metric in $\mathbb{R}^n$.

Physics papers about the construction of a Riemannian manifold from areas of minimal surfaces in context of AdS/CFT duality.

- S. Bilson, N. Bao, CJ. Cao, S. Fischetti, C. Keeler, V. Hubeny, N. Jokela, A. Pönni...

Recent advances in inverse problem for nonlinear equations in general:

- Kurylev, Lassas & Uhlmann (2018), inverse problem for $\Box_g u(x,t) + q(x,t)u^2(x,t) = 0$.
- A. Feizmohammadi & L. Oksanen and Lassas, T.L, Y-H. Lin & M. Salo (2019), inverse problems for $\Delta_g u + qu^m = 0$, $m \geq 2$.
- Other recent results for nonlinear *elliptic* by K. Krupchyk, T. Zhou, Y. Kian, R-Y. Lai, H. Liu, L. Tzou, S. Lu, B. Harrach, T. Tyni, L. Potenciano-Machado...

Proof of main theorem

How to recover an embedded minimal surface from the DN map (1)

The recovery is based on the higher order linearization method: Consider $f_j \in C^\infty(\partial\Sigma)$, $j = 1, 2, 3, 4$ and denote by $u = u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ the solution to the minimal surface equation with boundary data $\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4$. Denote $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_4)$.

By taking the derivative $\partial_{\varepsilon_j}|_{\varepsilon=0}$ of the solution $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$, we see that the function

$$v^j := \frac{\partial}{\partial \varepsilon_j}\Big|_{\varepsilon=0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$$

solves the first linearized equation

$$\Delta_g v + qv = 0, \tag{1}$$

where $q(x)$ is the quantity $\frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Tr}(g_u^{-1}\partial_s g_u)$. The DN map of (1) is known.

- In fact $q(x) = -|\eta|^2 - \mathrm{Ric}(N, N) \Rightarrow$ will obtain information about ambient space.

Under our assumption that Σ is topologically a domain in $\mathbb{R}^2$, it is possible to recover (Σ, g) from the DN map of (1) up to a conformal mapping by the result of O. Y. Imanuvilov, G. Uhlmann, and M. Yamamoto (2012).

How to recover an embedded minimal surface from the DN map (2)

Let's proceed to find the conformal factor and 2nd fundamental form $\eta(X,Y) = \langle \nabla_X N, Y\rangle_{\overline{g}}$.

- The conformal factor will be found from the third linearization. A conformal multiple of η will be found from the second linearization.

The function $w^{jk} := \frac{\partial^2}{\partial \varepsilon_j \partial \varepsilon_k}\Big|_{\varepsilon=0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ satisfies the second linearized equation

$$(\Delta_g + q) w^{jk} = \text{terms of the form } \eta(\nabla v^j, \nabla v^k) + \text{lower order terms.}$$

- The lower order terms are terms contain at most one gradient of a linearized solution v_j. They will be negligible in the "first layer" of the asymptotic analysis to follow.

Since we know the Cauchy data of w^{jk}, from the DN map of the minimal surface equation, it follows that the integral

$$\int_\Sigma v^1 \eta(\nabla v^2, \nabla v^3) dV + \int_\Sigma v^2 \eta(\nabla v^1, \nabla v^3) dV + \int_\Sigma v^3 \eta(\nabla v^1, \nabla v^2) dV$$
$$+ \text{lower order terms}$$

is known.

How to recover an embedded minimal surface from the DN map (3)

We recover the matrix field η (up to a conformal factor) next from the integral quantity above. This is done by choosing special CGO solutions for the linearized equation $(\Delta_g + q)v = 0$.

To recover η, we use as solutions v^k, $(\Delta_g + q)v^k = 0$, the CGOs constructed by C. Guillarmou and L. Tzou (2011, GAFA) of the form

$$e^{\Phi/h}(a + r_h),$$

where $\Phi = \phi + i\psi$ is a holomorphic Morse function, h small, a is a holomorphic function and r_h is a correction term given by

$$r_h = -\overline{\partial}_\psi^{-1} \sum_{j=0}^{\infty} T_h^j \overline{\partial}_\psi^{*-1}(qa),$$

where $\overline{\partial}_\psi^{-1}$ is defined (modulo localization) by $\overline{\partial}_\psi^{-1} f = \overline{\partial}^{-1}(e^{-2i\psi/h} f)$, where $\overline{\partial}^{-1}$ is the Cauchy-Riemann operator that solves $\overline{\partial}^{-1}\overline{\partial} = \mathrm{Id}$.

- The form of T_h is not important, but what is important to note is that the h-dependence of r_h is quite implicit and especially not polynomial: $r_h \neq ha_1 + ha_2 + \cdots + O_{H^k}(h^R)$.

How to recover an embedded minimal surface from the DN map (4)

Plugging in the CGOs $v^k = e^{\Phi_k/h}(a_k + r_{k,h})$, $k = 1, 2, 3$, to the integral identity

$$\int_\Sigma v^1\eta(\nabla v^2, \nabla v^3)dV + \int_\Sigma v^2\eta(\nabla v^1, \nabla v^3)dV + \int_\Sigma v^3\eta(\nabla v^1, \nabla v^2)dV + \cdots = 0,$$

yields the term

$$I_{\text{leading}} = \int_\Sigma \hat{v}^1\eta(\nabla\hat{v}^2, \nabla\hat{v}^3)dV + \int_\Sigma \hat{v}^2\eta(\nabla\hat{v}^1, \nabla\hat{v}^3)dV + \int_\Sigma \hat{v}^3\eta(\nabla\hat{v}^1, \nabla\hat{v}^2)dV,$$

where $\hat{v}^k = e^{\Phi_k/h}a_k$, and another term

$$I_{\text{other}} = \text{ other terms that contain integrals of products of } e^{\Phi_k/h}r_{k,h} \text{ and their gradients.}$$

We wish to use stationary phase for I_{leading} to recover η and consider I_{other} as a negligible term.

Stationary phase yields $I_{\text{leading}} = c_0 + O(h) = O(1)$, $c_0 \neq 0$, as $h \to 0$ while L^p estimates $\|r_{k,h}\|_{L^p}, \|\nabla r_{k,h}\|_{L^p} = O(h^{1/p})$ yield $I_{\text{other}} \sim O(h^{-1})$. Thus

$$I_{\text{other}} \sim O(h^{-1}) > O(1) = I_{\text{leading}}, \quad h \to 0$$

by just using L^p estimates. Thus L^p estimates are not enough. A problem.

Solution: Nonlinear CGO calculus

We call *nonlinear CGO calculus* a collection estimates that can be considered to be stationary phase type estimates for a class of h-dependent functions. The main estimate is:

Theorem (C. Carstea, M. Lassas, T. L, L. Tzou 2023)
Let f be C_c^∞ smooth outside a finite number of points and deg$(f) \geq l \geq 0$, then

$$\int e^{4i\psi/h} f \partial^l r_h = o(h^{\lfloor (\deg(f)-l)/2 \rfloor + 1}),$$

where r_h is the correction term of a CGO and $\lfloor \cdot \rfloor$ is the floor function.

- The degree $\deg(f)$ of a function f is roughly the order it vanishes at critical points of ψ: If $\deg(f) = l$, then $f(z) = z^k \overline{z}^m + O(|z|^{l+1})$, $k + m = l$.
- Example: If $f = \nabla \Phi_1 \cdot \nabla \Phi_2$ (times a cutoff), then $\int e^{4i\psi/h} f r_h = o(h^2)$, while $\int e^{4i\psi/h} f = O(h^2)$. The integral with r_h has slightly better decay.

End of proof

By the nonlinear CGO calculus, we obtain improved the estimate $I_{\text{other}} = o(1)$. Thus $I_{\text{other}} < O(1) = I_{\text{leading}}$ and we recover 2nd (conformal multiple) of the second fundamental form η.

- We continue to "lower layers" in the asymptotic analysis to recover the remaining quantities appearing in the second linearization.
- From the third order linearization we recover the conformal factor by similar methods as we used for the second order linearization.
- We have recovered the embedded minimal surface (Σ, g) up to an isometry.

Keypoints from the proof:

1. One has to work down with the linearizations to find the conformal factor.
2. Because L^p estimates do not see oscillation caused decay, usual L^p estimates were not enough. Nonlinear CGO calculus.

Back to mathematics of AdS/CFT duality

The geometrical situation of AdS/CFT is in the noncompact setting, where the Riemannian manifold is typically either asymptotically hyperbolic or conformally compact Einstein manifold.

- The conformal metric blows up when approaching the boundary in a specific way.
- Notable contributions to study geometry and scattering in this setting by R. Graham, M. Zworski, S.Y. A. Chang, J. Lee, S. Alexakis, R. Mazzeo, R. Melrose, M. Anderson...

To approach the generalized boundary rigidity in the above noncompact settings one needs to understand (for starters):

- Renormalized volumes of minimal surfaces embedded in, say, asymptotically hyperbolic spaces.
- Scattering map on embedded minimal surfaces in the above case, a la Graham-Zworski "Scattering matrix in conformal geometry (2001)".
- These are geometric and forward problems.

Summary

1. Recovery of a general 2D minimal surface from the DN map of minimal surface equation under a topological assumption.
   - Conformal factor can be recovered without the topological assumption.
   - Also recovered second fundamental form; information about the ambient space.
2. Introduction of the higher order linearization method to inverse problems with geometric data and to the AdS/CFT correspondence in physics.
   - Higher order variations of areas of minimal surfaces.
   - Entanglement entropies of CFT determine the DN map of minimal surface equation.
3. Nonlinear CGO calculus to handle contributions from products of CGOs needed in studies of nonlinear models in 2D.
   - CGO calculus is independent of the application to inverse problem for the minimal surface equation.
   - Likely useful not only in geometric settings, but also for nonlinear equations in $\mathbb{R}^2$.

Slides will be available at https://www.mv.helsinki.fi/home/tjliimat/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