Inverse problem for the minimal surface equation and nonlinear CGO calculus in dimension 2

Tony Liimatainen
University of Helsinki

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Based on a work with C. Carstea, M. Lassas and L. Tzou

Introduction

The minimal surface equation (1)

A minimal surface, which is given as a graph $\subset \mathbb{R}^{n+1}$ of a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ satisfies the minimal surface equation

$$\begin{cases} \nabla \cdot \left(\frac{\nabla u}{(1+|\nabla u|)^{1/2}} \right) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1}$$

- Quasilinear elliptic.
- A minimal surface has vanishing mean curvature; trace of the tensor $(X, Y) \mapsto \langle \nabla_X N, Y \rangle$ vanishes, X, Y tangential.

More generally, a minimal surface embedded in an $(n+1)$-dimensional Riemannian manifold $(M, \overline{g})$ can be defined to be an n-dimensional submanifold whose mean curvature vanishes.

- As mean curvature depends not only on the metric on the minimal surface, but also on the "ambient" metric $\overline{g}$, the form of the minimal surface equation will depend also on $\overline{g}$.

The minimal surface equation (2)

We consider $n = 2$. We work in Fermi coordinates relative to a surface Ω embedded in M, where the metric of $(M, \overline{g})$ reads

$$\overline{g}(s,x) = ds^2 + \sum_{k,l=1}^{2} g_{kl}(s,x)dx^k dx^l.$$

- Fermi coordinates always exist. Not a restriction of generality.
- The metric g on Ω is given by $g_{kl}(s,x)|_{s=0}$.
- If u is a function over Ω, we write $g_u(x) = g(u(x), x)$.

If a minimal surface embedded in $(M, \overline{g})$ is given as a graph of u over Ω, then u satisfies

$$-\frac{1}{\mathrm{Det}(g_u)^{1/2}} \nabla \cdot \left(g_u^{-1} \frac{\mathrm{Det}(g_u)^{1/2}}{\sqrt{1 + |\nabla u|_{g_u}^2}} \right) \nabla u + f(u, \nabla u) = 0,$$

where

$$f(u, \nabla u) = \frac{1}{2} \frac{1}{(1 + |\nabla u|_{g_u}^2)^{1/2}} (\partial_s g_u^{-1})(\nabla u, \nabla u) + \frac{1}{2}(1 + |\nabla u|_{g_u}^2)^{1/2} \mathrm{Tr}(g_u^{-1} \partial_s g_u).$$

The inverse problem and main results

Let us then assume $\Omega = (\Sigma, g)$ is itself a minimal surface and that the DN map Λ_g of the minimal surface equation is known. The problem is to determine the minimal surface (Σ, g).

Theorem (C. Carstea, M. Lassas, T. L, L. Tzou 2023)

Let $(\Sigma_1, g_1) \subset (M_1, \overline{g}_1)$ *and* $(\Sigma_2, g_2) \subset (M_2, \overline{g}_2)$ *be embedded* $2D$ *minimal surface surfaces with a mutual boundary* $\partial\Sigma$*. Assume that* Σ_1, Σ_2 *are diffeomorphic to a fixed domain in* $\mathbb{R}^2$*. (Assume also boundary determination.)*

If the DN maps of the associated minimal surface equations satisfy $\Lambda_{g_1} f = \Lambda_{g_2} f$*, for* $f \in C^\infty(\partial\Sigma)$ *sufficiently small, then there is an isometry* $F : \Sigma_1 \to \Sigma_2$*,*

$$F^* g_2 = g_1, \quad F|_{\partial\Sigma} = Id.$$

Also $F^*\eta_2 = \eta_1$*, where* η_β *are the second fundamental forms of* (Σ_β, g_β)*,* $\beta = 1, 2$*.*

- If we only consider recovering a conformal class by assuming a priori $g_2 = c g_1$, then the assumption that Σ_1 and Σ_2 are topologically a fixed domain in $\mathbb{R}^2$ can be dropped.

Motivation

Special motivations for the study:

1. Generalized boundary rigidity problem where the aim is to construct a manifold from the areas of minimal surfaces instead of lengths of minimal geodesics.
   - Areas of minimal surfaces determine the DN map of the minimal surface equation.
2. AdS/CFT duality conjecture in physics by Ryu and Takayanagi (2006, thousands of citations) states that "entanglement entropies" of a quantum field theory living on the boundary determine areas of related minimal surfaces.
   - Entanglement entropy is the experienced entropy (i.e. state of disorder) of a physical system for an observer who has only access to a subregion of a larger space.
   - Is a (static) spacetime determined by entanglement entropies of a QFT living on the (asymptotic) boundary?
   - Physicists give examples where this is true, i.e. examples where generalized boundary rigidity problem is solvable.

Earlier results

Inverse problems for the minimal surface equation:

- C. Carstea, Lassas, T. L, L. Oksanen (2022), determination of a minimal surface (Σ, g) embedded in $\Sigma \times \mathbb{R}$.
- J. Nurminen (2022, 2023), results for conformally Euclidean metric in $\mathbb{R}^n$.

Generalized boundary rigidity & manifold construction in AdS/CFT duality:

- S. Alexakis, T. Balehowsky & A. Nachman (2020) "How to determine a 3 dimensional manifold from the areas of their minimal surfaces".
- Physics papers by: S. Bilson, N. Bao et al, V. Hubeny, N. Jokela, A. Pönni...

Recent advances in inverse problem for nonlinear equations:

- Kurylev, Lassas & Uhlmann (2018), inverse problem for $\Box_g u(x,t) + q(x,t)u^2(x,t) = 0$.
- A. Feizmohammadi & L. Oksanen and Lassas, T.L, Y-H. Lin & M. Salo (2019), inverse problems for $\Delta_g u + qu^m = 0$, $m \geq 2$.
- Other recent results for nonlinear elliptic by K. Krupchyk, T. Zhou, Y. Kian, R-Y. Lai, H. Liu, L. Tzou, B. Harrach, T. Tyni, L. Potenciano-Machado...

Proof of main theorem

How to recover an embedded minimal surface from the DN map (1)

The recovery is based on the higher order linearization method: Consider $f_j \in C^\infty(\partial\Sigma)$, $j = 1,2,3,4$ and denote by $u = u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ the solution to the minimal surface equation with boundary data $\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4$. Denote $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_4)$.

By taking the derivative $\partial_{\varepsilon_j}|_{\varepsilon=0}$ of the solution $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$, we see that the function

$$v^j := \frac{\partial}{\partial \varepsilon_j}\Big|_{\varepsilon=0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$$

solves the first linearized equation

$$\Delta_g v + qv = 0,$$

where $q(x)$ is the quantity $\frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Tr}(g_u^{-1}\partial_s g_u)$.

- Under the assumption that Σ is topologically a domain in $\mathbb{R}^2$, it is possible to recover (Σ, g) up to a conformal mapping by the result of O. Y. Imanuvilov, G. Uhlmann, and M. Yamamoto (2012).
- The conformal factor will be found only from the third linearization.

How to recover an embedded minimal surface from the DN map (2)

Let us denote by $\eta(X,Y) = \langle \nabla_X N, Y\rangle_{\overline{g}}$ the (scalar) second fundamental form. The function $w^{jk} := \frac{\partial^2}{\partial \varepsilon_j \partial \varepsilon_k}\Big|_{\varepsilon=0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ satisfies the second linearized equation

$$(\Delta_g + q) w^{jk} = \text{terms of the form } \eta(\nabla v^j, \nabla v^k) + \text{lower order terms.}$$

Lower order terms are terms contain at most one gradient of a linearized solution v_j. Since we know the DN map of second linearization, it follows that the integral

$$\int_\Sigma v^1 \eta(\nabla v^2, \nabla v^3)\, dV + \int_\Sigma v^2 \eta(\nabla v^1, \nabla v^3)\, dV + \int_\Sigma v^3 \eta(\nabla v^1, \nabla v^2)\, dV + \text{lower order terms} \quad (2)$$

is known.

- The aim is to recover the matrix field η next from (2). This is done by choosing special CGO solutions for the linearized equation $(\Delta_g + q) v = 0$.

How to recover an embedded minimal surface from the DN map (3)

To recover η, we use as solutions v^k, $(\Delta_g + q)v^k = 0$, the CGOs constructed by C. Guillarmou and L. Tzou (2011, GAFA) of the form

$$e^{\Phi/h}(a + r_h),$$

where $\Phi = \phi + i\psi$ is a holomorphic Morse function, h small, a is a holomorphic function and r_h is a correction term given by

$$r_h = -\overline{\partial}_\psi^{-1} \sum_{j=0}^{\infty} T_h^j \overline{\partial}_\psi^{*-1}(qa),$$

where $\overline{\partial}_\psi^{-1}$ is defined (modulo localization) by $\overline{\partial}_\psi^{-1} f = \overline{\partial}^{-1}(e^{-2i\psi/h} f)$, where $\overline{\partial}^{-1}$ is the Cauchy-Riemann operator that solves $\overline{\partial}^{-1}\overline{\partial} = \mathrm{Id}$.

- The form of T_h is not important, but what is important to note is that the h-dependence of r_h is quite implicit and not polynomial.

How to recover an embedded minimal surface from the DN map (4)

Plugging in the CGOs $v^k = e^{\Phi_k/h}(a_k + r_{k,h})$, to the integral identity

$$\int_\Sigma v^1\eta(\nabla v^2, \nabla v^3)dV + \int_\Sigma v^2\eta(\nabla v^1, \nabla v^3)dV + \int_\Sigma v^3\eta(\nabla v^1, \nabla v^2)dV + \cdots = 0,$$

yields the term

$$I_{leading} = \int_\Sigma \hat{v}^1\eta(\nabla \hat{v}^2, \nabla \hat{v}^3)dV + \int_\Sigma \hat{v}^2\eta(\nabla \hat{v}^1, \nabla \hat{v}^3)dV + \int_\Sigma \hat{v}^3\eta(\nabla \hat{v}^1, \nabla \hat{v}^2)dV,$$

where $\hat{v}^k = e^{\Phi_k/h}a_k$ and integral I_{other} of other terms that contain products of $e^{\Phi_k/h}r_{k,h}$ and their gradients. We wish to use stationary phase for $I_{leading}$ to recover η and consider I_{other} as a negligible term.

Stationary phase yields $I_{leading} = O(1)$ as $h \to 0$ while L^p estimates $\|r_{k,h}\|_{L^p}, \|\nabla r_{k,h}\|_{L^p} = O(h^{1/p})$ yield $I_{other} \sim O(h^{-1})$. Thus

$$I_{other} > I_{leading}, \quad h \to 0$$

by just using L^p estimates. A problem.

Solution: Nonlinear CGO calculus

Nonlinear CGO calculus is a collection estimates that can be considered to be stationary phase type estimates for a class of h-dependent functions. The main estimate is

Theorem (C. Carstea, M. Lassas, T. L, L. Tzou 2023)
Let f be C_c^∞ smooth outside a finite number of points and deg$(f) \geq l \geq 0$, then

$$\int e^{4i\psi/h} f \partial^l r_h = o(h^{\lfloor (\deg(f)-l)/2 \rfloor + 1}).$$

- $\lfloor \cdot \rfloor$ is the floor function. The degree $\deg(f)$ of a function f is roughly the order it vanishes at critical points of ψ; if $\deg(f) = l$, then $f(z) = z^k \overline{z}^m + O(|z|^{l+1})$, $k + m = l$.
- The theorem yields improved estimate $I_{other} = o(1)$. Thus $I_{other} < O(1) = I_{leading}$, and we recover 2nd fundamental form η. 3rd order linearization recovers the conformal factor.
- Note that for $n \geq 3$, typical CGOs used in geometric inverse problems can be made to have correction terms R_h with $R_h = O_{H^k}(\tau^{-R})$, for any $k, R \in \mathbb{N}$, so that nonlinear CGO calculus is not needed. Recall that we have only $r_h = O_{L^p}(h^{1/p})$.

Summary

1. Recovery of a general 2D minimal surface from the DN map of minimal surface equation under a topological assumption.
   - Conformal factor can be recovered in without the topological assumption.
   - Also recovered second fundamental form; information about the ambient space.
2. Introduction of the recent higher order linearization method for inverse problems into the AdS/CFT correspondence in physics.
   - Entanglement entropies of CFT determine the DN map of minimal surface equation.
3. Nonlinear CGO calculus to handle contributions from products of CGOs needed in studies of nonlinear models in 2D.
   - CGO calculus is independent of the application to inverse problem for the minimal surface equation.
   - Needed not only in geometric settings, but useful in inverse problems for quasilinear elliptic equations in $\mathbb{R}^2$.

Slides will be available at https://www.mv.helsinki.fi/home/tjliimat/


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