TOEPLITZ OPERATORS WITH DISTRIBUTIONAL SYMBOLS ON FOCK SPACES

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ABSTRACT. We define and study Toeplitz operators T_a with distributional symbols in the setting of weighted Fock spaces of entire functions on the complex plane. Sufficient conditions for boundedness and compactness are presented in terms of the symbol belonging to a weighted Sobolev space $W_\omega^{-m,\infty}$ of negative order.

1. INTRODUCTION.

We consider Toeplitz operators T_a on weighted Fock spaces F_γ^p of entire functions on the complex plane. The Fock space is also known as the Segal-Bargmann space in the case $p = 2$, and various aspects of Toeplitz operators and their applications to canonical quantization theory of theoretical physics have been considered in that case e.g. in [4], [5], [6], [8], [9]. More recent studies are included in [2], [7], [10]. For related studies in Bergman spaces we refer to [3], [11], [14], [16], [17], [19].

We are concerned in this paper with the problems of finding sufficient conditions for the boundedness and compactness of $T_a : F_\gamma^p \to F_\gamma^p$, if $1 \leq p \leq \infty$, $\gamma > 0$. In the context of Bergman spaces, a weak sufficient condition was recently found in [12] even for distributional symbols a. Our purpose is to establish similar results in Fock spaces. The main result (Theorem 4.1) states that $T_a : F_\gamma^p \to F_\gamma^p$ is bounded, if the symbol a can be presented (2.3) as a sum of (in general, distributional) derivatives of functions vanishing rapidly enough at infinity (the order of vanishing being equal to the order of the derivative). This result is interesting already for function symbols, since it yields plenty of examples of unbounded functions which induce bounded Toeplitz operators. A quite dramatic case is presented in Example 4.2.

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Recall that, if $1 \le p < \infty$, F^p_γ is the closed subspace of

$$L^p_\gamma = L^p(\mathbb{C}, e^{-\gamma p|z|^2/2} dA)$$

consisting of analytic functions. The spaces are endowed with the norm

$$\|f; L^p_\gamma\|^p := \int\limits_{\mathbb{D}} |f(z)|^p e^{-\gamma p|z|^2/2} dA(z) =: \int\limits_{\mathbb{D}} |f(z)|^p dA_{p,\gamma}(z) \tag{1.1}$$

where $dA = dxdy$ is the area measure on the complex plane $\mathbb{C}$. The space F^∞_γ consists entire functions f such that $z \mapsto f(z)e^{-\gamma|z|^2/2}$ is bounded, and the norm is given by

$$\|f; L^\infty_\gamma\| := \sup_{z\in\mathbb{C}} |f(z)|e^{-\gamma|z|^2/2} \tag{1.2}$$

Note that for every $\gamma > \gamma' > 0$ and $1 \le p < q \le \infty$, we have

$$F^1_\gamma \subset F^p_\gamma \subset F^q_\gamma \subset F^\infty_\gamma \quad \text{and} \quad F^p_{\gamma'} \subset F^p_\gamma$$

with bounded inclusions.

The Fock projection P is the orthogonal projection of L^2_γ onto A^2_γ, and it has the integral representation

$$Pf(z) = \int\limits_{\mathbb{C}} f(w) e^{\gamma z\bar{w}} e^{-\gamma|w|^2} dA(w).$$

It is also known to be a bounded projection of L^p_γ onto A^p_γ for all $1 \le p \le \infty$. This fact depends on the correct form of the weight in (1.1): the boundedness is not true for the norm of $L^p(\mathbb{C}, e^{-\gamma|z|^2}dA)$. Note also that unlike in the Bergman spaces setting, things like the boundedness of the projection also hold in the cases $p = 1$ and $p = \infty$. Henceforth we will always assume that $p \in [1, \infty]$.

For an essentially bounded $a : \mathbb{C} \to \mathbb{C}$ and $f \in F^p_\gamma$, the Toeplitz operator T_a with symbol a is defined by setting

$$T_a f = P(af).$$

Since P is bounded, it follows easily that T_a is a bounded operator $F^p_\gamma \to F^p_\gamma$ for $1 \le p \le \infty$. If a is a compactly supported distribution on $\mathbb{C}$, one can easily define the corresponding Toeplitz operator by

$$T_a f(z) = \langle f(w) e^{\gamma z\bar{w}} e^{-\gamma|w|^2}, a\rangle_w \tag{1.3}$$

where $\langle \cdot, \cdot \rangle_w$ denotes the dual paring of the test function and distribution spaces and the test function is considered as a function of the variable w with z being a parameter. Analogously to the Bergman space case, we give below a more general definition for distributional symbols without compactness restrictions. Sufficient conditions for boundedness and compactness of Toeplitz operators are given in Theorem 4.1, Section 4, and Proposition 5.1, Section 5, respectively. The distributional symbol class will be a weighted

Sobolev space of negative order, the analysis of which is the object of Section 2. New integral estimates, specific to the Fock space case, are derived in Section 3.

2. Preliminaries.

Concerning notation and basic definitions, we follow the terminology of [13] for general theory of distributions, [1] for Sobolev spaces, and [18] for operator theory and analytic function spaces. In the following we consider various function and distribution spaces, all of which are defined on $\mathbb{C}$. For the norm of an element f of a Banach function space X we use the notation $\|f; X\|$; for the operator norm of a bounded linear operator $T : X \to Y$ we write $\|T : X \to Y\|$. The standard space of infinitely smooth compactly supported test function in the plane is denoted by $C_0^\infty = C_0^\infty(\mathbb{C})$, and its dual, the space of distributions on $\mathbb{C}$, is $\mathcal{D}' = \mathcal{D}'(\mathbb{C})$. The order of a multi–index $\alpha \in \mathbb{N}^2$, where $\mathbb{N} := \{0, 1, 2, , \ldots\}$, is denoted by $|\alpha| := \alpha_1 + \alpha_2$. The notation $\alpha \geq \beta$ for the multi–indices α, β means that $\alpha_j \geq \beta_j$ for $j = 1, 2$. As for derivatives, the notation $D^\alpha f$ stands for

$$\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}} f,$$

if f is a function of $z = x + iy$, where x, $y \in \mathbb{R}$, and α is a multi–index. The same notation is used for both classical and distributional derivatives. We also write $D_w^\alpha f$, if it is necessary to indicate the differentiation of a function f with respect to its variable w. For an analytic function f of the variable $z \in \mathbb{C}$, we denote by $f^{(l)}$ the l:th derivative with respect to z, for all $l \in \mathbb{N}$. By C, C', C_1, c etc. (respectively, C_n etc.) we mean positive constants independent of functions, variables or indices occurring in the given calculations (respectively, depending only on n). These may vary from place to place, but not in the same group of inequalities.

We define $\omega : \mathbb{D} \to \mathbb{R}^+$ to be the standard weight function

$$\omega(z) = 1 + |z|. \tag{2.1}$$

Given $m \in \mathbb{N}$ and $1 \leq p < \infty$ we denote by $W_\omega^{m,1} = W_\omega^{m,1}(\mathbb{C})$ the weighted Sobolev space consisting of measurable functions f on $\mathbb{C}$ such that

$$\|f; W_\omega^{m,1}\| := \sum_{|\alpha| \leq m} \int_{\mathbb{C}} |D^\alpha f(z)| \omega(z)^{-|\alpha|} dA(z) < \infty. \tag{2.2}$$

The following fact is known; the proof is easier than the one in the Bergman space setting, [12], Lemma 2.2, since now it is possible to define suitable cut-off functions χ_n with bounded derivatives, instead of those in the citation.

Lemma 2.1. *The subspace C_0^∞ of compactly supported infinitely smooth functions on $\mathbb{C}$ is dense in $W_\omega^{m,1}$.*

Definition 2.2. Given $m \in \mathbb{N}$ we denote by $W_\omega^{-m,\infty} = W_\omega^{-m,\infty}(\mathbb{C})$ the weighted Sobolev space consisting of distributions a on $\mathbb{C}$ which can be written in the form

$$a = \sum_{0 \le |\alpha| \le m} (-1)^{|\alpha|} D^\alpha b_\alpha, \tag{2.3}$$

where $b_\alpha \in L_\alpha^\infty := L^\infty(\mathbb{C}, \omega(z)^{|\alpha|})$, i.e.,

$$\|b_\alpha; L_\alpha^\infty\| := \operatorname*{ess\,sup}_{\mathbb{C}} \omega(z)^{|\alpha|} |b_\alpha(z)| < \infty. \tag{2.4}$$

Here every b_α is considered as a distribution like a locally integrable function, and the identity (2.3) contains distributional derivatives. Note that if $b_\alpha \in L_\alpha^\infty$, then

$$|b_\alpha(z)| \le \|b_\alpha; L_\alpha^\infty\| (1 + |z|)^{-|\alpha|}, \tag{2.5}$$

for almost every $z \in \mathbb{C}$. In particular $\|b_\alpha; L^\infty\| \le \|b_\alpha; L_\alpha^\infty\|$.

We remark that using a representation (2.3) even for smooth symbols yields interesting result on boundedness of T_a; see Example 4.2.

The representation (2.3) is not unique in general. Hence, we define the norm of a by

$$\|a\| := \|a; W_\omega^{-m,\infty}\| := \inf \max_{0 \le |\alpha| \le m} \|b_\alpha; L_\alpha^\infty\|, \tag{2.6}$$

where the infimum is taken over all representations (2.3).

Lemma 2.3. *The dual of $W_\omega^{m,1}$ is isometrically isomorphic to $W_\omega^{-m,\infty}$ with respect to the dual paring*

$$\langle f, a \rangle := \sum_{0 \le |\alpha| \le m} \int_{\mathbb{C}} (D^\alpha f) b_\alpha dA, \tag{2.7}$$

where the functions b_α are as in (2.3).

The proof uses Lemma 2.1 and the arguments of [1], Sections 3.8–3.10. See [12] for some more explanations.

Remark 2.4. If $a \in W_\omega^{-m,\infty}$, the value of the expression on the right hand side of (2.7) is unique, although the representation (2.3) is not. Namely, for every $\varphi \in C_0^\infty$, the value of

$$\sum_{0 \le |\alpha| \le m} \int_{\mathbb{C}} (D^\alpha \varphi) b_\alpha dA$$

coincides with $\langle \varphi, a \rangle$, by the standard definition of distributional derivative, and the uniqueness of (2.7) follows from Lemma 2.1.

We shall need a pointwise estimate for Fock functions, which follows from [10], Lemma 1 in Section 2.

Lemma 2.5. *Let $1 \leq p \leq \infty$ and $\gamma > 0$, and let $f \in F^p_\gamma$. There exists a constant $C > 0$ such that*

$$|f(z)| \leq Ce^{\gamma|z|^2/2} \|f; L^p_\gamma\|$$

for all $z \in \mathbb{C}$.

Proof. When $p = \infty$, this is trivial because of the definition of the norm. When $p < \infty$, enlarge the integration domain from the disc $B(z,r)$ to the entire plane and take the p:th root in the reference. □

3. Integral estimates in Fock spaces.

In this section we present some integral estimates, which imply a weighted norm estimate for the differentiation operator in the Fock spaces, see Lemma 3.4. estimates. The proofs in this section rely on a simple but useful integral splitting trick.

Denote by $\bar{P}$ the maximal projection;

$$\bar{P}f(z) = \int\limits_{\mathbb{C}} |f(w)|\, |e^{\gamma z\bar{w}}| e^{-\gamma|w|^2} dA(w).$$

The following fact is proven in [7]. Again, the validity of this result depends on the particular form of the Fock norm for $p \neq 2$.

Lemma 3.1. *The maximal projection $\bar{P}$ is bounded $L^p_\gamma \to F^p_\gamma$, i.e. $\|\bar{P}f; L^p_\gamma\| \leq C\|f; L^p_\gamma\|$.*

The boundedness of the maximal projection gives us more freedom when dealing with integral estimates. We use this to prove the following.

Lemma 3.2. *Let $\alpha \in \mathbb{N}^2$ be any multi-index and $1 \leq p \leq \infty$. The operator*

$$T_k f(z) := z^k \int\limits_{\mathbb{C}} f(w) e^{\gamma z\bar{w}} e^{-\gamma|w|^2} b_\alpha(w) dA(w)$$

is bounded $L^p_\gamma \to F^p_\gamma$, whenever $k \leq |\alpha|$ and $b_\alpha \in L^\infty_\alpha$. Moreover, $\|T\| \leq C\|b_\alpha; L^\infty_\alpha\|$ for some positive constant $C = C(p,\gamma,\alpha)$.

Proof. Let first $p < \infty$. Write $T_k = T^1_k + T^2_k$, where

$$T^1_k f(z) := z^k \int\limits_{\{|w|\leq|z|/4\}} f(w) e^{\gamma z\bar{w}} e^{-\gamma|w|^2} b_\alpha(w) dA(w)$$

and

$$T^2_k f(z) := z^k \int\limits_{\{|w|>|z|/4\}} f(w) e^{\gamma z\bar{w}} e^{-\gamma|w|^2} b_\alpha(w) dA(w).$$

For T_k^1 we note that if $|w| \le |z|/4$, then $|e^{\gamma z\bar{w}}| \le e^{\gamma|z|^2/4}$. This, lemmas 2.5 and 3.1 and the remark following (2.5) imply for $f \in L^p_\gamma$

$$\begin{aligned}
\|T_k^1 f; L^p_\gamma\|^p &= \int_{\mathbb{C}} |T_k^1 f(z)|^p e^{-\gamma p|z|^2/2} dA(z) \\
&\le \int_{\mathbb{C}} |z|^{kp} e^{\gamma p|z|^2/4} \Big(\int_{\mathbb{C}} |f(w)||b_\alpha(w)| e^{-\gamma|w|^2} dA(w) \Big)^p e^{-\gamma p|z|^2/2} dA(z) \\
&= \int_{\mathbb{C}} |z|^{kp} e^{\gamma p|z|^2/4} \Big(P(|f b_\alpha|)(0) \Big)^p e^{-\gamma p|z|^2/2} dA(z) \\
&\le C' \int_{\mathbb{C}} |z|^{kp} e^{\gamma p|z|^2/4} \|b_\alpha f; L^p_\gamma\|^p e^{-\gamma p|z|^2/2} dA(z) \\
&\le C' \int_{\mathbb{C}} |z|^{kp} e^{\gamma p|z|^2/4} \|b_\alpha; L^\infty\|^p \|f; L^p_\gamma\|^p e^{-\gamma p|z|^2/2} dA(z) \\
&\le C'' \|f; L^p_\gamma\|^p \|b_\alpha; L^\infty\|^p \int_{\mathbb{C}} |z|^{kp} e^{-\gamma p|z|^2/4} dA(z) \\
&\le C''' \|f; L^p_\gamma\|^p \|b_\alpha; L^\infty_\alpha\|^p.
\end{aligned}$$

This shows that T_k^1 is bounded.

To show that T_k^2 is bounded, just note that if $|w| > |z|/4$, then

$$|T_k^2 f(z)| \le \int_{\mathbb{C}} 4^k |w^k f(w) e^{\gamma z\bar{w}} b_\alpha(w)|\, e^{-\gamma|w|^2} dA(w).$$

Using the fact that

$$|w|^k |b_\alpha(w)| \le \|b_\alpha; L^\infty_\alpha\|,$$

whenever $k \le |\alpha|$, and the boundedness of the maximal projection, we conclude that T_k^2 is also bounded.

The above things also work when $p = \infty$; the reasoning is completely similar, even though the norm is not defined as an integral. The details are left as an easy exercise for the reader.

Combining the above estimates we see that

$$\|T_k\| \le C(p, \gamma, \alpha) \|b_\alpha; L^\infty_\alpha\|,$$

as claimed. Note that $T_k f$ is clearly analytic. $\Box$

The following is a corollary of the proof rather than the lemma itself. The norm on the left hand side is for the function of the variable z.

Corollary 3.3. *If the assumptions of Lemma 3.2 are satisfied, we have the bound*

$$\Big\| \int_{\mathbb{C}} \omega(z)^k \omega(w)^{-|\alpha|} \, |f(w) e^{\gamma z \bar{w}}| \, e^{-\gamma |w|^2} dA(w) \; ; L^p_\gamma \Big\| \le C \|f; L^p_\gamma\|$$

for $f \in L^p_\gamma$. The same is still true, if $\omega(z)^k$ is replaced by z^k.

The proof of the following result uses similar ideas.

Lemma 3.4. *The operator S_k defined by*

$$S_k f(z) := \omega^{-|\alpha|}(z) f^{(k)}(z)$$

is bounded $F^p_\gamma \to L^p_\gamma$, whenever $k \le |\alpha|$.

Proof. Let first $p < \infty$. Assume that $f \in F^p_\gamma$. Using the reproducing property, we have

$$f(z) = \int_{\mathbb{C}} f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} dA(w).$$

Differentiating k times under the integral sign, we see that S_k has the representation

$$S_k f(z) = \int_{\mathbb{C}} (\gamma \bar{w})^k \omega(z)^{-|\alpha|} f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} dA(w).$$

Now $S_k = S^1_k + S^2_k$, where

$$S^1_k f(z) = \int_{\{|w| \ge 4|z|\}} (\gamma \bar{w})^k \omega(z)^{-|\alpha|} f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} dA(w)$$

and

$$S^2_k f(z) = \int_{\{|w| < 4|z|\}} (\gamma \bar{w})^k \omega(z)^{-|\alpha|} f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} dA(w).$$

For S^1_k, note that $|e^{\gamma z \bar{w}}| \le e^{\gamma |w|^2/4}$, since $|z| \le |w|/4$. Using the pointwise estimate in Lemma 2.5 for f, we get

$$\begin{aligned}
\|S^1_k f; L^p_\gamma\|^p &= \int_{\mathbb{C}} |S^1_k f(z)|^p e^{-\gamma p |z|^2/2} dA(z) \\
&\le \int_{\mathbb{C}} \Big(\int_{\{|w| \ge 4|z|\}} |\gamma \bar{w}|^k \|f; L^p_\gamma\| e^{\gamma |w|^2/2} e^{\gamma |w|^2/4} e^{-\gamma |w|^2} dA(w) \Big)^p e^{-\gamma p |z|^2/2} dA(z) \\
&\le C \|f; L^p_\gamma\|^p \int_{\mathbb{C}} \Big(\int_{\mathbb{C}} |\bar{w}|^k e^{-\gamma |w|^2/5} dA(w) \Big)^p e^{-\gamma p |z|^2/2} dA(z) \\
&\le C' \|f; L^p_\gamma\|.
\end{aligned}$$

For the operator S_k^2, note that $|z| > |w|/4$ and $k \le |\alpha|$, hence, $(\bar{w}\gamma)^k \omega(z)^{-|\alpha|}$ is bounded and the claim easily follows from the boundedness of the maximal projection.

Again, the case $p = \infty$ bears no additional difficulties. $\square$

4. Boundedness of Toeplitz operators with distributional symbols.

The main result about the boundedness of T_a can be stated as follows.

Theorem 4.1. *Assume the symbol $a \in \mathcal{D}' = \mathcal{D}'(\mathbb{C})$ belongs to $W_\omega^{-m,\infty}$ for some m. Then the Toeplitz operator T_a, defined by the formula*

$$(4.1) \quad T_a f(z) = \sum_{0 \le |\alpha| \le m} \int_{\mathbb{C}} D_w^\alpha \Big(f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} \Big) b_\alpha(w) dA(w) \quad , \; f \in F_\gamma^p,$$

is well defined and bounded $F_\gamma^p \to F_\gamma^p$ for all $1 \le p \le \infty$, $\gamma > 0$. The resulting operator is independent of the choice the representation (2.3). Moreover, there is a constant $C_1 > 0$ such that

$$\|T_a\| \le C_1 \|a; W_\omega^{-m,\infty}\|. \tag{4.2}$$

Proof. Fix a representation (2.3) for a such that $\|a; W_\omega^{-m,\infty}\| \ge \max_{|\alpha| \le m} \|b_\alpha, L_\alpha^\infty\|/2$. For $|\alpha| \le m$, we have

$$\begin{aligned} &\big| D_w^\alpha \big(f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} \big) b_\alpha(w) \big| \\ \le &C \sum_{\beta \le \alpha} | D_w^{\alpha - \beta} (f(w)) D_w^\beta (e^{\gamma z \bar{w}} e^{-\gamma |w|^2}) b_\alpha(w)|. \end{aligned}$$

Furthermore, for each multi-index β we have

$$|D_w^\beta (e^{\gamma z \bar{w}} e^{-\gamma |w|^2})| \le C_\beta \sum_{\sigma \le \beta} \omega(z)^{|\sigma|} \omega(w)^{|\beta| - |\sigma|} \left| e^{\gamma z \bar{w}} \right| e^{-\gamma |w|^2},$$

for some positive constant C_β. This can be seen by a direct calculation. We arrive at

$$\begin{aligned} &\big| D_w^\alpha \big(f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} \big) b_\alpha(w) \big| \\ &\le C' \sum_{\beta \le \alpha} \sum_{\sigma \le \beta} \omega(z)^{|\sigma|} \omega(w)^{|\beta| - |\sigma|} \left| f^{(|\alpha| - |\beta|)}(w) e^{\gamma z \bar{w}} \right| e^{-\gamma |w|^2} \left| b_\alpha(w) \right| \\ &\le C'' \|b_\alpha; L_\alpha^\infty\| \sum_{\beta \le \alpha} \sum_{\sigma \le \beta} \left| \omega(w)^{-|\alpha| + |\beta|} f^{(|\alpha| - |\beta|)}(w) \right| \\ &\quad \cdot \omega(z)^{|\sigma|} \omega(w)^{-|\sigma|} \left| e^{\gamma z \bar{w}} \right| e^{-\gamma |w|^2}. \end{aligned} \tag{4.3}$$

Notice that by Lemma 3.4 the function

$$w \mapsto |\omega(w)^{-|\alpha| + |\beta|} f^{(|\alpha| - |\beta|)}(w)|$$

belongs to L^p_γ, with norm bounded by a constant times $\|f; L^p_\gamma\|$. Hence, integrating with respect to w and applying Corollary 3.3 to each of the terms (4.3) separately, yields the bound

$$\|D^\alpha_w\big(f(w)e^{\gamma z\bar{w}}e^{-\gamma|w|^2}\big)b_\alpha; L^p_\gamma\| \le C\|f; L^p_\gamma\|.$$

The operator above is also unambiguously defined; Using Lemmas 3.2 and 3.4 and some simple estimates it is not difficult to see that if $f \in F^p_\gamma$ for any $p \in [1, \infty]$, then

$$w \mapsto f(w)e^{\gamma z\bar{w}}e^{-\gamma|w|^2}$$

is an element of the space $W^{m,1}_\omega$. This, by the reasoning in Remark 2.4, implies the uniqueness of T_a. $\square$

Example 4.2. $1°$. Consider the function

$$b_{(1,0)}(z) = \frac{\sin(\exp^{100}(x^2))}{\omega(z)}$$

where $z = x + iy$ and $\exp^k(x) := \exp(\exp^{k-1}(x))$ for all natural numbers k . The function $a := D^{(1,0)}b_{(1,0)} = \partial b_{(1,0)}/\partial x$ is very far from being L^1, but, by Theorem 4.1, T_a is still bounded.

$2°$. We provide an example of a distributional symbol with noncompact support. Consider

$$a := b_{(0,0)} + D^{(1,0)}b_{(1,0)} = \frac{\partial}{\partial x}b_{(1,0)}, \tag{4.4}$$

where

$$b_{(0,0)}(z) = \begin{cases} 0\ , & \text{if } x \le 0 \\ -2x/(1+r^2)^2\ , & \text{if } x > 0 \end{cases}$$

and

$$b_{(1,0)}(z) = \begin{cases} 0\ , & \text{if } x \le 0 \\ 1/(1+r^2)\ , & \text{if } x > 0 \end{cases}$$

with $z = x + iy$, $r = |z|$. We have $b_{(1,0)} = Y(x)/(1 + r^2)$, where $Y(x)$ is the usual step function of one variable, and moreover

$$a = b_{(0,0)} + D^{(1,0)}b_{(1,0)} = \delta_0/(1 + y^2).$$

Here δ_0 denotes the Dirac measure in variable $x \in \mathbb{R}$, so the symbol a is a weighted Dirac measure of the imaginary axis. Since $a \in W^{-1,\infty}_\omega$, it defines a bounded Toeplitz operator $T_a : F^p_\gamma \to F^p_\gamma$ (it is even compact, see below).

5. Compactness of Toeplitz operators with distributional symbols.

The observation in Proposition 4.1 of [12] remains true also in the Fock space case.

Proposition 5.1. *An arbitrary compactly supported distribution $a \in \mathcal{D}'$ belongs to the Sobolev space $W_\omega^{-m,\infty}$ and defines a compact Toeplitz operator $F_\gamma^p \to F_\gamma^p$.*

The proof is the same as in the citation.

Theorem 5.2. *Let $a \in \mathcal{D}'$ belong to $W_\omega^{-m,\infty}$ for some m. The Toeplitz operator T_a,* (4.1)*, is compact, if a has a representation* (2.3) *such that the functions b_α satisfy*

$$\lim_{r\to\infty} \operatorname{ess\,sup}_{|z|\geq r} \omega(z)^{-|\alpha|}|b_\alpha(z)| = 0. \tag{5.1}$$

Proof. We pick up functions b_α, $0 \leq |\alpha| \leq m$, as in (5.1). For $0 < r < \infty$, we define for all α the compactly supported functions

$$b_{\alpha,r}(z) = \begin{cases} b_\alpha(z)\ , & \text{if } |z| \leq r, \\ 0\ , & \text{if } |z| > r \end{cases}$$

Let also $a_r = \sum_{0\leq|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha b_{\alpha,r}$, where the derivatives are distributional. Of course, a_r is a distribution with compact support, hence, by the remark above, the Toeplitz operator $T_{a_r} : F_\gamma^p \to F_\gamma^p$ is compact for every r. On the other hand, due to the definition (2.6), the property (5.1) and the norm estimate (4.2), the operator norm $\|T_a - T_{a_r}\|$ can be made arbitrarily small choosing r close enough to 1. Consequently, T_a must be a compact operator. $\square$

Example 5.3. Returning to the example 4.2, the symbol

$$D^{(1,0)}b_{(1,0)}(z) = D^{(1,0)}\Big(\frac{\sin(\exp^{100}(x^2))}{\omega(z)^2}\Big)$$

defines a Toeplitz operator which is even compact.

In view of Theorem 5.2, also the symbol (4.4) defines a compact Toeplitz operator.

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