TOEPLITZ OPERATORS ON BERGMAN SPACES AND HARDY MULTIPLIERS

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Abstract. We study Toeplitz operators T_a with radial symbols in weighted Bergman spaces A^p_μ, $1 < p < \infty$, on the disc. Using a decomposition of A^p_μ into finite dimensional subspaces the operator T_a can be considered as a coefficient multiplier. This leads to new results on boundedness of T_a and also shows a connection to Hardy space multipliers. Using another method we also prove a necessary and suffient condition for the boundedness of T_a for a satisfying an assumption on the positivity of certain indefinite integrals.

1. Introduction and notation.

We study Toeplitz operators T_a with radial symbols in weighted Bergman spaces $A^p_\mu = A^p_\mu(\mathbb{D})$, $1 < p < \infty$, of the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. In the Hilbert space case $p = 2$, the article [8] (see also [22]) contains a study of T_a as the Taylor coefficient multiplier

$$T_a : \sum_{k\in\mathbb{N}} f_k z^k \mapsto \sum_{k\in\mathbb{N}} \gamma_k f_k z^k. \tag{1.1}$$

The multiplier coefficients γ_k are weighted moments γ_k of the symbol a, see (3.1), so the boundedness of $T_a : A^2 \to A^2$ can be characterized in terms of the boundedness of the sequence $(\gamma_k)_{k=1}^\infty$. The reference [8] contains the unitary equivalence of T_a to a multiplication operator even in a more general setting, thus completely clarifiying the basic properties of T_a.

The Toeplitz operator can still be considered as a multiplier even in the case $p \neq 2$, but this is a less useful point of view, since the monomials do not form an unconditional Schauder basis in A^p, $p \neq 2$, not to speak about weighted norms. However, in this paper we prove (Theorem 3.5) the fact that even with quite general weighted norms the Bergman space can still be decomposed into finite dimensional subspaces $A^{(n)}$, $n \in \mathbb{N}$, spanned by monomials with degrees in certain subintervals $\mathbb{N}_n$ of $\mathbb{N} = \{0, 1, 2, \ldots\}$. The boundedness and compactness of T_a can be characterized in terms of its behaviour on these blocks, see Theorem 3.3.

In fact this leads to the following result (for the assumptions on μ, see this section below, and for γ_k and $\mathbb{N}_n$, see Section 3).

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Theorem 1.1. *Let $1 < p < \infty$. The Toeplitz operator $T_a : A^p_\mu \to A^p_\mu$ is bounded, if and only if the coefficient multipliers*

$$T_a^{(n)} : \sum_{k \in \mathbb{N}} f_k e^{ik\theta} \mapsto \sum_{k \in \mathbb{N}_n} \gamma_k f_k e^{ik\theta} \quad , \ \theta \in [0, 2\pi], \tag{1.2}$$

are for all $n \in \mathbb{N}$ uniformly bounded operators $H^p \to H^p$, where H^p is the Hardy space on the disc. Moreover, $T_a : A^p_\mu \to A^p_\mu$ is compact, if and only if the sequence formed by the operator norms of $T_a^{(n)} : H^p \to H^p$ converges to 0.

The reason is that on every $A^{(n)}$, the Bergman norm is actually equivalent to a Hardy-type norm, a result which is contained in Theorem 3.5. We remark that a complete characterization of the boundedness of Hardy multipliers is not known. The operator norms of $T_a^{(n)} : H^p \to H^p$ will later be denoted by $\mathcal{M}_p(\tau_n)$.

As for other result, we observe in Proposition 3.1 that the boundedness of the multiplier sequence (γ_k) is still necessary for the boundedness of the operator $T_a : A^p_\mu \to A^p_\mu$ in the case $p \neq 2$, for general weights. Hence, we obtain the result that boundedness of the operator $T_a : A^p_\mu \to A^p_\mu$, $p \neq 2$, implies the boundedness $T_a : A^2_\mu \to A^2_\mu$, see Theorem 3.2.

The above approach of Toeplitz operators as multipliers is contained in Section 3. In Section 2 we provide another type of necessary and sufficienct condition for the boundedness of T_a for special a. Namely, making a rather weak assumption on the positivity of certain indefinite nth integral $I_a(n)$ of the symbol a, the boundedness of $T_a : A^2 \to A^2$ was characterized in [9] in terms of the boundary behaviour of $I_a^{(n)}$. In Theorem 2.1 we generalize this to the case $p \neq 2$. The proof uses some estimates of the kernel of the Berezin transform and it is considerably more complicated than in the Hilbert space case.

We recall the basic definitions and notation. An introduction to Bergman spaces on $\mathbb{D}$ and Toeplitz operators can be found in [28]. For Hardy spaces we also refer to [19]. By C, C', C_1 etc. we mean positive constants independent of given functions or indices, but which may vary from place to place. The Toeplitz operator T_a is defined as the product of pointwise multiplication and Bergman projection operators,

$$T_a f(z) = \int_{\mathbb{D}} \frac{a(w) f(w)}{(1 - z\bar{w})^2} dA(w),$$

where dA is the normalized two-dimensional Lebesgue measure on $\mathbb{D}$ and $a : \mathbb{D} \to \mathbb{C}$ is the symbol of T_a. In the following we restrict to the case $a \in L^1(\mathbb{D})$ is radial: $a(z) = a(|z|)$. Then also the one real variable function $a(r)$ belongs to $L^1(0, 1)$.

We shall work in the context of weighted Bergman spaces with rather general radial weights. Let μ be a nonatomic, bounded positive measure on $[0, 1[$ such that $\mu([1 - \varepsilon, 1[) > 0$ for every $0 < \varepsilon < 1$. We define the space $L^p_\mu = L^p_\mu(\mathbb{D})$ using the norm

$$\|f\|_{p,\mu} := \Big(\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta}|^p r d\theta d\mu(r) \Big)^{1/p}. \tag{1.3}$$

The corresponding weighted Bergman space (closed subspace of L^p_μ consisting of analytic functions) is denoted by $A^p_\mu := A^p_\mu(\mathbb{D})$. The most classical case of unweighted Bergman space $A^p := A^p(\mathbb{D})$ corresponds to the measure $d\mu = dr$; in this case we omit the index μ in the notation. Also the weighted cases with measures $d\mu = (1-r^2)dr$ are standard in the literature.

Remark 1.2. 1°. It is not really important to assume the measure to be non-atomic, since, for any bounded positive measure μ on $[0,1[$ and every $\varepsilon > 0$, there exists a non-atomic (bounded positive) measure μ_0 such that the weighted Bergman spaces corresponding to μ and μ_0 are the same and

$$(1-\varepsilon)\|f\|_{p,\mu} \le \|f\|_{p,\mu_0} \le \|f\|_{p,\mu} \tag{1.4}$$

for all $f \in A_{p,\mu}$. This fact is essentially contained in Proposition 1.2. of [11].

2°. With our assumptions on the measure μ, polynomials form a dense subspace of A^p_μ. See [14], Proposition 2.1.

The basic problem of characterizing the boundedness of Toeplitz operators on Bergman spaces is still open. Well-known partial results on the topic are included in [12, 16, 25, 26, 27, 29]. Recently, quite weak sufficient conditions for the boundedness were given in [21, 17]. The latter reference also contains the definition of Toeplitz operators with distributional symbols. The recent review [18] contains some results on radial symbols, related to the present paper. Many other topics, like matrices or products of Toeplitz-operators, operators on general domains, Fredholm properties and so on, have attained a lot of attention; as for the literature selection, see [1, 2, 3, 4, 5, 6, 7, 11, 20, 23, 29] and others.

2. Symbols with positive integrals.

The problem of characterizing the boundedness of T_a was solved for positive a in [16] (a good presentation of the topic with references to later developments and generalizations can be found in [28]): the Toeplitz operator $T_a : A^p \to A^p$ is bounded if and only if the Berezin transform of a is bounded. Our aim here is to consider the radial case for all $1 < p < \infty$ and in particular weaken the positivity assumption on the symbol a as follows. For all $n \in \mathbb{N}$, $n \ge 1$, we define the n:th indefinite integrals by

$$I_a(r) := I_a^{(1)}(r) = \int_r^1 a(s)s\,ds \ , \quad I_a^{(n+1)}(r) := \int_r^1 I_a^{(n)}(s)s\,ds \ \ , \ r \in [0,1[. \tag{2.1}$$

It is clear that for positive a all indefinite integrals $I_a^{(n)}$ are positive functions; moreover, if $I_a^{(n)}$ is positive, then the same is true for all $I_a^{(k)}$ with $k > n$. On the other hand, $I_a^{(n)}$ may very well be positive, though a is not. We shall solve the boundedness problem under the assumption that $I_a^{(n)}$ is positive for some n:

Theorem 2.1. *Assume that for some $n \ge 1$ the function I_a^n is nonnegative. Then the Toeplitz operator $T_a : A^p \to A^p$ is bounded, if and only if*

$$|I_a^{(n+1)}(r)| \le C(1-r)^{n+1} \tag{2.2}$$

for all $r \in [0,1[$.

In the case $p=2$ this result was proven in [8], [22].

The proof requires some preparations. The normalized Bergman kernel and the kernel of the Berezin transform are denoted, respectively, by

$$k_z(w) = \frac{(1-|w|^2)}{(1-z\bar{w})^2} \quad , \quad B_z(w) = \frac{(1-|z|^2)^2}{|1-z\bar{w}|^4} = k_w(z)\overline{k_w(z)} \quad , \quad z,w \in \mathbb{D} \tag{2.3}$$

the Berezin transform of a function $f \in L^1(\mathbb{D})$ thus being $\tilde{f}(z) := \int_{\mathbb{D}} f(w) B_z(w) dA(w)$.

Lemma 2.2. *For every* $k \in \mathbb{N}$, $k \geq 1$, *there exists a constant* $C_k > 0$ *such that*

$$\frac{1}{C_k}\frac{r^{k-1}(1-r^2)^2}{(1-r\varrho)^{k+3}} \leq \int_0^{2\pi} \frac{\partial^k}{\partial \varrho^k}\big(\varrho B_z(\varrho e^{i\theta})\big) d\theta \leq C_k \frac{r^{k-1}(1-r^2)^2}{(1-r\varrho)^{k+3}}, \tag{2.4}$$

where $r := |z|$ *and* $\varrho \in [0,1[$.

Proof. Writing $w = \varrho e^{i\theta}$,

$$\begin{aligned}
\frac{1}{|1-z\bar{w}|^4} &= \frac{1}{(1-z\bar{w})^2}\frac{1}{(1-\bar{z}w)^2} \\
&= \Big(\sum_{n=0}^{\infty}(n+1)(z\bar{w})^n\Big)\Big(\sum_{n=0}^{\infty}(n+1)(\bar{z}w)^n\Big) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(n-m+1)(m+1)(z\bar{w})^{n-m}(\bar{z}w)^m \\
&= \sum_{n=0}^{\infty}\varrho^n\sum_{m=0}^{n}(n-m+1)(m+1)z^{n-m}\bar{z}^m e^{-i\theta(n-2m)} \qquad (2.5)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\partial^k}{\partial\varrho^k}\frac{\varrho}{|1-z\bar{w}|^4} \\
&= \sum_{n=k-1}^{\infty}\frac{(n+1)!}{(n-k+1)!}\varrho^{n-k+1}\sum_{m=0}^{n}(n-m+1)(m+1)z^{n-m}\bar{z}^m e^{-i\theta(n-2m)}, \qquad (2.6)
\end{aligned}$$

and, since only the terms with even n and $m = n/2$ are nonzero after integration, we get

$$\begin{aligned}
\int_0^{2\pi}\frac{\partial^k}{\partial\varrho^k}\frac{\varrho}{|1-z\bar{w}|^4}d\theta &= 2\pi\sum_{\substack{n\geq k-1\\ n \text{ even}}}\frac{(n+1)!}{(n-k+1)!}\Big(\frac{n}{2}+1\Big)^2 z^{n/2}\bar{z}^{n/2}\varrho^{n-k+1} \\
&= \frac{\pi}{2}r^{k-1}\sum_{\substack{n\geq 0\\ n+k \text{ odd}}}\frac{(n+k)!}{n!}(n+k+1)^2(r\varrho)^n \qquad (2.7)
\end{aligned}$$

From the identity

$$\frac{d^{k+2}}{dx^{k+2}}\frac{1}{1-x}=\sum_{n=0}^{\infty}\frac{(n+k+2)!}{n!}x^n \quad , \quad l\in\mathbb{N}, \tag{2.8}$$

one can develope the Taylor series for the expressions $(1-r\varrho)^{-k-3}$; the result follows from (2.7), using the estimates

$$\frac{1}{C}\frac{(n+k+2)!}{n!}\leq\frac{(n+k)!}{n!}(n+k+1)^2\leq\frac{C(n+k+2)!}{n!}, \tag{2.9}$$

and the observation that the missing even, say $(n+k+1)$st, degree term in (2.7) is proportional to the existing $(n+k)$th degree term. □

Notice that though the ϱ-derivatives of the Berezin kernel are not positive functions, the angular integrals of the derivatives are, by the previous lemma. This will be of crucial importance for the proof of the main result.

The following lemma is well known, and follows from the boundedness of T_a by considering the expression $\langle T_a k_\zeta, k_\zeta\rangle$, and using the definition of the operator norm and duality of Bergman spaces.

Lemma 2.3. *If $T_a : A^p \to A^p$ is bounded, then*

$$\Big|\int_{\mathbb{D}} a(w)B_z(w)dA(w)\Big|\leq C. \tag{2.10}$$

Proof of Theorem 2.1. Let us consider the necessity of the condition (2.2). Assuming $T_a : A^p \to A^p$ bounded, Lemma 2.3 and repeated integrations by parts imply

$$\begin{aligned} C \;\geq\; & \Big|\int_{\mathbb{D}} a(w)B_z(w)dA(w)\Big| = \Big|\int_0^{2\pi}\int_0^1 a(\varrho)B_z(\varrho e^{i\theta})\varrho d\varrho d\theta\Big| \\ = \; & \Big|\int_0^{2\pi}\Big(\sum_{k=1}^{n}(-1)^{k+1}\Big[I_a^{(k)}(\varrho)\frac{\partial^{k-1}}{\partial\varrho^{k-1}}\big(\varrho B_z(\varrho e^{i\theta})\big)\Big]_{\varrho=0}^{\varrho=1} \\ & + (-1)^n\int_0^1 I_a^{(n)}(\varrho)\frac{\partial^n}{\partial\varrho^n}\big(\varrho B_z(\varrho e^{i\theta})\big)d\varrho\Big)d\theta\Big| \end{aligned} \tag{2.11}$$

Since $a\in L^1(\mathbb{D},dA)$ and it is constant with respect to θ, the function a restricted to the unit interval actually belongs to $L^1(0,1)$ with respect to the one dimensional measure. Hence,

$$\lim_{\varrho\to 1}\int_{\varrho}^{1} a(s)ds = 0 \tag{2.12}$$

and by induction $\lim_{\varrho\to 1} I_a^{(k)}(\varrho) = 0$ for every k. This implies that the substitution $\varrho = 1$ on the last line of (2.11) is null, since for any fixed z the expression

$\partial^k(\varrho B_z(\varrho))/\partial\varrho^k$ is even bounded. Moreover, the substitution $\varrho = 0$ is bounded by a constant times the $L^1(0,1)$-norm of a, hence, the same applies to the whole term

$$\Big[I_a^{(k)}(\varrho) \frac{\partial^{k-1}}{\partial \varrho^{k-1}} \big(\varrho B_z(\varrho e^{i\theta}) \big) \Big]_{\varrho=0}^{\varrho=1} \tag{2.13}$$

Concerning the last term of (2.11), its modulus can be bounded from below, using the positivity of $I_a^{(n)}$ and $r \geq 1/2$, and (2.4), as follows:

$$\int_0^1 I_a^{(n)}(\varrho) \int_0^{2\pi} \frac{\partial^n}{\partial\varrho^n} \big(\varrho B_z(\varrho e^{i\theta}) \big) d\theta d\varrho \geq C_n \int_0^1 I_a^{(n)}(\varrho) \frac{r^{n-1}(1-r^2)^2}{(1-r\varrho)^{n+3}} d\varrho$$

$$\geq C_n'(1-r)^2 \int_r^1 I_a^{(n)}(\varrho) \frac{1}{(1-r\varrho)^{n+3}} d\varrho$$

$$\geq \frac{C_n''}{(1-r)^{n+1}} \int_r^1 I_a^{(n)}(\varrho) d\varrho = \frac{C_n'' I_a^{(n+1)}}{(1-r)^{n+1}}. \tag{2.14}$$

This, together with (2.11) and the boundedness of (2.13), imply the result that the condition (2.2) is necessary for the boundedness of the Toeplitz operator.

The sufficiency part follows using the method of [17], Theorem 3.1. Some details are however different, so we present the proof. Let $n \in \mathbb{N}$ be as in the assumption, and let f be an arbitrary polynomial and $z \in \mathbb{D}$. Repeated integrations by parts yields

$$\begin{aligned} T_a f(z) &= \int_0^{2\pi} \int_0^1 a(\varrho e^{i\theta}) \frac{f(\varrho e^{i\theta})}{(1 - z\varrho e^{-i\theta})^2} \varrho d\varrho d\theta \\ &= \int_0^{2\pi} \Bigg(\sum_{k=1}^n (-1)^{k+1} \Big[I_a^{(k)}(\varrho) \frac{\partial^{k-1}}{\partial\varrho^{k-1}} \frac{\varrho f(\varrho e^{i\theta})}{(1 - z\varrho e^{-i\theta})^2} \Big]_{\varrho=0}^{\varrho=1} \\ &\quad + (-1)^n \int_0^1 I_a^{(n)}(\varrho) \frac{\partial^n}{\partial\varrho^n} \frac{\varrho f(\varrho e^{i\theta})}{(1 - z\varrho e^{-i\theta})^2} d\varrho \Bigg) d\theta. \end{aligned} \tag{2.15}$$

Since f is a polynomial, we deduce as around (2.12) that the substitution terms with $\varrho = 1$ vanish and those with $\varrho = 0$ are bounded by the L^1-norm of a times $\|f\|_p$. Moreover, it is plain that

$$\Big| \int_0^{2\pi} \int_0^{1/2} I_a^{(n)}(\varrho) \frac{\partial^n}{\partial\varrho^n} \frac{\varrho f(\varrho e^{i\theta})}{(1 - z\varrho e^{-i\theta})^2} d\varrho d\theta \Big|$$

can be bounded by $C\|f\|_p$. Hence, we can add the Jacobian ϱ and bound the last term of (2.15) by $C\|f\|_p$ plus

$$
\begin{aligned}
&\int_0^{2\pi}\int_{1/2}^{1} |I_a^{(n)}(\varrho)| \Big| \frac{\partial^n}{\partial \varrho^n} \frac{f(\varrho e^{i\theta})}{(1-z\varrho e^{-i\theta})^2} \Big| \varrho d\varrho d\theta \\
&\le C \int_0^{2\pi}\int_{1/2}^{1} |I_a^{(n)}(\varrho)| \sum_{m=0}^{n} \Big| \frac{\partial^m f(\varrho e^{i\theta})}{\partial \varrho^m} \frac{\partial^{n-m}}{\partial \varrho^{n-m}} \frac{1}{(1-z\varrho e^{-i\theta})^2} \Big| \varrho d\varrho d\theta \\
&\le C \int_0^{2\pi}\int_{1/2}^{1} |I_a^{(n)}(\varrho)| (1-\varrho)^{-n} \\
&\quad \cdot \sum_{m=0}^{n} (1-\varrho)^m |f^{(m)}(\varrho e^{i\theta})| \frac{(1-\varrho)^{n-m}}{|1-z\varrho e^{-i\theta}|^{2+n-m}} \varrho d\varrho d\theta \\
&\le C' \int_0^{2\pi}\int_0^1 \sum_{m=0}^{n} (1-\varrho)^m |f^{(m)}(\varrho e^{i\theta})| \frac{1}{|1-z\varrho e^{-i\theta}|^2} \varrho d\varrho d\theta, \qquad (2.16)
\end{aligned}
$$

where (2.2) and the identity $\partial^m f(\varrho e^{i\theta})/\partial\varrho^m = f^{(m)}(\varrho e^{i\theta})e^{im\theta}$ were used. The rest goes as in [21]: the last expression of (2.16) is the maximal Bergman projection of the $L^p(dA)$-function

$$\sum_{m=0}^{n} (1-|z|)^m |f^{(m)}(z)|.$$

As a conclusion, the $L^p(dA)$-norm of (2.16) is bounded by a constant times $\|f\|_p$, proving that $T_a : A^p \to A^p$ is bounded, since polynomials form a dense subspace of A^p. □

3. Toeplitz operators as multipliers.

In case $p = 2$ and radial symbols it was proven in [8] that the Toeplitz operator T_a is unitarily equivalent to the ℓ^2-multiplier operator. We provide here a partial generalization for the case $p \neq 2$, though a complete analogue is not possible due to the fact that monomials do not form an unconditional Schauder basis in this case. However, it is still possible to decompose the weighted Bergman space into finite dimensional blocks such that the space is the ℓ^p-sum of the blocks, with some consequences to the boundedness and compactness propertier of T_a (see Theorem 3.3). This approach is due to the first named author in a series of papers, see [13, 14, 15]. Moreover, our methods work in Bergman spaces with quite general radial weights: we use the weighted norms $\|\cdot\|_{p,\mu}$ as defined in (1.3).

We start with some easy observations. For all k, the quantity

$$\gamma_k = 2\int_0^1 (k+1) r^{2k+1} a(r) dr, \tag{3.1}$$

is well-defined, since $a(r) \in L^1(0,1)$ by our basic assumptions (see Section 1).

Proposition 3.1. *Let $1 < p < \infty$. If the Toeplitz operator $T_a : A^p_\mu \to A^p_\mu$ is bounded, then it is a coefficient multiplier, i.e.,*

$$(T_a f)(z) = \sum_{n\in\mathbb{N}} \gamma_n f_n z^n \tag{3.2}$$

and moreover,

$$\sup_{k\in\mathbb{N}} |\gamma_k| \le \|T_a\| < \infty, \tag{3.3}$$

where $\|T_a\|$ is the operator norm.

Proof. We first recall that the Taylor series

$$\frac{1}{(1-z\bar{w})^2} = \sum_{n=0}^{\infty} (n+1) z^n r^n e^{-in\theta}$$

of the Bergman kernel converges uniformly for z, $w = re^{i\theta}$ in compact subsets of $\mathbb{D}$. For $f(w) = \sum_n f_n r^n e^{-in\theta} \in A^p_\mu$ we thus get

$$\begin{aligned} T_a f(z) &= \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (k+1) z^k f_n \int_0^1 \int_0^{2\pi} r^{n+k} e^{-ik\theta} e^{in\theta} a(r) r d\theta dr \\ &= \sum_{n=0}^{\infty} f_n z^n \int_0^1 2(n+1) r^{2n+1} a(r) dr, \end{aligned}$$

i.e. (3.2) holds. Furthermore, for any k, define $h_k(z) = C_k z^k$, where $C_k > 0$ is chosen such that $\|h_k\|_{p,\mu} = 1$. Since $T_a h_k = \gamma_k h_k$, we obtain $|\gamma_k| \le \|T_a\|$ for all k. □

If f is a polynomial, then $T_a f$ is well-defined by the integral formula for all radial $a \in L^1(\mathbb{D})$. We remark that in this case (3.2) holds irrespectively whether T_a is a bounded operator.

An interesting consequence of the proposition is the following:

Theorem 3.2. *If $T_a : A^p_\mu \to A^p_\mu$ is bounded for some $1 < p < \infty$, then T_a is bounded $A^2_\mu \to A^2_\mu$.*

This will follow from (3.8), Theorem 3.3.a) and Proposition 4.1.(ii), below. In the unweighted case this already follows from [22], Corollary 6.1.2.: it was proven there that (3.3) is sufficient for the boundedness of $T_a : A^2 \to A^2$.

The main result Theorem 3.3 is based on a natural decomposition of A^p_μ into finite dimensional subspaces spanned by monomials of degrees in the intervals $\mathbb{N}_n := \mathbb{N} \cap]m_n, m_{n+1}]$, where $(m_n)_{n=1}^\infty$ is a positive strictly increasing sequence to be defined soon. For analytic functions $f(z) = \sum_{k\in\mathbb{N}} f_k z^k$ and $h(z) = \sum_{k\in\mathbb{N}} h_k z^k$ on $\mathbb{D}$ we define

$$Q_n f(z) = \sum_{k\in\mathbb{N}_n} f_k z^k, \tag{3.4}$$

$$M_p(f,r) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p} \quad \text{for } r \in [0,1[. \tag{3.5}$$

If f is analytic in a neighbourhood of the closed unit disc, say, a polynomial, the quantity $M_p(f,1)$ can be defined by putting $r = 1$ in (3.5). Moreover, for polynomials $h = \sum_k h_k z^k$ we can define

$$\mathcal{M}_p(h) = \sup_{\substack{M_p(f,1)\le 1\\ f \text{ polynomial}}} M_p(h * f, 1)\ , \quad \text{where} \tag{3.6}$$

$$h * f = \sum_k h_k f_k z^k. \tag{3.7}$$

Finally, for all n let τ_n be the polynomial

$$\tau_n(z) = \sum_{k\in\mathbb{N}_n} \gamma_k z^k, \tag{3.8}$$

So, if we denote by $A^{(n)}$ the finite dimensional space of the polynomials $f = \sum_{k\in\mathbb{N}_n} f_k z^k$, the mapping Q_n is the canonical projection, say, from A^p_μ onto $A^{(n)}$. Moreover, if $f \in A^{(n)}$, then $T_a f = T_a^{(n)} f = \tau_n * f$; see (1.2) and the remark after Proposition 3.1.

Theorem 3.3. *The following statements hold true:*
a) $T_a : A^p_\mu \to A^p_\mu$ *is bounded, if and only if*

$$\sup_{n\in\mathbb{N}} \mathcal{M}_p(\tau_n) < \infty. \tag{3.9}$$

Moreover, if T_a *is bounded, then its operator norm satisfies for some constant* $C > 0$

$$\frac{1}{C}\|T_a\| \le \sup_{n\in\mathbb{N}} \mathcal{M}_p(\tau_n) \le C\|T_a\|.$$

b) $T_a : A^p_\mu \to A^p_\mu$ *is compact, if and only if*

$$\lim_{n\to\infty} \mathcal{M}_p(\tau_n) = 0. \tag{3.10}$$

Proof of Theorem 1.1. Working out the definitions of $\mathcal{M}_p$ and τ_n one finds that the operator norm of the multiplier $T_a^{(n)}$ on H^p is equal to $\mathcal{M}_p(\tau_n)$. So Theorem 3.3 is a reformulation of Theorem 1.1. □

The proof of Theorem 3.3 will be given in Section 4 . The rest of this section is devoted to the study of the finite dimensional decomposition of the weighted

Bergman space. First we need to specify the numerical sequence appearing in the above definitions.

Definition 3.4. We fix a number $b > 2$ and set $m_1 = 0$, and assume that for some $n \geq 1$ the numbers $m_1 < m_2 < m_3 < \ldots m_n$ and $0 =: s_0 < s_1 < s_2 < s_3 < \ldots s_{n-1} < 1$ have been chosen. Let s_n be such that

$$\int_0^{s_n} r^{m_n p} r d\mu = b \int_{s_n}^1 r^{m_n p} r d\mu \tag{3.11}$$

(this is possible by what was assumed on $d\mu$, see (1.3)), and then find $m_{n+1} > m_n$ such that

$$\int_0^{s_n} r^{m_{n+1} p} r d\mu = \int_{s_n}^1 r^{m_{n+1} p} r d\mu. \tag{3.12}$$

We denote

$$\omega_n := \Big(\int_0^{s_n} \Big(\frac{r}{s_n}\Big)^{m_n p} r d\mu + \int_{s_n}^1 \Big(\frac{r}{s_n}\Big)^{m_{n+1} p} r d\mu \Big)^{1/p} \tag{3.13}$$

The following result shows that the weighted Bergman space is an ℓ^p-sum of finite dimensional subspaces; this is a substitute for the property that the monomials form an orthogonal basis of A^2.

Theorem 3.5. *If every space $A^{(n)} = Q_n(A^p_\mu)$, $n \in \mathbb{N}$, is endowed with the norm*

$$\|g\|_{p,\mu,n} := \omega_n M_p(g, s_n) = \omega_n \Big(\frac{1}{2\pi} \int_0^{2\pi} |g(s_n e^{i\theta})|^p d\theta \Big)^{1/p}, \tag{3.14}$$

then A^p_μ is the ℓ^p-sum of the spaces $A^{(n)}$, more precisely, there are constants C_1, $C_2 > 0$ such that

$$C_1 \|f\|_{p,\mu} \leq \Big(\sum_{n=1}^{\infty} \|Q_n f\|^p_{p,\mu,n} \Big)^{1/p} \leq C_2 \|f\|_{p,\mu} \tag{3.15}$$

for all $f \in A^p_\mu$.

For the proof we first provide a collection of lemmas. The notation $[s]$ denotes the largest integer not larger than $s \in \mathbb{R}$.

Lemma 3.6. *If $m < n$, $0 < r \leq s$ and $g = \sum_{k=[m]+1}^{[n]} g_k z^k$, then*

$$\Big(\frac{r}{s}\Big)^n M_p(g, s) \leq M_p(g, r) \leq \Big(\frac{r}{s}\Big)^m M_p(g, s). \tag{3.16}$$

This follows from the monotonicity of $M_p(f, r)$ as a function of r.

Lemma 3.7. *For all* $g \in A^{(n)} = Q_n(A^p_\mu)$,

$$\int_0^1 M_p^p(g,r)rd\mu \le \omega_n^p M_p^p(g,s_n) \le b\int_0^1 M_p^p(g,r)rd\mu \tag{3.17}$$

Proof. One can estimate, using Lemma 3.6 and then (3.11), (3.12),

$$\begin{aligned}\int_0^1 M_p^p(g,r)rd\mu &\le M_p^p(g,s_n)\int_0^{s_n}\left(\frac{r}{s_n}\right)^{m_n p} rd\mu + M_p^p(g,s_n)\int_{s_n}^1\left(\frac{r}{s_n}\right)^{m_{n+1}p} rd\mu \\ &\le M_p^p(g,s_n)b\int_{s_n}^1\left(\frac{r}{s_n}\right)^{m_n p} rd\mu + M_p^p(g,s_n)\int_0^{s_n}\left(\frac{r}{s_n}\right)^{m_{n+1}p} rd\mu \\ &\le b\int_{s_n}^1 M_p^p(g,r)rd\mu + \int_0^{s_n} M_p^p(g,r)rd\mu \\ &\le b\int_0^1 M_p^p(g,r)rd\mu. \quad \square\end{aligned}$$

Lemma 3.8. *If* $g \in A^{(n)}$, *we have*

$$\int_0^1 M_p^p(g,r)rd\mu \le \frac{2b}{b-2}\int_{s_{n-1}}^{s_{n+1}} M_p^p(g,r)rd\mu. \tag{3.18}$$

Proof. Again, by (3.11) and (3.12)

$$\int_0^{s_{n-1}} M_p^p(g,r)rd\mu \le M_p^p(g,s_{n-1})\int_0^{s_{n-1}}\left(\frac{r}{s_{n-1}}\right)^{m_n p} rd\mu$$

$$\le M_p^p(g,s_{n-1})\int_{s_{n-1}}^1\left(\frac{r}{s_{n-1}}\right)^{m_n p} rd\mu \le \int_{s_{n-1}}^1 M_p^p(g,r)rd\mu,$$

hence,

$$\frac{1}{2}\int_0^1 M_p^p(g,r)rd\mu \le \int_{s_{n-1}}^1 M_p^p(g,r)rd\mu. \tag{3.19}$$

Moreover,

$$\int_{s_{n+1}}^1 M_p^p(g,r)rd\mu \le M_p^p(g,s_{n+1})\int_{s_{n+1}}^1\left(\frac{r}{s_{n+1}}\right)^{m_{n+1}p} rd\mu$$

$$\leq \frac{1}{b}\int_0^{s_{n+1}} M_p^p(g,r)rd\mu \leq \frac{1}{b}\int_0^1 M_p^p(g,r)rd\mu. \tag{3.20}$$

As a consequence,

$$\int_{s_{n-1}}^{s_{n+1}} M_p^p(g,r)rd\mu = \int_{s_{n-1}}^1 M_p^p(g,r)rd\mu - \int_{s_{n+1}}^1 M_p^p(g,r)rd\mu$$

$$\geq \Big(\frac{1}{2}-\frac{1}{b}\Big)\int_0^1 M_p^p(g,r)rd\mu. \ \square$$

We proceed with the proof of Theorem 3.5. First, using Lemma 3.8 and $M_p(Q_nf;r) \leq C_oM_p(f;r)$ for some constant $C_0 > 0$ (see [24]),

$$\int_0^1 M_p^p(f,r)rd\mu \leq \sum_{n=0}^\infty \int_0^1 M_p^p(Q_nf,r)rd\mu$$

$$\leq \sum_{n=0}^\infty \frac{2b}{b-2}\int_{s_{n-1}}^{s_{n+1}} M_p^p(Q_nf,r)rd\mu$$

$$\leq \sum_{n=0}^\infty \frac{2Cb}{b-2}\int_{s_{n-1}}^{s_{n+1}} M_p^p(f,r)rd\mu \leq \frac{4Cb}{b-2}\int_0^1 M_p^p(f,r)rd\mu \tag{3.21}$$

Use of Lemmas 3.7 and 3.8 yields:

$$\int_{s_{n-1}}^{s_{n+1}} M_p^p(Q_nf,r)rd\mu \leq M_p^p(Q_nf,s_n)\omega_n^p \leq \frac{2b^2}{b-2}\int_{s_{n-1}}^{s_{n+1}} M_p^p(Q_nf,r)rd\mu, \tag{3.22}$$

so combining this with (3.21) proves the theorem with $C_1 = ((b-2)/(2b))^{1/p}$ and $C_2 = ((4C_0b^2)/(b-2))^{1/p}$, C_0 as above. $\square$

4. Proof of Theorem 3.3

We consider the boundedness statements. Assume (3.9) holds, and let $f \in A^p_\mu$ first be a polynomial. Since T_a is a multiplier, $Q_nT_af = T_aQ_nf = \tau_n * Q_nf$, by the remark just before Theorem 3.3.

Put $F_n(z) = (Q_nf)(s_nz)$ for $z \in \mathbb{D}$. Then $F_n \in A^{(n)}$ and we obtain

$$M_p^p(Q_nT_af,s_n) = M_p^p(\tau_n * F_n,1)$$
$$\leq \mathcal{M}_p^p(\tau_n)M_p^p(F_n,1) = \mathcal{M}_p^p(\tau_n)M_p^p(Q_nf,s_n),$$

hence,

$$\|Q_nT_af\|_{p,\mu,n} \leq \Big(\sup_l \mathcal{M}_p(\tau_l)\Big)\|Q_nf\|_{p,\mu,n},$$

for all n, hence Theorem 3.5 yields

$$\|T_a f\|_{p,\mu} \le C\Big(\sup_{n\in\mathbb{N}} \mathcal{M}_p(\tau_n)\Big)\|f\|_{p,\mu}$$

for some constant C. This bound also holds for general $f \in A^p_\mu$ due to the density of polynomials in the Bergman space.

Assume next that the Toeplitz operator is bounded. For all n, let h_n be an analytic function such that $M_p(h_n,1) = 1$ and $\mathcal{M}_p(\tau_n) = M_p(\tau_n * h_n, 1)$. Set

$$\tilde{h}_n = Q_n h_n \in A^{(n)}.$$

Then we have $M_p(\tau_n * h_n, 1) = M_p(\tau_n * \tilde{h}_n, 1)$, and $M_p(\tilde{h}_n, 1) \le C$ for a constant C. Define

$$f(z) = \omega_n^{-1}\tilde{h}_n(s_n^{-1}z) \in A^{(n)} \ \ , \ z \in \mathbb{D}.$$

Then

$$\|f\|_{p,\mu,n} = M_p(\tilde{h}_n, 1) \le C \ \ , \ \|T_a f\|_{p,\mu,n} = M_p(\tau_n * \tilde{h}_n, 1) = \mathcal{M}_p(\tau_n) \tag{4.1}$$

Theorem 3.5 yields

$$\|f\|_{p,\mu} \le C' \ \ , \ \ \|T_a f\|_{p,\mu} \ge \mathcal{M}_p(\tau_n)/C',$$

for a constant $C' > 0$, hence, $\|T_a\| \ge C''\mathcal{M}_p(\tau_n)$ for a constant $C'' > 0$, for all n. The proof of the case $a)$ is complete.

As for compactness, if (3.10) holds, T_a can be approximated in the operator norm by finite rank operators (use Theorem 3.5), hence, it must be compact. Conversely, if (3.10) fails, we find a constant $C > 0$ and an increasing subsequence $(n_l)_{l\in\mathbb{N}}$ such that $\mathcal{M}_p(\tau_{n_l}) \ge C$. For each l we thus find a polynomial $f_l \in A^{(n_l)}$ such that

$$M_p(\tau_{n_l} * f_l, 1) \ge C M_p(f_l, 1)/2. \tag{4.2}$$

Normalizing the polynomials such that $\|f_l\|_{p,\mu} = 1$ we find that the unit ball of A^p_μ is not mapped into a precompact set by T_a, since $T_a f_l = \tau_{n_l} * f_l \in A^{(n_l)}$ and $\|T_a f_l\|_{p,\mu}$ is bounded from below by a positive constant, by Theorem 3.5 and (4.2). □

In view of the above results it is useful to consider the functional $\mathcal{M}_p$ in some more detail. We remark that the following statement (ii) completes the proof of Theorem 3.2.

Proposition 4.1. *Let $1/p + 1/q = 1$ and g be a polynomial.*
(i) We have $\mathcal{M}_p(g) \le M_q(g,1)$. In particular, if $\sup_{n\in\mathbb{N}} M_q(\tau_n,1) < \infty$, then $T_a : A^p_\mu \to A^p_\mu$ is bounded.
(ii) If $p = q = 2$, then $\mathcal{M}_p(g) = \sup_k |g_k|$, where $g = \sum_k g_k z^k$
(iii) If $p = 1$, then $\mathcal{M}_p(g) = M_1(g,1)$.

Proof. As for (i),

$$\mathcal{M}_p^p(g) = \sup_{M_p(f,1)\le 1} \frac{1}{(2\pi)^2}\int_0^{2\pi}\Big|\int_0^{2\pi} f(e^{i(\theta-\varphi)})g(e^{i\varphi})\Big|^p d\varphi$$

$$
\begin{aligned}
&\le \sup_{M_p(f,1)\le 1} \frac{1}{2\pi}\int_0^{2\pi} M_p^p(f,1)M_q^p(g,1)d\varphi \\
&= \sup_{M_p(f,1)\le 1} M_p^p(f,1)M_q^p(g,1) \\
&\le M_q^p(g,1),
\end{aligned}
$$

and for (ii),

$$\mathcal{M}_2^2(g) = \sup_{\sum |f_n|^2 \le 1} \sum_n |g_n|^2 |f_n|^2 = \sup_{n\in\mathbb{N}} |g_n|^2,$$

where $f = \sum_n f_n z^n$.

For the proof of (iii), we have

$$\mathcal{M}_1(g) = \sup_{M_1(f,1)\le 1} M_1(f * g, 1) \le M_1(g,1).$$

Moreover, defining for each $0 < r < 1$ the bounded analytic function f_r as

$$f_r(z) = \frac{1-|z|^2 r^2}{|1-zr|^2}\ ,\ z \in \mathbb{D}, \tag{4.3}$$

we have by the properties of the Poisson kernel $M_1(f_r, 1) = 1$, and

$$\sup_r M_1(f_r * g, 1) = \sup_r \frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|\, d\theta = M_1(g,1).\square \tag{4.4}$$

Example 4.2. We calculate the indices s_n, m_n for $d\mu = dr$, $b = 3$. Here $\int_0^s r^m r d\mu = s^{m+2}/(m+2)$, $\int_s^1 r^m r d\mu = (1-s^{m+2})/(m+2)$. Hence, (3.11) means

$$s_n^{m_n p+2} = 3(1 - s_n^{m_n p+2}),$$

and we obtain

$$s_n = \left(\frac{3}{4}\right)^{1/(m_n p+2)}.$$

Also (3.12) leads to

$$s_n^{m_{n+1}p+2} = 1 - s_n^{m_{n+1}p+2},$$

hence,

$$s_n^{m_{n+1}p+2} = \frac{1}{2} \quad \text{or} \quad m_{n+1}p + 2 = \frac{\log(1/2)}{\log(s_n)} = \frac{\log(1/2)}{\log(3/4)}(m_n p + 2). \tag{4.5}$$

From these equalities we find the indices m_n and s_n:

$$m_n = \frac{2}{p}(a^{n-1}-1)\ ,\ s_n = \left(\frac{3}{4}\right)^{a^{-n+1}/2} \quad \text{with} \quad a := \frac{\log 2}{\log(4/3)} > 1.$$

References

[1] A. Alexandrov and G. Rozenblum, Finite rank Toeplitz operators: some extensions of D. Luecking's theorem. J. Funct. Anal. 256 (2009), no. 7, 2291–2303.
[2] S. Axler, Z. Cuckovic, and N. V. Rao, Commutants of analytic Toeplitz operators on the Bergman space. Proc. Amer. Math. Soc. 128 (2000), no. 7, 1951–1953.
[3] S. Axler, D. Zheng, Compact operators via the Berezin transform. Indiana Univ. Math. J. 47 (1998), 387–400.
[4] B. R. Choe and K. Nam, Berezin transform and Toeplitz operators on harmonic Bergman spaces. J. Funct. Anal. 257 (2009), no. 10, 3135–3166.
[5] A. Böttcher and S. M. Grudsky, Spectral properties of banded Toeplitz matrices. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.
[6] G. F. Cao, Toeplitz operators on Bergman spaces of multiply connected domains. (Chinese. English, Chinese summary) Chinese Ann. Math. Ser. A 29 (2008), no. 5, 671–678.
[7] M. Englis, Toeplitz operators and weighted Bergman kernels. J. Funct. Anal. 255 (2008), no. 6, 1419–1457.
[8] S. Grudsky, A. Karapetyants, and N. Vasilevski, Toeplitz operators on the unit ball in $\mathbb{C}^n$ with radial symbols. J. Operator Th. 49 (2003), 325–346.
[9] S. Grudsky, A. Karapetyants, and N. Vasilevski, Dynamics of properties of Toeplitz operators with radial symbols. Integr. Eq. Operator Th. 50 (2004), 217–253.
[10] A. Harutyunyan, W. Lusky, Toeplitz operators on weighted spaces of holomorphic functions, Math. Scan 103 (2008), 40–52.
[11] A. Harutyunyan, W. Lusky, on L_1-subspaces of holomorphic functions, Studia Math. 198 (2010), no. 2, 157-175.
[12] D. H. Luecking, Trace ideal criteria for Toeplitz operators. J. Funct. Anal. 73 (1987), no. 2, 345–368.
[13] W. Lusky, On weighted spaces of harmonic and holomorphic functions, J. London Math. Soc. 51 (1995), no. 2, 309–320.
[14] W. Lusky, On generalized Bergman spaces, Studia Math. 119 (1996), no. 1, 77–95.
[15] W. Lusky, On the isomorphism classes of weighted spaces of harmonic and holomorphic functions, Studia Math. 175 (2006), no. 1, 19–45.
[16] G. McDonald, and C. Sundberg, Toeplitz operators on the disk, Indiana Univ. Math. J., 28 (1979), 595–611.
[17] A. Perälä, J. Taskinen, and J. A. Virtanen, Toeplitz operators with distributional symbols on Bergman spaces, Proc. Edinb. Math. Soc, to appear.
[18] A. Perälä, J. Taskinen, and J. A. Virtanen, manuscript.
[19] W. Rudin, Real and complex analysis.
[20] K. Stroethoff, Compact Toeplitz operators on weighted harmonic Bergman spaces. J. Austral. Math. Soc. Ser. A 64 (1998), no. 1, 136–148.
[21] J. Taskinen and J. A. Virtanen, Toeplitz operators on Bergman spaces with locally integrable symbols, Rev. Mat. Iberoamericana, to appear.
[22] N. Vasilevski, Commutative algebras of Toeplitz operators on the Bergman space, Operator Theory: Advances and Applications, Vol. 185, Birkhäuser Verlag, 2008.
[23] U. Venugopalkrishna, Fredholm operators associated with strongly pseudoconvex domains in C^n, J. Functional Analysis 9 (1972) 349–373.
[24] P. Wojtasczcyk, Banach spaces for analysts, Cambridge University Press, 1991.
[25] Z. Wu, R. Zhao, and N. Zorboska, Toeplitz operators on analytic Besov spaces. Integral Equations Operator Theory 60 (2008), no. 3, 435–449.
[26] K. Zhu, VMO, ESV, and Toeplitz operators on the Bergman space, Trans. Amer. Math. Soc. 302 (1987), no. 2, 617–646
[27] K. Zhu, Multipliers of BMO in the Bergman metric with applications to Toeplitz operators. J. Funct. Anal. 87 (1989), no. 1, 31–50.

[28] K. Zhu, Operator Theory in Function Spaces. Marcel Dekker, 1990.
[29] N. Zorboska Toeplitz operators with BMO symbols and the Berezin transform. Int. J. Math. Math. Sci. 2003, no. 46, 2929–2945..

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