TOEPLITZ OPERATORS ON DIRICHLET-BESOV SPACES

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Abstract. We study Toeplitz operators T_a on the Besov spaces $\mathcal{B}^p$ in the case of the open unit disk $\mathbb{D}$ and $1 < p < \infty$. We prove that a symbol a satisfying a weak Lipschitz type condition induces a bounded operator T_a. Such symbols do not need to be bounded functions or have continuous extensions to the boundary of $\mathbb{D}$. We discuss the problem of the existence of nontrivial compact Toeplitz operators and also consider Fredholm properties and prove an index formula.

1. Introduction, main theorems.

The purpose of our work is to present new, weak sufficient conditions for boundedness of Toeplitz operators T_a on analytic Dirichlet and Besov spaces of the open unit disk. This main result is contained in Theorem 1.1. We also give examples of such operators having symbols which are not bounded or even L^1 in any neighbourhood of the unit circle. We also consider Fredholm theory for Toeplitz operators on analytic Besov spaces.

Toeplitz operators constitute one of the most important classes of non-selfadjoint integral operators on function spaces, and they are defined in general as follows. Let X be a Banach function space on some domain Ω, and let $Y \subset X$ be a closed subspace. Given a suitable function a on Ω, denote the pointwise multiplier operator by $M_a : Y \to X$, and let P be a bounded projection operator from X onto Y. Then, the Toeplitz operator with symbol a is defined by $T_a f = PM_a f$ for $f \in Y$. In practise, M_a can be allowed to map to a larger space than X, as long as T_a still makes sense. A typical choice for X is an L^p-space on

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the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane or on the boundary $\{|z| = 1\}$, and for Y the subspace of analytic functions, i.e., a Bergman or Hardy space.

In this paper we study a more complicated case, where X is an L^p-Sobolev space on the unit disk $\mathbb{D}$, $1 < p < \infty$, and the corresponding subspace of analytic functions Y is the Besov space $\mathcal{B}^p$. Let us start from the definition of the latter. An analytic $f : \mathbb{D} \to \mathbb{C}$ belongs to $\mathcal{B}^p$, if $f(0) = 0$ and

$$\|f\|_{\mathcal{B}^p}^p := (p-1) \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty, \tag{1.1}$$

where $dA(z) = \pi^{-1} dxdy$ is the normalised area measure, $z = x+iy \in \mathbb{D}$. The space $\mathcal{B}^2$ is also known as the Dirichlet space $\mathcal{D}$, and the corresponding norm is the square root of the area of $f(\mathbb{D})$ taking into account the multiplicities. Note that we accept the condition $f(0) = 0$ only for technical convenience, although this choice also has the drawback that the space $\mathcal{B}^p$ loses its conformal invariance. If one wishes to include analytic functions with $f(0) \neq 0$ in the definition of the Besov space, one has to add a term like $|f(0)|$ to the right hand side of (1.1). We leave the resulting changes completely to the reader.

The superspace X is easiest to describe in the case $p = 2$: it is the Sobolev space $W^{1,2}$ consisting of those $f \in L^2$ for which all first order partial derivatives in the sense of distributions also belong to L^2; see e.g. [1] or [13]. (Here and in the following we always mean by L^p the Lebesgue space with respect to the normalised area measure dA.) Instead of the conventional norm of $W^{1,2}$ we use the equivalent norm coming from the inner product

$$\langle f, g \rangle_{W^{1,2}} = \int_{\mathbb{D}} f dA \int_{\mathbb{D}} \bar{g} dA + \int_{\mathbb{D}} \left(\partial f \overline{\partial g} + \bar{\partial} f \overline{\bar{\partial} g} \right) dA, \tag{1.2}$$

where $\partial = (\partial_x - i\partial_y)/2$ and $\bar{\partial} = (\partial_x + i\partial_y)/2$ are the Wirtinger derivatives; see for example Theorem 1.1.16 of [13]. Recall that for an analytic f we have $\partial f = f'$ and $\bar{\partial} f = 0$.

In case $1 \leq p < \infty$, $p \neq 2$, we need weighted Sobolev spaces $W_w^{1,p}$. The definition is similar to the case $p = 2$: a function $f \in L^1$ belongs to $W_w^{1,p}$, if its first order derivatives belong to the weighted L^p-space defined by the weighted measure $(1 - |z|^2)^{p-2} dA(z)$. The norm of $W_w^{1,p}$ is defined by

$$\|f\|_{W_w^{1,p}}^p = \Big| \int_{\mathbb{D}} f dA \Big|^p + (p-1) \int_{\mathbb{D}} \left(|\partial f(z)|^p + |\bar{\partial} f(z)|^p \right) (1 - |z|^2)^{p-2} dA(z)$$

All spaces $W^{1,p}_w$ are complete, hence, $W^{1,2}$ is a Hilbert space.

When $1 < p < \infty$, the spaces $\mathcal{B}^p$ are closed subspaces of the respective $W^{1,p}_w$ spaces, and the orthogonal projection $Q : W^{1,2} \to \mathcal{B}^2$ can be written as the integral

$$Qf(z) = \int_{\mathbb{D}} \frac{z \partial f(w)}{1 - z\bar{w}} \, dA(w). \tag{1.3}$$

This is known to be a bounded projection $W^{1,p}_w \to \mathcal{B}^p$ for all $1 < p < \infty$.

The formula for Q suggests the following definition for the Besov-Toeplitz operators:

$$T_a f(z) = \int_{\mathbb{D}} \frac{z \partial (af)(w)}{1 - z\bar{w}} \, dA(w). \tag{1.4}$$

This is a well-defined, bounded operator on $\mathcal{B}^p$ if, for instance, a and ∂a are both in L^∞, see [6, 12].

The existing literature on the present topic includes the paper of Cao [6] and the more recent works of Lee [9, 10] and Lee-Zhu [12] on the case of the Dirichlet space, i.e., $p = 2$. These papers investigate Toeplitz operators generated by the Sobolev-Dirichlet projection Q of (1.3), and the operator symbols belong to the Sobolev space $W^{1,\infty}$. Note that $W^{1,\infty}$ agrees almost everywhere with functions that are Lipschitz continuous on the closed unit disk, see Theorem 4.1 of [8]; in particular, the symbols are bounded functions. We also want to mention the paper of Rochberg and Wu [17], where a Dirichlet-Toeplitz operators are defined in a different manner.

There exists another approach to Toeplitz operators on Besov spaces: using the notation of the beginning of Section 2, we denote by L^p_α the L^p-space with respect to the area measure $(1-|z|^2)^\alpha dA(z)$ and by P the classical Bergman projection. As is known, $P : L^p_\alpha \to L^p_\alpha$ is bounded whenever $p > \alpha + 1$. When $\alpha \leq -1$, the spaces of analytic functions with respect to the measure $(1 - |z|^2)^\alpha dA(z)$ degenerate. However, by passing to derivatives, the case $\alpha = -2$ can be identified with the Besov space $\mathcal{B}^p$, and the Bergman projection is bounded and onto $L^p_{-2} \to \mathcal{B}^p$. One can then also study more general operators P_β for suitable weight parameters β.

It makes sense to study Bergman-Toeplitz operators in the described setting, as is done in the paper by Wu, Zhao and Zorboska [21] and in the recent work of Tchoundja [20]. The drawback of this approach is the fact that $\mathcal{B}^p$ is not a norm subspace of L^p_α, so the Bergman projection is not a projection, strictly speaking.

The setting of the present paper, a general p and T_a defined in terms of Q, does not seem to appear in the literature at all, in spite of the obvious inclusion $\mathcal{B}^p \subset W^{1,p}_w$. It is not so surprising that many of our results are new even for $p = 2$, and thus the recording of such results in this paper is well motivated.

We introduce the local Sobolev space $W^{1,1}_{\rm loc}$ consisting of measurable functions $a : \mathbb{D} \to \mathbb{C}$ such that a and the distributional derivatives ∂a and $\bar\partial a$ are locally integrable functions. By the Sobolev embedding theorem, [1], Theorem 5.4, the elements of $W^{1,1}_{\rm loc}$ are locally L^2-functions, but they do not need to be continuous, since for example $|z|^{-\delta} \in W^{1,1}_{\rm loc}$ for all $0 < \delta < 1$. However, as a consequence of Lemma A.5.2 of [4], for instance, integrals over circular arcs and line segments of such functions make sense almost everywhere.

We aim to prove the following result, which contains a sufficient condition for the boundedness of T_a. We shall show in Remark 3.2 that there are unbounded symbols, which satisfy this condition and which cannot be continuously extended to the boundary of $\mathbb{D}$. We also remark that if the symbol a belongs to the Sobolev space $W^{1,\infty}$, then it satisfies the assumptions of Theorem 1.1. The condition (1.5) appears in the main theorem of [18], see also (2.3) below, and the other two conditions are differentiated versions of it.

Theorem 1.1. *Assume that $a \in W^{1,1}_{\rm loc}$ and that there is a constant $C > 0$ such that for all $0 \le r < 1$ and $\theta \in [0, 2\pi]$,*

$$\Big| \int\limits_r^s \int\limits_\theta^\phi a(\varrho e^{i\varphi}) \varrho \, d\varphi \, d\varrho \Big| \le C(1-r)^2 \ , \tag{1.5}$$

$$\Big| \int\limits_r^s \big(\varrho a(\varrho e^{i\theta}) - \varrho a(\varrho e^{i\phi}) \big) d\varrho \Big| \le C(1-r)^2 \ , \ \ and \tag{1.6}$$

$$\Big| \int\limits_\theta^\phi \big(r^2 a(re^{i\varphi}) - s^2 a(se^{i\varphi}) \big) d\varphi \Big| \le C(1-r)^2 \tag{1.7}$$

for all s with $r \le s \le (1+r)/2$ and all ϕ with $\theta \le \phi \le \theta + \pi(1-r)$. Then T_a is a well defined and bounded operator $\mathcal{B}^p \to \mathcal{B}^p$, $1 < p < \infty$.

It seems possible to formulate an estimate for the operator norm on T_a in terms of the infimum of all constants C appearing in (1.5)–(1.7). However, we skip the details, since this would require the checking of several steps including the proof of Theorem 2.3 of [18].

We also refrain from formulating a "little-o" theorem on compact Toeplitz operators. In fact, it is known that for symbols $a \in W^{1,\infty}$ there exist no nontrivial compact Toeplitz operators on $\mathcal{B}^2$, see [11]. In Section 4 we shall prove an analogue of this fact for any $p \in (1, \infty)$ and all compactly supported distributional symbols. It remains an open question whether there are any $W^{1,1}_{\rm loc}$-symbols that generate nontrivial compact Toeplitz operators, although this seems improbable.

2. Preliminaries on Bergman spaces.

Sections 2 and 3 are devoted to the proof of Theorem 1.1. The idea is to return the proof to the Bergman space setting and to use the results of [18]. In this section we review the Bergman space theory and the result of the citation, which we actually need in a slightly more general form.

For every $1 < p < \infty$ and $\alpha > -1$ we denote by L^p_α the weighted L^p-space, which is endowed with the norm

$$\|f\|^p_{p,\alpha} := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z), \tag{2.1}$$

where we note that $(\alpha+1)(1-|z|^2)^\alpha dA(z)$ is a probability measure on $\mathbb{D}$. The weighted Bergman space A^p_α is the closed subspace of L^p_α consisting of analytic functions. The Bergman projection P is the orthogonal projection of L^2 onto A^2, and it has the integral representation

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} dA(w).$$

It is also a bounded projection from L^p onto A^p, when $1 < p < \infty$, but much stronger results are known to hold true. Namely, the maximal Bergman projection $\mathbf{P}$ is a nonlinear operator defined by

$$\mathbf{P}f(z) = \int_{\mathbb{D}} \frac{|f(w)|}{|1 - z\bar{w}|^2} dA(w),$$

and it has the following boundedness property.

Lemma 2.1. *Given $1 < p < \infty$ and $\alpha > -1$ such that $p > \alpha + 1$, there exists a constant $C = C(p, \alpha)$ with*

$$\|\mathbf{P}f\|_{p,\alpha} \le C\|f\|_{p,\alpha} \tag{2.2}$$

for all $f \in L^p_\alpha$.

We shall apply this lemma to the case $\alpha = p - 2$. The lemma is proven in [23], which generalises the original work of Forelli and Rudin, [7]. It also follows by applying the Schur test (see Theorem

3.2.2 of [26]) to the kernel $K(z,w) := (1-|w|^2)^{-\alpha}|1-z\bar{w}|^{-2}$, measure $d\mu(w) := (1-|w|^2)^{\alpha}dA(w)$ and test function $h(w) := (1-|w|^2)^{-pq}$.

The standard definition of Toeplitz operators $T(a)$ in Bergman spaces is

$$T(a)f = P(af) = \int_{\mathbb{D}} \frac{a(w)f(w)}{(1-z\bar{w})^2} dA(w),$$

where $a : \mathbb{D} \to \mathbb{C}$ is at least measurable and the integral is assumed to converge for $f \in A^p_\alpha$. In general, $T_a \neq T(a)$. Since P is a bounded operator, $T(a)$ extends to a bounded operator $A^p_\alpha \to A^p_\alpha$ for $1 < p < \infty$ and $p-1 > \alpha > -1$, whenever a is a bounded function.

Remark 2.2. Concerning Toeplitz operators in weighted Bergman spaces A^p_α, one usually defines them using the orthogonal projection $P_\alpha : L^2_\alpha \to A^2_\alpha$,

$$P_\alpha f(z) = (\alpha+1)\int_{\mathbb{D}} \frac{f(w)(1-|w|^2)^\alpha dA(w)}{(1-z\overline{w})^{2+\alpha}},$$

$\mathbf{P}_\alpha$ and $T_\alpha(a)$. However, since our primary objects are Toeplitz operators on Besov spaces, we are forced to use the unweighted P in weighted Bergman spaces, which leads to some difficulties in the last section.

It is known and easy to see, that a does not need to be bounded in order to induce a bounded Toeplitz operator. In order to present our results in as general form as possible, we need to improve the main result of [18] for the boundedness of $T(a)$ in Bergman spaces.

Theorem 2.3. *Assume that $b : \mathbb{D} \to \mathbb{C}$ is a locally integrable function, and let $1 < p < \infty$, $\alpha > -1$ and $p > \alpha + 1$. Then $T(b) : A^p_\alpha \to A^p_\alpha$ is well defined and bounded, if there exists a constant $C > 0$ such that, for all $0 \le r < 1$, $\theta \in [0, 2\pi]$,*

$$\Big| \int_r^s \int_\theta^\phi b(\varrho e^{i\varphi})\varrho d\varphi d\varrho \Big| \le C(1-r)^2 \tag{2.3}$$

for all s, ϕ with $r \le s \le (1+r)/2$ and $\theta \le \phi \le \theta + \pi(1-r)$.

In the case the condition (2.3) holds, the operator $T(b)$ is defined by expressing the value of $T(b)f(z)$ as an absolutely convergent numerical series for every $z \in \mathbb{D}$. The definition is described in more detail in Remark 3.1.

Proof. In the case $\alpha = 0$ this result is Theorem 2.3 of [18]. To prove the present, more general version we notice that the formulas (3.13) and (3.6) of [18] show that

$$|T(b)f(z)| \le C \int_{\mathbb{D}} \frac{|g(w)|}{|1-z\bar{w}|^2} dA(w) = C\mathbf{P}g(z), \tag{2.4}$$

where

$$g(z) := |f(z)| + |f'(z)|(1-|z|^2) + |f''(z)|(1-|z|^2)^2. \tag{2.5}$$

As in [18], this leads to the proof of the theorem, since we can conclude from (2.4) and Lemma 2.1 that the modulus of the function $T(b)f$ is pointwise majorised by a function $F \in L^p_\alpha$ with $\|F\|_{p,\alpha} \le C\|f\|_{p,\alpha}$. To this end we only need the following observation: for all $f \in A^p_\alpha$ there exists a constant C such that

$$\begin{aligned} &\|f'(z)(1-|z|^2)\|_{p,\alpha} \le C\|f\|_{p,\alpha} \quad \text{and} \\ &\|f''(z)(1-|z|^2)^2\|_{p,\alpha} \le C\|f\|_{p,\alpha}. \end{aligned} \tag{2.6}$$

But this is well known: using the Bergman projection formula and differentiating under the integral sign we get for $f \in A^p \cap A^p_\alpha$

$$|f'(z)|(1-|z|^2) \le C \int_{\mathbb{D}} \frac{|f(w)|(1-|z|^2)}{|1-z\bar{w}|^3} dA(w)$$

$$\le C \int_{\mathbb{D}} \frac{|f(w)|}{|1-z\bar{w}|^2} dA(w) = C'\mathbf{P}f(z),$$

so that $|f'(z)|(1-|z|^2)$ is pointwise bounded by the function $C\mathbf{P}f(z)$, and this has a bounded L^p_α-norm, again by Lemma 2.1. The case of the second derivative is treated in the same way. Finally, the subspace $A^p \cap A^p_\alpha$ is dense in A^p_α. $\square$

At the end of this section we still recall the following lemma.

Lemma 2.4. *For every $1 < p < \infty$ we have $\mathcal{B}^p \subset A^p_{p-2}$. Moreover, the identity map $\mathbf{i} : \mathcal{B}^p \to A^p_{p-2}$ is compact.*

Proof. Obviously, $\partial : \mathcal{B}^p \to A^p_{p-2}$ is bounded. Also, the Volterra integration operator

$$\mathbf{V}g(z) = \int_0^z g(t)dt \tag{2.7}$$

is compact on A^p_{p-2}, which, for instance, is a special case of a result of Aleman and Siskakis [2]. Now, the claim follows since $\mathbf{i} = \mathbf{V}\partial$. $\square$

3. Proof of the main result.

Let us complete the proof of Theorem 1.1. The idea is to use an isomorphy between Besov and Bergman spaces and then apply Theorem 2.3. An analytic $f : \mathbb{D} \to \mathbb{C}$ vanishing at 0 belongs to $\mathcal{B}^p$ if and only if f' belongs to A^p_{p-2}. In fact, the operator $\partial : \mathcal{B}^p \to A^p_{p-2}$ is an isometric isomorphism, and its inverse is given by $\mathbf{V}$ defined in (2.7). It is crucial that $f(0) = 0$ for $\partial : \mathcal{B}^p \to A^p_{p-2}$ to be injective.

With this in mind we make the following formal calculation:

$$\begin{aligned} \partial T_a f(z) &= \int_{\mathbb{D}} \frac{d}{dz}\Big(\frac{z}{1-z\bar{w}}\Big)\partial(af)(w)dA(w) \\ &= \int_{\mathbb{D}} \frac{1}{(1-z\bar{w})^2}\big(f(w)\partial a(w) + a(w)\partial f(w)\big)dA(w) \\ &= T(\partial a)f(z) + T(a)f'(z). \end{aligned} \tag{3.1}$$

Hence,

$$T_a = \mathbf{V}\big(T(\partial a)\mathbf{i} + T(a)\partial\big). \tag{3.2}$$

As a conclusion, if we know that $T(\partial a)$ and $T(a)$ are bounded on A^p_{p-2}, the operator T_a will be bounded on $\mathcal{B}^p$. Moreover, the boundedness of $T(a) : A^p_{p-2} \to A^p_{p-2}$ follows directly from (1.5) and Theorem 2.3, cf. (2.3).

It remains to consider the boundedness of $T(\partial a)$ in A^p_{p-2}. Instead of directly proving this, we still introduce simplifying technical tricks. Let us denote by $\chi : \mathbb{D} \to \mathbb{R}$ the characteristic function, which vanishes in the set $\{|z| < 1/2\}$ and equals one in $\{1/2 \le |z| \le 1\}$. It is quite clear that $T(\partial a) : A^p_{p-2} \to A^p_{p-2}$ is bounded, if and only if $T(\chi\partial a) : A^p_{p-2} \to A^p_{p-2}$ is bounded. Moreover, the multiplier $f(z) \mapsto z^{-1}\chi(z)f(z)$ is a bounded operator $A^p_{p-2} \to L^2_{p-2}$, hence, in order to prove that $T(\partial a)$ is bounded on A^p_{p-2} it is enough to show that

$$T(z\chi\partial a) : A^p_{p-2} \to A^p_{p-2}$$

is bounded. Since the conditions (2.3) and (1.5)–(1.7) are written in polar coordinates, we recall that

$$\partial_r = \cos\theta\partial_x + \sin\theta\partial_y, \qquad \partial_\theta = -r\sin\theta\partial_x + r\cos\theta\partial_y,$$

hence, for $|z| \ge 1/2$,

$$\partial = \frac{e^{-i\theta}}{2}\Big(\partial_r - \frac{i}{r}\partial_\theta\Big) \tag{3.3}$$

So, integrating by parts with respect to ϱ, the integral (2.3) for $b(z) = \chi(z)z\partial a(z)$, for $r \geq 1/2$, becomes

$$
\begin{aligned}
&\int_r^s \int_\theta^\phi \varrho e^{i\varphi}(\partial a)(\varrho e^{i\varphi})\varrho d\varphi d\varrho \\
&= \frac{1}{2}\int_r^s \int_\theta^\phi \Big(\varrho^2(\partial_\varrho a)(\varrho e^{i\varphi}) - i\varrho(\partial_\varphi a)(\varrho e^{i\varphi})\Big) d\varphi d\varrho \\
&= \frac{1}{2}\int_\theta^\phi \left(\Big[\varrho^2 a(\varrho e^{i\varphi})\Big]_{\varrho=r}^{\varrho=s} - \int_r^s 2\varrho a(\varrho e^{i\varphi}) d\varrho\right) d\varphi \\
&\quad -\frac{i}{2}\int_r^s \Big[\varrho a(\varrho e^{i\varphi})\Big]_{\varphi=\theta}^{\varphi=\phi} d\varrho
\end{aligned} \tag{3.4}
$$

Obviously, the moduli of the first, second, and third integrals on the right hand side are bounded by $C(1-r)^2$ due to (1.7), (1.5), and (1.6), respectively. This yields (2.3) for $\chi(z)z\partial a$, and the desired boundedness follows. $\square$

Remark 3.1. It is worthwhile to describe the actual definition of T_a in detail. By the identity (3.2), it is enough to explain the definition in the case of Bergman space Toeplitz operators, so, let b be a symbol satisfying the condition of Theorem 2.3, that is, one of the symbols a or $z\chi\partial a$. The proof of Theorem 2.3 of [18] uses a decomposition of the disk $\mathbb{D}$ into infinitely many standard hyperbolic rectangles D_n, $n \in \mathbb{N}$, and it shows that in the expression

$$
T(b)f(z) = \sum_{n=1}^{\infty} \int_{D_n} \frac{f(w)b(w)}{(1-z\bar{w})^2} dA(w)
$$

every integral over the set D_n converges, and moreover, the sum over n converges absolutely at least for almost every $z \in \mathbb{D}$. In other words, the Toeplitz operator is defined as a sum of integrals rather than by a usual integral formula.

Remark 3.2. There exist symbols a satisfying the assumptions of Theorem 1.1, which are not bounded or even L^1 on the unit disk and which do not have boundary values. Such examples can be constructed based on cancellation in the integrals (1.5)–(1.7), if a happens to be a rapidly oscillating function. Indeed, given any integer N greater than 2, let us consider the function

$$(3.5)\quad a(re^{i\theta}) = \frac{\chi(\theta)}{r(1-r)^{N-2}} \exp\Big(i\frac{\theta}{(1-r)^N}\Big)\,,\ \ r \ge \frac{1}{2},\, \theta \in [0, 2\pi].$$

Here, the C^∞-function $\chi : [0, 2\pi] \to [0, 1]$ is defined such that the function a becomes smooth on the positive real axis, say, $\chi(\theta) = 0$ for $0 \le \theta \le 1$ and $2\pi - 1 \le \theta \le 2\pi$ and $\chi(\theta) = 1$ for $\pi/2 \le \theta \le 3\pi/2$. Obviously, a is continuously differentiable in its domain of definition and it can be extended to be continuously differentiable in $\mathbb{D}$. Hence, a belongs to $W^{1,1}_{\rm loc}$, but not to L^1, due to the factor $(1 - r)^{2-N}$.

We show that a satisfies the conditions (1.5)–(1.7). Given r and θ as in Theorem 1.1 we make the change of variable $y = \theta(1 - \varrho)^{-N}$ with

$$\frac{1}{(1-\varrho)^{N-2}} d\varrho = -\frac{\theta^{3/N}}{N} \frac{dy}{y^{3/N}}.$$

We denote $Y = \theta(1 - r)^{-N}$ and use $0 \le \chi(\theta) \le 1$ and $\int_x^{x+2\pi} e^{iy} dy = 0$ to obtain, for $r \ge 1/2$,

$$\begin{aligned}
&\Big|\int\limits_r^1 a(\varrho e^{i\theta})\varrho d\varrho\Big| = \chi(\theta)\Big|\int\limits_r^1 \frac{1}{(1-\varrho)^{N-2}} \exp\Big(i\frac{\theta}{(1-\varrho)^N}\Big) d\varrho\Big| \\
&\le C\theta^{3/N}\Big|\int\limits_Y^\infty \frac{1}{y^{3/N}}\, e^{iy}\, dy\Big| = C\theta^{3/N}\Big|\sum_{n=0}^\infty \int\limits_{Y+2\pi n}^{Y+2\pi(n+1)} \frac{1}{y^{3/N}} e^{iy}\, dy\Big| \\
&= C\theta^{3/N}\Big|\sum_{n=0}^\infty \int\limits_{Y+2\pi n}^{Y+2\pi(n+1)} \Big(\frac{1}{y^{3/N}} - \frac{1}{(Y+2\pi n)^{3/N}}\Big) e^{iy}\, dy\Big| \\
&\le C\theta^{3/N} \sum_{n=0}^\infty \int\limits_{Y+2\pi n}^{Y+2\pi(n+1)} \Big|\frac{1}{y^{3/N}} - \frac{1}{(Y+2\pi n)^{3/N}}\Big| dy \\
&\le C'\theta^{3/N} \sum_{n=J(r,\theta)}^\infty \frac{1}{n^{1+3/N}}
\end{aligned}$$

$$(3.6)\quad \le C''\theta^{3/N} Y^{-3/N} \le C''(1-r)^3 \le C''(1-r)^2,$$

where $J(r, \theta)$ is the largest integer not larger than Y. This yields (1.5) and also (1.6), since we obtain from (3.6) the bound

$$(3.7)\qquad \Big|\int\limits_r^s a(\varrho e^{i\theta})\varrho d\varrho\Big| \le \Big|\int\limits_r^1 a\varrho d\varrho\Big| + \Big|\int\limits_s^1 a\varrho d\varrho\Big| \le C(1-r)^2,$$

and the same for the term with $a(\varrho e^{i\phi})$. Moreover, (1.7) is also true, since

$$\Big|\int_\theta^\phi \exp\Big(i\frac{\varphi}{(1-r)^N}\Big)d\varphi\Big| \le 2(1-r)^N;$$

this implies, using integration by parts and the bound $|\chi'(\theta)| \le C$,

$$|\int_\theta^\phi r^2 a(re^{i\varphi})d\varphi| = \frac{r}{(1-r)^{N-2}}\Big|\int_\theta^\phi \chi(\varphi)\exp\Big(i\frac{\varphi}{(1-r)^N}\Big)d\varphi\Big|$$

$$\le C(1-r)^{-N+2}\Big|\chi(\phi)\int_\theta^\phi \exp\Big(i\frac{\varphi}{(1-r)^N}\Big)d\varphi\Big|$$

$$+C(1-r)^{-N+2}\Big|\int_\theta^\phi \chi'(\varphi)\int_\theta^\varphi \exp\Big(i\frac{\psi}{(1-r)^N}\Big)d\psi d\varphi\Big|$$

$$\le C'(1-r)^2$$

and the same for s replacing r, cf. (1.7).

4. Compactness and Fredholm theory

In this final section we discuss the existence of nontrivial compact Toeplitz operators and also the Fredholm theory of Toeplitz operators on Besov spaces.

While [11] provides an example of rank 1 Dirichlet-Toeplitz operator on a general domain, it is known that there are no symbols $a \in W^{1,\infty}$, which induce non-zero compact Toeplitz operators on $\mathcal{B}^2$ on the disk, see Corollary 13 of [11]. We next show that the same is true, if the symbol is a compactly supported distribution in $\mathbb{D}$. We shall use a recent approach to Bergman-Toeplitz operators of the papers [3] and [14]. Given a compactly supported distribution a, the natural generalisation of the Bergman-Toeplitz operator is the following,

$$T(a)f(z) = \langle f(w)(1-z\bar{w})^{-2}, a\rangle_w, \tag{4.1}$$

where $\langle\cdot,\cdot\rangle_w$ denotes dual pairing of the space $C^\infty = C^\infty(\mathbb{D})$ and its dual, the space of compactly supported distributions in $\mathbb{D}$. The dual is taken with respect to the variable w in the subscript. Of course, the pair (4.1) is well defined, since the function $w \mapsto f(w)(1-z\bar{w})^{-2}$ is infinitely smooth, the number $z \in \mathbb{D}$ being considered as a parameter.

For all compactly supported L^1-functions a, the above definition of $T(a)$ coincides with the classical one. Now, note that for a smooth f

and a compactly supported distribution a, the distribution $\partial(af)$ with compact support acts on C^∞ by

$$\langle g, \partial(af)\rangle = -\langle f\partial g, a\rangle,$$

according to the standard definition of distributional derivatives. Thus, it is clear for us that a natural definition for Besov-Toeplitz operator is

$$T_a f(z) = \langle z/(1-z\bar{w}), \partial(af)\rangle_w, \quad z \in \mathbb{D}, \tag{4.2}$$

if a is a distribution with compact support.

Proposition 4.1. *Assume that a is a compactly supported distribution. Then the Besov-Toeplitz operator T_a given by (4.2) is the zero operator.*

Proof. By definition,

$$T_a f(z) = \langle z/(1-z\bar{w}), \partial(af)\rangle_w = -\langle f\partial(z/(1-z\bar{w})), a\rangle_w$$

for all $z \in \mathbb{D}$. The rightmost expression is always zero because $w \mapsto z/(1-z\bar{w})$ is conjugate analytic. □

We finally turn to the Fredholm properties of Besov-Toeplitz operators. Recall that a bounded operator T acting on the Banach space X is Fredholm if its kernel $\ker T$ and cokernel $X/T(X)$ are both finite dimensional. In this case, its index is given by

$$\operatorname{ind} T = \dim\ker T - \dim\big(X/T(X)\big).$$

Our idea is to demonstrate how the representation

$$T_a = \mathbf{V}(T(\partial a)\mathbf{i} + T(a)\partial)$$

yields the Fredholm properties for T_a, $a \in W^{1,\infty}$, as a consequence of Bergman space case. To our knowledge, the only paper dealing with this subject is the paper of Cao [6], which establishes Fredholm theory for the case $p = 2$. The symbol class $W^{1,\infty}$ is larger than the classes considered in Cao's paper, and the present setting works also for the whole range of $p \in (1,\infty)$.

By [8], Theorem 4.1, all functions $a \in W^{1,\infty}$ are Lipschitz continuous on $\overline{\mathbb{D}}$; in particular, the boundary values are well-defined. Let us denote by $a^* : \partial\mathbb{D} \to \mathbb{C}$ the continuous function defined by these boundary values.

Theorem 4.2. *Let $a \in W^{1,\infty}$. Then $T_a : \mathcal{B}^p \to \mathcal{B}^p$ is a Fredholm operator, if and only if a^* has no zeroes. In the case T_a is Fredholm, its index is given by the formula*

$$\operatorname{ind} T_a = -\operatorname{wind} a^*.$$

Here, $\operatorname{wind} a^*$ denotes the winding number of the function a^* with respect to 0.

Proof. We recall that since $a \in W^{1,\infty}$, all the assumptions of Theorem 1.1 are satisfied and we can thus use the representation

$$T_a = \mathbf{V}(T(\partial a)\mathbf{i} + T(a)\partial),$$

see (3.2). By Proposition 2.4, the operator $\mathbf{V}T(\partial a)\mathbf{i}$ is compact. Thus, the Fredholm properties of T_a are the same as those of $\mathbf{V}T(a)\partial$. Since both $\mathbf{V} : A^p_{p-2} \to \mathcal{B}^p$ and $\partial : \mathcal{B}^p \to A^p_{p-2}$ are isometric isomorphisms, the Fredholm properties of T_a coincide with those of $T(a) : A^p_{p-2} \to A^p_{p-2}$. Thus, the proof is completed by Lemma 4.4, which we prove shortly. $\square$

The essential spectrum $\sigma_{\mathrm{ess}}(T)$ of a bounded linear operator T is defined by

$$\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}.$$

Corollary 4.3. *Let $a \in W^{1,\infty}$. Then $\sigma_{\mathrm{ess}}(T_a) = a^*(\partial\mathbb{D})$.*

Recently in [10] it was proved that Coburn's lemma remains true for Toeplitz operators on the Dirichlet space; that is, if T_a is nonzero, then either T_a is one-to-one or T_a^* is one-to-one. It follows that if $a \in W^{1,\infty}$,

$$\sigma(T_a) = a^*(\partial\mathbb{D}) \cup \{\lambda \in \mathbb{C} \setminus a^*(\partial\mathbb{D}) : \operatorname{wind}(a - \lambda) \neq 0\}.$$

It is also important to determine when the points $\lambda \in \sigma(T_a)$ are eigenvalues of T_a. If $\lambda \notin \sigma_{\mathrm{ess}}(T_a)$, by a standard argument as in Hardy spaces, it follows from Coburn's lemma that λ is an eigenvalue if and only if $\operatorname{wind}(\lambda - a^*) > 0$. A considerably more difficult question is to determine when $\lambda \in \sigma_{\mathrm{ess}}(T_a)$ is an eigenvalue. The approach in Hardy spaces may offer some ideas of how to deal with this question—see [16] and the references therein.

The Fredholm properties are well-known, when the Toeplitz operator is defined using the reproducing kernel of A^2_{p-2} (see Remark 2.2 and [15]), while here the kernel is that of the (unweighted) space A^2_0. However, as we will see, the proof for this case is not that difficult:

Lemma 4.4. *Let $\alpha > -1$, $p \in (1, \infty)$ and $a \in C(\overline{\mathbb{D}})$. Then the operator $T(a)$ is Fredholm on A^p_α, if and only if a has no zeroes on the boundary. In this case, we have the index formula*

$$\operatorname{ind} T(a) = -\operatorname{wind} a^*.$$

Proof. We sketch the steps of the proof, which is quite close to the classical one.

Step 1. Suppose first that a has no zeroes on the boundary. Then, there exists $b \in C(\overline{\mathbb{D}})$ such that $ab = 1$ on the boundary. Thus, denoting by $M(f)$ multiplication by the function f, we obtain

$$T(a)T(b) = T(ab) + PM(a)(I - P)M(b) = I + K_1 + PM(a)H(b),$$

and similarly for $T(b)T(a)$. To show that $T(a)$ is Fredholm, it therefore suffices to show that the unweighted Hankel operator $H(b) = (I-P)M(b)$ is compact on A^p_α. This follows from Theorem 20 of [24] by taking $\alpha = 0$ and $\lambda = p-2$. Note that, in our particular case, this can be proven without Zhu's result by simply showing that $H(\overline{z}^n)$ is compact (as $H(z^k\overline{z}^n) = H(\overline{z}^n)M(z^k)$ and we can use the Stone-Weierstrass theorem).

Step 2. By Step 1, the operators $T(z^n)$ and $T(\overline{z}^n)$ are Fredholm on A^p_α. Clearly, $T(z^n)$ has a trivial kernel and its image has codimension n. By an easy calculation based on orthogonality, the dimension of the kernel of $T(\overline{z}^n)$ is n. Moreover, monomials z^{n+k} are mapped to the linear span of z^k. Thus, the image of $T(\overline{z}^n)$ contains all polynomials. By Step 1, the operator is Fredholm and therefore has closed range. Hence $T(\overline{z}^n)$ is surjective. In conclusion,

$$\operatorname{ind} T(z^n) = -n = -\operatorname{ind} T(\overline{z}^n).$$

Step 3. The general index formula is obtained by homotopy, see either [15] or [19]. Finally, to see that the Fredholmness of $T(a)$ implies that a has no zeroes on the boundary, use the standard perturbation argument. □

Remark 4.5. The main difference between Lemma 4.4 and the more standard case of [15] lies, rather surprisingly, on Step 2. Indeed, note that with the present definition the adjoint of $T(a)$ is not $T(\overline{a})$, since the operator P is not self-adjoint, unless $\alpha = 0$.

We note that in addition to Fredholm properties of Bergman-Toeplitz operators, the proof of Theorem 4.2 only needs the assumption that the symbol a satisfies the requirements of Theorem 1.1. This observation opens an approach for generalisations of Theorem 4.2. However, we refrain from presenting such results, since is might be not so easy to find a satisfactory, simple enough formulation, like in Theorem 4.2.

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