ON GENERALIZED TOEPLITZ AND LITTLE HANKEL OPERATORS ON BERGMAN SPACES

JARI TASKINEN AND JANI VIRTANEN

Abstract. We find a concrete integral formula for the class of generalized Toeplitz operators T_a in Bergman spaces A^p, $1 < p < \infty$, studied in an earlier work by the authors. The result is extended to little Hankel operators. We give an example of an L^2-symbol a such that $T_{|a|}$ fails to be bounded in A^2, although $T_a : A^2 \to A^2$ is seen to be bounded by using the generalized definition. We also confirm that the generalized definition coincides with the classical one whenever the latter makes sense.

1. Introduction.

Consider the space $L^p := (L^p(\mathbb{D}, dA), \| \cdot \|_p)$, where $1 < p < \infty$ and dA is the normalized area measure on the unit disc $\mathbb{D}$ of the complex plane, and the Bergman space A^p, which is the closed subspace of L^p consisting of analytic functions. The Bergman projection P is the orthogonal projection of L^2 onto A^2, and it has the integral representation

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta).$$

It is also known to be a bounded projection of L^p onto A^p for every $1 < p < \infty$. For an integrable function $a : \mathbb{D} \to \mathbb{C}$ and, say, bounded analytic functions f, the Toeplitz operator T_a with symbol a is defined by

$$T_a f = P(af) = \int_{\mathbb{D}} \frac{a(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta). \tag{1.1}$$

Since P is bounded, it follows easily that T_a extends to a bounded operator $A^p \to A^p$ for $1 < p < \infty$, whenever a is a bounded measurable function. The question of the boundedness of T_a on A^p with unbounded symbols is a long-standing problem. Examples of unbounded symbols inducing bounded Toeplitz operators can be easily constructed, since the behaviour of the symbol inside any compact subset of $\mathbb{D}$ is not important for the boundedness of the operator. Also it is not difficult to find unbounded symbols a for which the integral in (1.1) converges, say, for all $f \in A^2$ but the operator is not bounded; see Section 3 for an interesting example. We refer to the papers [1], [2], [3], [4], [5], [6], [8], [10], [11], [12], [14], [15], [16], [18] for classical and recent results on the boundedness and compactness of Toeplitz operators on Bergman spaces.

In the paper [13] we have given a generalized definition of Toeplitz operators, which we denote here $\mathbf{T}_a$. The definition takes efficiently into account the possible cancellation phenomena of a symbol. This leads to very weak sufficient conditions for the boundedness of Toeplitz operators. More precisely, in the reference it was

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shown that $\mathbf{T}_a$ is bounded under an averaging condition for the symbol itself rather than for its modulus (the result is repeated and also extended to little Hankel operators in Theorem 1.2, below). However, the presentation of the result in [13] has some shortcomings and accordingly the purpose of this paper is to make some improvements, which will be described in detail at the end of this section.

The results of [13] show that cancellation phenomena may be essential in order to have a bounded operator T_a. Here, we give an example which emphasizes this: in Section 3 we study the radial symbol $a \in L^2$, where $a(z) = |z|^{-1}(1-|z|)^{-1/4}\sin((1-|z|)^{-1})$ for $|z| \geq 1/2$, and prove that the operator T_a is bounded in A^p, although $T_{|a|}$ is obviously not. Thus, the boundedness of T_a cannot be proven by conventional methods that only take into account the modulus of the symbol. We can actually construct such a symbol in any given space L^q with $q < \infty$.

Given $a \in L^1$, the little Hankel operator h_a with symbol a is defined as

$$h_a f(z) = \int_{\mathbb{D}} \frac{a(\zeta) f(\zeta)}{(1-\bar{z}\zeta)^2} dA(\zeta)$$

for $f \in A^p$ such that this integral converges. In this paper we make the observation that the generalized definition of a Toeplitz operator and the results of [13] can be extended to the little Hankel case as well. The results for h_a are presented in parallel with Toeplitz operators.

As for the notation used in this paper, all function spaces are defined on the open unit disc $\mathbb{D}$. In particular H^∞ denotes the Hardy space of bounded analytic functions on $\mathbb{D}$. If $0 < \rho < 1$, we denote $\mathbb{D}_\rho = \{|z| \leq \rho\}$. We also denote the standard weight by $W(z) = 1-|z|^2$, the kernel functions by $K_\lambda(z) = (1-z\bar{\lambda})^{-2}$ and $k_\lambda = K_\lambda / \|K_\lambda\|_2 = W(\lambda) k_\lambda$, and the Möbius transform by $\varphi_\lambda(z) = (\lambda - z)/(1-z\bar{\lambda})$, where $z, \lambda \in \mathbb{D}$. By C, C' etc. we mean generic constants, the exact values of which may change from place to place. We will deal with symbols a, which always at least belong to the space $L^1_{\rm loc}$ of locally integrable functions on $\mathbb{D}$. For other notation and definitions we refer to the book [17].

Let us first describe briefly the sufficient condition for the boundedness of generalized Toeplitz operators given in [13].

Definition 1.1. Denote by $\mathcal{D}$ the family of the sets $D := D(r,\theta)$, where

$$D = \{\rho e^{i\phi} \mid r \leq \rho \leq 1 - \frac{1}{2}(1-r) \ , \ \theta \leq \phi \leq \theta + \pi(1-r)\} \tag{1.2}$$

for all $0 < r < 1$, $\theta \in [0, 2\pi]$. We denote $|D| := \int_D dA$ and, for $\zeta = \rho e^{i\phi} \in D(r,\theta)$,

$$\hat{a}_D(\zeta) := \frac{1}{|D|} \int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi}) \varrho \, d\varphi \, d\varrho, \tag{1.3}$$

where $a \in L^1_{\rm loc}$. In the following we will study symbols a for which there exists a constant $C > 0$ such that

$$|\hat{a}_D(\zeta)| \leq C \tag{1.4}$$

for all $D \in \mathcal{D}$ and all $\zeta \in D$.

It turns out that one can proceed to a generalized definition of bounded Toeplitz operators just by using the condition (1.4). However, for the proofs we need to recall some more definitions from [13]. The countably many sets $D\big(1-2^{-m+1}, 2\pi(\mu -$

$1)2^{-m}\big) \in \mathcal{D}$, where $m \in \mathbb{N}, \mu = 1, \dots, 2^{-m}$, form a decomposition of the disc $\mathbb{D}$. We index these sets somehow into a family $(D_n)_{n=1}^{\infty}$, so that every D_n is of the form

$$D_n = \{\ z = re^{i\theta} \mid r_n < r \leq r_n', \theta_n < \theta \leq \theta_n'\} \tag{1.5}$$

where, for some m and μ,

$$r_n = 1 - 2^{-m+1}\ , r_n' := 1 - 2^{-m}, \quad \theta_n = \pi(\mu - 1)2^{-m+1}\ ,\ \theta_n' := \pi\mu 2^{-m+1}. \tag{1.6}$$

Let $f \in A^p$. For all $n = n(m, \mu)$ we write

$$F_n f(z) = \int\limits_{D_n} \frac{a(\zeta)f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta)\ \ ,\ \ H_n f(z) = \int\limits_{D_n} \frac{a(\zeta)f(\zeta)}{(1 - \bar{z}\zeta)^2} dA(\zeta) \qquad \forall\, z \in \mathbb{D}, \tag{1.7}$$

so that F_n can actually be considered as a conventional, bounded Toeplitz operator on A^p; similarly for H_n.

Item 1° of the following theorem is the main result Theorem 2.3 of [13]. Also, 2° is an immediate consequence of its proof: we leave to the reader the completely straightforward task to verify that the change $z\bar{\zeta} \to \bar{z}\zeta$ in the denominator does not affect the proof.

Theorem 1.2. *Let $1 < p < \infty$ and assume that $a \in L^1_{\rm loc}$ satisfies the condition* (1.4). *Then, the following hold true.*
1°. *Given $f \in A^p$, the series $\sum_{n=1}^{\infty} F_n f(z)$ converges pointwise, absolutely for almost all $z \in \mathbb{D}$, and the generalized Toeplitz operator $\mathbf{T}_a : A^p \to A^p$, defined by*

$$\mathbf{T}_a f(z) = \sum_{n=1}^{\infty} F_n f(z) \tag{1.8}$$

is bounded for all $1 < p < \infty$, and there is a constant C such that

$$\|\mathbf{T}_a\| \leq C \sup_{D \in \mathcal{D}, \zeta \in D} |\hat{a}_D(\zeta)|.$$

2°. *For $f \in A^p$, the series $\sum_{n=1}^{\infty} H_n f(z)$ converges pointwise, absolutely, for almost all $z \in \mathbb{D}$. We define the generalized little Hankel operator by*

$$\mathbf{h}_a f(z) = \sum_{n=1}^{\infty} H_n f(z) \tag{1.9}$$

Then, $\mathbf{h}_a : A^p \to A^p$ is bounded for all $1 < p < \infty$, and there is a constant C such that

$$\|\mathbf{h}_a\| \leq C \sup_{D \in \mathcal{D}, \zeta \in D} |\hat{a}_D(\zeta)|.$$

In this paper we improve Theorem 1.2 in the following ways.

1°. The definition (1.8) of a generalized Toeplitz operator seems to depend on the geometry of a fixed decomposition (1.5) of the unit disc. (No doubt, other decompositions of $\mathbb{D}$, say with different choices of the points r_n and θ_n, could be used as well, and it is not a priori clear, if the generalized operator defined in that way coincides with (1.8). In fact, an approach using Whitney decompositions with Euclidean rectangles for simply connected domains was presented in [7].) In this paper, formula (2.2), we show that the definition (1.8) coincides with a natural radial limit of conventional Toeplitz operators, and thus the dependence of the definition on the decomposition of the disc vanishes.

2°. It is not difficult to see that the generalized definition (1.8) of a Toeplitz operator coincides with the usual definition, whenever the latter gives a bounded operator and condition (1.4) holds. This simple proof was omitted from [13], but we present it here in Proposition 3.1.

3°. The terms F_n in the series (1.8) are actually conventional, bounded Toeplitz operators. In [13] it is only shown that the series (1.8) converges in the very weak sense mentioned in Theorem 1.2 above. Here, we show in Theorem 2.1 that the operator series $\sum_n F_n$ converges in the strong operator topology, and the same is true for the new limit representation (2.2). Theorem 2.1 also contains an immediate application of this result to transposed operators.

4°. The proof of Theorem 2.3 of [13] contains a small error: the inequality (3.8) of the citation is not true as such, since the point $r_n' e^{i\theta_n'}$ there is actually on the boundary of the set D_n. It is however not difficult to fix the flaw, and indeed in the course of the proof of Theorem 2.1 we do this by replacing the set D_n by a bit larger set denoted by U_n, see (2.6).

2. Main result.

We now give a simplified expression of the generalized Toeplitz operator $\mathbf{T}_a$, (1.8), and also treat the little Hankel operator as well as the transposed operators. Given $a \in L^1_{\text{loc}}$ and $0 < \rho < 1$ we define the function $a_\rho : \mathbb{D} \to \mathbb{C}$ by $a_\rho(z) = a(z)$, if $|z| \le \rho$ and $a_\rho(z) = 0$ otherwise. It is plain that the Toeplitz and little Hankel operators

$$(2.1)\quad T_{a_\rho} f(z) = \int\limits_{\mathbb{D}_\rho} \frac{a(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) \quad , \quad h_{a_\rho} f(z) = \int\limits_{\mathbb{D}_\rho} \frac{a(\zeta) f(\zeta)}{(1 - \bar{z}\zeta)^2} dA(\zeta) \quad , \quad z \in \mathbb{D},$$

are bounded $A^p \to A^p$.

Theorem 2.1. *Let $1 < p < \infty$ and $1/p + 1/q = 1$, and assume that $a \in L^1_{\text{loc}}$ and that (1.4) holds. Then, the generalized Toeplitz operators $\mathbf{T}_a : A^p \to A^p$ and little Hankel operators $\mathbf{h}_a : A^p \to L^p$, defined in (1.8) and (1.9), respectively, can be written as*

$$(2.2)\qquad \mathbf{T}_a f = \lim_{\rho \to 1} T_{a_\rho} f \ ,$$

$$(2.3)\qquad \mathbf{h}_a f = \lim_{\rho \to 1} h_{a_\rho} f$$

for all $f \in A^p$. The limits converge with respect to the strong operator topology (SOT).

Moreover, the transposed operators (with respect to the standard complex dual pairing) $\mathbf{T}_a^* : A^q \to A^q$ and $\mathbf{h}_a^* : L^q \to A^q$ can be written as

$$(2.4)\qquad \mathbf{T}_a^* f = \lim_{\rho \to 1} T_{\bar{a}_\rho} f \ ,$$

$$(2.5)\qquad \mathbf{h}_a^* g = \lim_{\rho \to 1} h_{\bar{a}_\rho} f$$

for $f \in A^q$ and $g \in L^q$, for almost all $z \in \mathbb{D}$, and the limits here also converge in the SOT.

Remark. In the course of the proof we also show that the sum in (1.8) converges in the SOT and thus improve the result of [13] also in this sense. Of course, the limit on the right hand side of (2.2) cannot in general converge in the operator norm, since the operators T_{a_ρ} are compact.

Proof. The proof will be given in a few steps. Moreover, we prove the statement (2.2) only for the Toeplitz operator, but the reader is asked to observe the necessary changes for the little Hankel case (2.3).

(i) In the first step we review and strengthen the proof of Theorem 2.3 in [13] concerning the sum in (1.8). Let $f \in A^p$ be arbitrary.

For all $n \in \mathbb{N}$ we define the collection of all sets D_ν which touch the given D_n, more precisely,

$$\mathcal{D}_n = \{D_\nu \,:\, \nu \in \mathbb{N},\ \overline{D}_\nu \cap \overline{D}_n \neq \emptyset\}.$$

By the definition of the sets D_n, see (1.2)–(1.6), there exist constants N, $M \in \mathbb{N}$ such that any set $\mathcal{D}_n$ contains at most N elements D_ν and on the other hand, any set D_ν belongs to at most M sets $\mathcal{D}_n$. Moreover, given D_n and $w \in D_n$, the subdomain

$$\bigcup\nolimits_{D\in\mathcal{D}_n} D =: U_n \tag{2.6}$$

always contains a Euclidean disc $D(w, R)$ with center w and radius $R = R(n) > 0$ such that $|D(w, R)| \geq C|D_n|$ (use again the choice of the sets D_n to see this).

We claim that for each n and $w \in D_n$,

$$|f(w)| \leq \frac{C}{|D_n|} \sum_{D\in\mathcal{D}_n} \int\limits_D |f(\zeta)| dA(\zeta). \tag{2.7}$$

To prove (2.7), let $D(w, R) \subset U_n$ be as above. Then, (2.7) follows from the usual subharmonicity property for $D(w, R)$:

$$|f(w)| \leq \frac{C}{|D(w, R)|} \int\limits_{D(w,R)} |f(\zeta)| dA(\zeta) \leq \frac{C'}{|D_n|} \int\limits_{U_n} |f(\zeta)| dA(\zeta).$$

From now on we replace the incorrect inequality (3.8) of [13] by (2.7).

The proof of [13], which uses the integration by parts -trick and the assumption (1.4), yields the estimate

$$|F_n f(z)| \leq C \sum_{D\in\mathcal{D}_n} G_D(z), \qquad \text{where} \tag{2.8}$$

$$G_D(z) = \int\limits_D \frac{|f(\zeta)| + |f'(\zeta)|\, W(\zeta) + |f''(\zeta)|\, W(\zeta)^2}{|1 - z\bar{\zeta}|^2} dA(\zeta).$$

We observe by Theorem 4.28 of [17] that the function $g := |f| + |f'|\, W + |f''|\, W^2$ in the integrand belongs to L^p. Following the argument in [13], the positive term series

$$\sum_{n=1}^{\infty} G_{D_n}(z) \tag{2.9}$$

converges for almost all z and defines a function which belongs to L^p, since it it is pointwise bounded by the maximal Bergman projection $|P|$ of g. Thus we see that also the series

$$\sum_{n=1}^{\infty} \sum_{D\in\mathcal{D}_n} G_D(z) \tag{2.10}$$

converges for almost all z, and the sum belongs to L^p. This follows from the convergence of (2.9), since the terms of (2.10) consist of the positive expressions G_{D_n}, and any single G_{D_n} can occur at most MN times in (2.10), by the definition of the numbers N and M.

By (2.8), the convergence of (2.10) implies the absolute convergence of the series $\sum_n F_n f(z)$ a.e.. We claim that the operator sequence $(T^{(m)})_{m=1}^\infty$ defined by

$$T^{(m)} f = \sum_{n=1}^{m} F_n f \tag{2.11}$$

converges to $\mathbf{T}_a$ in the SOT, as $m \to \infty$. Indeed, given $f \in A^p$ and any $z \in \mathbb{D}$, the difference

$$\big|\mathbf{T}_a f(z) - T^{(m)} f(z)\big| = \Big|\sum_{n>m} F_n f(z)\Big| = \Big|\int\limits_{V_m} \frac{f(\zeta)}{(1-z\bar\zeta)^2} dA(\zeta)\Big|, \tag{2.12}$$

where $V_m = \cup_{n>m} D_n$, has by (2.8) the upper bound

$$C \int\limits_{V_\mu} \frac{g(\zeta)}{|1-z\bar\zeta)^2|} dA(\zeta) = C \int\limits_{\mathbb{D}} \frac{\chi_{V_\mu}(\zeta) g(\zeta)}{|1-z\bar\zeta|^2} dA(\zeta) = C|P|(\chi_{V_\mu} g)(z); \tag{2.13}$$

here, μ is some positive integer with $\mu \to \infty$ as $m \to \infty$, and χ_{V_μ} is the characteristic function of the set V_μ. But we have $\|\chi_{V_\mu} g\|_p \to 0$ as $\mu \to \infty$, by Lebesgue's dominated convergence theorem. Since $|P|$ is a bounded operator, there also holds $\||P|(\chi_{V_\mu} g)\|_p \to 0$ as $\mu \to \infty$. Combining this with the estimates (2.12)–(2.13) we get that $\|\mathbf{T}_a f - T^{(m)} f\|_p \to 0$ as $m \to \infty$, which proves the claim.

(ii) We next consider the relation of the limit in (2.2) with the sum (1.8).

Let us fix n for a moment. Inspecting the proof of [13] we see that given any $(\tilde r, \tilde\theta)$ such that $r_n < \tilde r < r_n'$ and $\theta_n < \tilde\theta < \theta_n'$, the expression

$$G_n(z, \tilde r, \tilde\theta) := \int\limits_{r_n}^{\tilde r} \int\limits_{\theta_n}^{\tilde\theta} \frac{a(\varrho e^{i\varphi}) f(\varrho e^{i\varphi})}{(1 - z\varrho e^{-i\varphi})^2} \varrho d\varrho d\varphi$$

has the same upper bound as $F_n f(z)$ in (2.8) (cf. (1.7)), namely

$$|G_n(z, \tilde r, \tilde\theta)| \le C \sum_{D \in \mathcal{D}_n} G_D(z). \tag{2.14}$$

To see this one has to make the straightforward changes to the upper limits of integrals in (3.6)–(3.11) of [13] and also use (2.7). This is left to the reader as an easy task.

Given ρ, the integral in (2.2) can be written as

$$\int\limits_{\mathbb{D}_\rho} \frac{a(\zeta) f(\zeta)}{(1-z\bar\zeta)^2} dA(\zeta) = \sum_{n=1}^{m} F_n f(z) + \sum_{n=m+1}^{K} G_n(z, \rho, \theta_n') \tag{2.15}$$

for some integers m and $K > m$, and moreover, $m \to \infty$ as $\rho \to 1$. It is then obvious from the estimate (2.14) and the convergence (2.10) that for almost all z, the limit in (2.2) must exist and, by (2.15), it has to coincide with $\sum_n F_n f(z) = \mathbf{T}_a f(z)$, (1.8).

Concerning the convergence in the SOT, we use (2.14) and (2.15) and the argument around (2.12)–(2.13) to estimate the difference

$$|\mathbf{T}_a f(z) - T_{a_\rho} f(z)| \le C|P|(\chi_{V_\mu} g)(z) \tag{2.16}$$

where $\mu \to \infty$ as $\rho \to 1$. Convergence in the SOT follows in the same way as at the end of part (i).

(iii) Let us consider (2.4); let $f \in A^p$ and $g \in A^q$ be given. Denoting by $\langle \cdot, \cdot \rangle$ the standard complex dual paring of A^p and A^q, we have

$$\langle f, \mathbf{T}_a^* g \rangle = \langle \mathbf{T}_a f, g \rangle = \int_{\mathbb{D}} \bar{g} \lim_{\rho \to 1} T_{a_\rho} f dA = \lim_{\rho \to 1} \int_{\mathbb{D}} \bar{g}\, T_{a_\rho} f \, dA, \tag{2.17}$$

where the limit and the integral could be commuted because of the convergence of (2.2) in the SOT. Then, (2.17) equals

$$\begin{aligned}
&\lim_{\rho \to 1} \int_{\mathbb{D}} \int_{\mathbb{D}_\rho} \frac{a(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) \overline{g(z)} dA(z) \\
&= \lim_{\rho \to 1} \int_{\mathbb{D}} f(\zeta) a_\rho(\zeta) \overline{Pg(\zeta)} dA(\zeta) = \lim_{\rho \to 1} \int_{\mathbb{D}} f(\zeta) a_\rho(\zeta) \overline{g(\zeta)} dA(\zeta) \\
&= \lim_{\rho \to 1} \int_{\mathbb{D}} f(\zeta) \overline{P(\overline{a_\rho} g)(\zeta)} dA(\zeta) = \lim_{\rho \to 1} \int_{\mathbb{D}} f\, \overline{T_{\bar{a}_\rho} g} \, dA = \int_{\mathbb{D}} f \lim_{\rho \to 1} \overline{T_{\bar{a}_\rho} g} \, dA,
\end{aligned}$$

where at the end we used the fact that $\bar{a}$ obviously also satisfies condition (1.4) and the convergence of (2.2) in the SOT.

That the limit exist in the SOT follows from the treatment of the limit (2.2), since $\bar{a}$ satisfies (1.4). The proof of the little-Hankel case (2.5) is similar, with obvious changes. □

3. Concluding remarks.

The following observation can be summarized as saying that $T_a f$ and $\mathbf{T}_a f$ coincide, whenever the former operator is bounded and condition (1.4) holds.

Proposition 3.1. *Let $1 < p < \infty$. Assume that $a \in L^1$, the integral* (1.1) *converges for all $f \in A^p$ and $T_a : A^p \to A^p$ is bounded; assume moreover that* (1.4) *is satisfied so that also $\mathbf{T}_a$ is bounded in A^p. Let $f \in A^p$ be arbitrary and then let $(f_n)_{n=1}^\infty \subset H^\infty$ be such that $f_n \to f$ in A^p as $n \to \infty$. Then, $T_a f_n \to \mathbf{T}_a f$ in A^p, and, consequently, $T_a f = \mathbf{T}_a f$ for all $f \in A^p$.*

The statement remains true for little Hankel operators, with h_a replacing T_a and $\mathbf{h}_a$ replacing $\mathbf{T}_a$.

Proof. Since $\mathbf{T}_a$ is a bounded operator $A^p \to A^p$, we have $\mathbf{T}_a f_n \to \mathbf{T}_a f$ in A^p, and thus it is enough to show that $T_a g = \mathbf{T}_a g$ for all $g \in H^\infty$. But for such g, the integral

$$\int_{\mathbb{D}} \frac{a(\zeta) g(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta).$$

converges, since $a \in L^1$ and the kernel function $\zeta \mapsto (1 - z\bar{\zeta})^2$ is bounded. Then it is clear, see e.g. [9], Theorem 1.27, that

$$\sum_{n=1}^\infty \int_{D_n} \frac{a(\zeta) g(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) = \int_{\mathbb{D}} \frac{a(\zeta) g(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) = T_a g(z).$$

This proves the result, since

$$\sum_{n=1}^{N} \int_{D_n} \frac{a(\zeta)g(\zeta)}{(1-z\bar{\zeta})^2} dA(\zeta) \to \mathbf{T}_a g(z) \quad \text{as} \quad N \to \infty$$

by what is mentioned around (1.8).

The proof in the case of little Hankel operators is the same. □

The sufficient condition (1.4) and the definitions (1.8), (2.2) of Toeplitz operators are formulated for quite general locally integrable symbols, but the following example shows that the condition and the boundedness result are useful already in very simple, concrete cases. A well known sufficient condition for the boundedness of T_a is that

$$\sup_{D \in \mathcal{D}} M_a(D) < \infty \ \text{ where } M_a(D) := \frac{1}{|D|} \int\limits_D |a| \, dA, \tag{3.1}$$

and this condition is also necessary, if $\mathbb{R} \ni a(z) \geq 0$ for all $z \in \mathbb{D}$. See [17].

For every $0 < b \leq 1/2$ we define the symbol

$$a_b(re^{i\theta}) := \begin{cases} \dfrac{1}{r(1-r)^b} \sin \dfrac{1}{1-r} \ , & r \geq \dfrac{1}{2} \\ 1 \ , & r < \dfrac{1}{2} \end{cases} \tag{3.2}$$

which obviously belongs to L^q, if $bq < 1$. Then, in particular, $a_{1/4} \in L^2$ and the defining integral formula of $T_{a_{1/4}}$ converges for every $f \in A^2$. Obviously, the defining formula of $T_{|a_{1/4}|}$ also converges for every $f \in A^2$. However, we have the following result.

Proposition 3.2. *(i) The Toeplitz operator $T_{|a_b|}$ is not bounded in A^p for any $1 < p < \infty$ and $0 < b \leq 1/2$.*
(ii) The Toeplitz operator T_{a_b} is bounded in A^p for all $1 < p < \infty$ and $0 < b \leq 1/2$.

Proof. Let us first deal with $T_{|a_b|}$. Given $0 < r < 1$ and any $\theta \in [0, 2\pi]$, we consider the behaviour of a_b in the set $D = D(r, \theta)$, see Definition 1.1. It is plain from the definition of a_b and the elementary properties of the sinus that for some universal constant $C > 0$ we have

$$|a_b(z)| \geq \frac{1}{4}(1-|z|)^{-b}$$

in a subset of D with area measure at least $C(1-r)^2$ (recall that $|D|$ is proportional to $(1-r)^2$). Then, of course $M_a(D) \geq C'(1-r)^{-b}$ for another constant $C' > 0$, and thus condition (3.1) cannot hold, and the operator $T_{|a_b|}$ is unbounded.

The symbol satisfies (1.4), since given $D = D(1-2\delta, \theta)$ with a small enough δ and $\zeta = \rho e^{i\phi} \in D$, we have, using the change of variable $y = 1/(1-\varrho)$ (so that $d\varrho = y^{-2} dy$)

$$|D||\hat{a}_D(\zeta)| = \int\limits_{\theta}^{\phi} d\varphi \Big| \int\limits_{1-2\delta}^{\rho} \frac{1}{(1-\varrho)^b} \sin \frac{1}{1-\varrho} d\varrho \Big| = \pi\delta \Big| \int\limits_{1/(2\delta)}^{1/(1-\rho)} y^{b-2} \sin y \, dy \Big|$$

Let us divide the integration interval to subintervals $J_n := [2\pi n, 2\pi(n+1)]$, $n \in \mathbb{N}$. On J_n we integrate as follows:

$$\Big| \int_{J_n} y^{b-2} \sin y \, dy \Big| = \Big| \int_{J_n} y^{b-2} \sin y \, dy - (2\pi(n+1))^{b-2} \int_{J_n} \sin y \, dy \Big|$$
$$\leq \int_{J_n} \Big| y^{b-2} - (2\pi(n+1))^{b-2} \Big| \, dy \leq C n^{b-3}.$$

Hence,

$$|D||\hat{a}_D(\zeta)| \leq \pi\delta \Big| \int_{1/(2\delta)}^{1/(1-\rho)} y^{b-2} \sin y \, dy \Big| \leq C\delta \sum_{n=[1/(4\pi\delta)]}^{\infty} n^{b-3} \leq C' \delta^{3-b},$$

where $[x]$ denotes the integer part of a number $x \in \mathbb{R}$. Since $|D|$ is proportional to δ^2, the condition (1.4) holds true, and T_{a_b} is bounded, by Theorem 1.2 and Proposition 3.1. □

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References

[1] M. Engliš, Toeplitz operators and weighted Bergman kernels. J. Funct. Anal. 255, 6 (2008), 1419–1457.
[2] S. Grudsky, A. Karapetyants, N. Vasilevski, Toeplitz operators on the unit ball in $\mathbb{C}^N$ with radial symbols, J. Oper. Theory 49 (2003), 325–346.
[3] S. Grudsky, N. Vasilevski, Bergman-Toeplitz operators: radial component influence, Integr. Eq. Oper. Theory 40 (2001), 16–33.
[4] D. H. Luecking, Trace ideal criteria for Toeplitz operators. J. Funct. Anal. 73 (1987), no. 2, 345–368.
[5] D. H. Luecking, Finite rank Toeplitz operators on the Bergman space. Proc. Amer. Math. Soc. 136 (2008), no. 5, 1717–1723.
[6] W. Lusky, J. Taskinen, Toeplitz operators on Bergman spaces and Hardy multipliers. Studia Math. 204 (2011), 137–154.
[7] P. Mannersalo, Toeplitz operators with locally integrable symbols on Bergman spaces of bounded simply connected domains, Compl.Variables Elliptic Eq. 61,6 (2016), 854–874.
[8] A. Perälä, J. Taskinen, and J. A. Virtanen, Toeplitz operators with distributional symbols on Bergman spaces, Proc. Edinb. Math. Soc. 54, 2 (2011), 505–514.
[9] W. Rudin, Real and Complex analysis, 3rd ed., McGraw-Hill, New York, 1987.
[10] K. Stroethoff, Compact Toeplitz operators on Bergman spaces, Math. Proc. Cambridge Philos. Soc. 124(1998), 151–160.
[11] K. Stroethoff, D. Zheng, Toeplitz and Hankel operators on Bergman spaces, Trans. AMS 329, 2 (1992), 773–794.
[12] D. Suárez, The essential norm of operators in the Toeplitz algebra on $A^p(B_n)$, Indiana Univ. Math. J. 56, no. 5, (2007) 2185–2232.
[13] J. Taskinen, J. A. Virtanen, Toeplitz operators on Bergman spaces with locally integrable symbols. Rev.Math.Iberoamericana 26,2 (2010), 693–706.

[14] N.L. Vasilevski, Bergman type spaces on the unit disk and Toeplitz operators with radial symbols, Reporte Interno 245, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 1999.
[15] N.L. Vasilevski, Commutative algebras of Toeplitz operators on the Bergman space, Operator Theory: Advances and Applications, Vol. 185, Birkhäuser Verlag, 2008.
[16] K. Zhu, Positive Toeplitz operators on weighted Bergman space, J. Operator Theory 20 (1988), 329–357.
[17] K. Zhu, Operator Theory in Function Spaces, 2nd edition, Mathematical Surveys and Monographs, 138, American Mathematical Society, Providence, RI, 2007.
[18] N. Zorboska, Toeplitz operators with BMO symbols and the Berezin transform. Int. J. Math. Math. Sci. 46 (2003), 2929–2945.

Department of Mathematics, University of Helsinki, 00014 Helsinki, Finland
E-mail address: jari.taskinen@helsinki.fi

Department of Mathematics, University of Reading, Whiteknights, P.O. Box 220, Reading RG6 6AX, UK
E-mail address: j.a.virtanen@reading.ac.uk