

# ON BOUNDEDNESS AND COMPACTNESS OF TOEPLITZ OPERATORS IN WEIGHTED $H^\infty$ -SPACES.

JOSÉ BONET, WOLFGANG LUSKY, AND JARI TASKINEN

ABSTRACT. We characterize the boundedness and compactness of Toeplitz operators  $T_a$  with radial symbols  $a$  in weighted  $H^\infty$ -spaces  $H_v^\infty$  on the open unit disc of the complex plane. The weights  $v$  are also assumed radial and to satisfy the condition (B) introduced by the second named author. The main technique uses Taylor coefficient multipliers, and the results are first proved for them. We formulate a related sufficient condition for the boundedness and compactness of Toeplitz operators in reflexive weighted Bergman spaces on the disc.

We also construct a bounded harmonic symbol  $f$  such that  $T_f$  is not bounded in  $H_v^\infty$  for any  $v$  satisfying mild assumptions. As a corollary, the Bergman projection is never bounded with respect to the corresponding weighted sup-norms. However, we also show that, for normal weights  $v$ , all Toeplitz operators with a trigonometric polynomial as the symbol are bounded on  $H_v^\infty$ .

## 1. INTRODUCTION AND PRELIMINARIES ON TOEPLITZ OPERATORS.

In this paper we consider the boundedness and compactness of Toeplitz operators on weighted sup-normed spaces of holomorphic functions  $H_v^\infty$  on the open unit disc  $\mathbb{D}$  of the complex plane. By a weight  $v$  we mean here a continuous function  $\mathbb{D} \rightarrow ]0, \infty[$  which is radial, vanishing on the boundary and decreasing with the radius, i.e. there holds  $v(z) = v(|z|)$  for all  $z \in \mathbb{D}$ ,  $\lim_{|z| \rightarrow 1} v(z) = 0$  and  $v(r) \geq v(s)$  if  $1 > s > r > 0$ . Moreover let  $\mu$  be the area measure  $dA$  on  $\mathbb{D}$  multiplied with  $v$  as density, i.e.  $d\mu(re^{i\varphi}) = v dA := v(r)rdrd\varphi$ , where  $r, \varphi$  are the polar coordinates of the complex plane. For  $1 \leq p < \infty$  consider first the spaces

$$L_v^p = \left\{ g : \mathbb{D} \rightarrow \mathbb{C} \text{ measurable} : \|g\|_{p,v}^p := \int_{\mathbb{D}} |g|^p d\mu < \infty \right\} \text{ and}$$

$$A_v^p = \{ h \in L_v^p : h \text{ holomorphic} \},$$

which are denoted by  $L^p = (L^p, \|\cdot\|_p)$  and  $A^p$ , respectively in the non-weighted case ( $v$  is replaced by the constant 1). The *Bergman space*  $A_v^p$  is a closed subspace (see below) of  $L_v^p$ , and the *Bergman projection*  $P_v$  is defined as the orthogonal projection of  $L_v^2$  onto  $A_v^2$ . In the non-weighted case it has the integral representation

$$Pg(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{g(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta).$$

For an integrable function  $f \in L^1$ , the *Toeplitz operator*  $T_f$  with symbol  $f$  is defined in the space  $A_v^p \ni h$  by

$$(1.1) \quad T_f h(z) = P_v(fh)(z),$$

if the expression on the right-hand side makes sense. (We will soon comment this in detail for our case.) If, for example, the projection  $P_v$  is also a bounded operator  $L_v^p \rightarrow A_v^p$ ,  $1 < p < \infty$ , it follows that  $T_f : A_v^p \rightarrow A_v^p$  is bounded, whenever  $f$  is a

bounded measurable function. The question of the boundedness of  $T_f$  on  $A_v^p$  with unbounded symbols is a long-standing, still open problem. Examples of unbounded symbols inducing bounded Toeplitz operators can be easily constructed, since the behaviour of the symbol inside any compact subset of  $\mathbb{D}$  is not important for the boundedness of the operator. We refer to the papers [6], [7], [8], [10], [11], [16], [17], [19], [21], [22], [23], [24], [25], [26], [27], [28], [29] for classical and recent results on the boundedness and compactness of Toeplitz operators on Bergman spaces. In particular a solution to the boundedness problem is known in the case of radial symbols, if  $p = 2$ , also in many weighted cases and higher dimensional domains in place of  $\mathbb{D}$ . Also, the case of a positive symbol  $f(z) \geq 0 \forall z \in \mathbb{D}$  can be treated with the help of the Berezin transform. The references above mostly concern unweighted Bergman spaces or spaces  $A_v^p$  with standard weights like  $v(z) = (1 - |z|^2)^\alpha$ ,  $\alpha > 0$ .

In the present paper, Theorem 3.6, we will provide a characterization of the boundedness and compactness of  $T_f$  in the case  $p = \infty$  and for a general class of weights. Thus, in the case of radial symbols this question remains unsolved only for  $1 \leq p < 2$  and  $2 < p < \infty$ ; however, our necessary and sufficient condition (3.7) seems quite a lot more complicated for  $p = \infty$  than for  $p = 2$  (see (5.1)). Anyway, both conditions involve the Hausdorff moments of the symbol on the unit interval, see (3.23), (3.24). (We do not have an idea how to interpolate between the two cases. The situation is somewhat analogous to the problem of describing the solid hulls and cores of the spaces  $A_v^p$  and  $H_v^\infty$ , see [2]. Both of these objects can be described in the cases  $p = 2$  and  $\infty$ , but for other  $p$  there are only partial results. Actually, this phenomenon is not completely unrelated with Toeplitz operators.)

We define the Banach spaces to be considered by

$$h_v^\infty = \{h : \mathbb{D} \rightarrow \mathbb{C} : h \text{ harmonic, } \|h\|_v := \sup_{z \in \mathbb{D}} |h(z)|v(|z|) < \infty\}$$

and

$$H_v^\infty = \{h \in h_v^\infty : h \text{ analytic}\};$$

we use the standard notation  $H^\infty = (H^\infty, \|\cdot\|_\infty)$  in the non-weighted case. We need to comment the definition of Toeplitz operators in the case of  $H_v^\infty$ . In the Hilbert spaces  $L_v^2$  and  $A_v^2$  we denote the inner product by

$$\langle f, g \rangle = \int_{\mathbb{D}} f \bar{g} d\mu$$

Then, the functions  $e_k(z) = \Gamma_{2k}^{-1/2} z^k$ , where  $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$  and

$$(1.2) \quad \Gamma_k = 2\pi \int_0^1 r^{k+1} v(r) dr \quad \text{for } k \in \mathbb{N}_0,$$

form an orthonormal basis of  $A_v^2$ . We remark that the numbers  $\Gamma_k$  satisfy for all  $0 < \varrho < 1$  and some constant  $C_{v,\varrho} > 0$  the following lower bound

$$(1.3) \quad \Gamma_k \geq C_{v,\varrho} \varrho^k$$

for every  $k \in \mathbb{N}_0$ . This follows from (1.2) by considering the integral e.g. over the interval  $[\varrho, 1 - (1 - \varrho)/2]$  only.

Convergence in the space  $A_v^p$ ,  $1 < p < \infty$ , with respect to the norm  $\|\cdot\|_{p,v}$  implies pointwise convergence (hence  $A_v^p$  is a closed subspace of  $L_v^p$ ), and thus the point evaluation functionals at any point of  $\mathbb{D}$  are bounded functionals on  $A_v^p$ . Consequently,

we find the reproducing kernel, i.e. a family of functions  $K_z \in A_v^2$ ,  $z \in \mathbb{D}$ , such that

$$(1.4) \quad g(z) = \langle g, K_z \rangle = \int_{\mathbb{D}} g(w) \overline{K_z(w)} d\mu(w)$$

for all  $g \in A_v^2$ . The integral operator defined by the right hand side can be extended to  $L_v^2$ , and it actually defines the orthogonal projection from  $L_v^2$  onto  $A_v^2$ , i.e. the Bergman projection  $P_v$ ; see [4], [5]. Using the orthonormal basis we can write for all  $z \in \mathbb{D}$

$$(1.5) \quad P_v g(z) = \sum_{k=0}^{\infty} \langle g, e_k \rangle e_k(z) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{z^k \overline{w^k}}{\Gamma_k} g(w) d\mu(w).$$

Here, the order of the summation and the integral can be changed, because (1.3) leads for any fixed  $z \in \mathbb{D}$  to the estimate

$$(1.6) \quad \left| \frac{z^k \overline{w^k}}{\Gamma_k} \right| \leq c_{v,\varrho} \left( \frac{|z|}{\varrho} \right)^k,$$

and we can choose here  $\varrho > |z|$  so that the sum on the right-hand side of (1.5) converges well enough. Moreover, the estimate (1.6) implies that for every  $z \in \mathbb{D}$  the Bergman kernel  $K_z$  is a bounded function:

$$(1.7) \quad |K_z(w)| \leq C_z \quad \text{for all } w \in \mathbb{D}.$$

Now let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a function which belongs to  $L^1$ . We define the Toeplitz operator  $T_f$  with symbol  $f$  on  $H_v^\infty$  by

$$(1.8) \quad T_f(h) = \int_{\mathbb{D}} f(w) h(w) \overline{K_z(w)} d\mu(w).$$

It follows from (1.7) that the integral converges for all  $z \in \mathbb{D}$  and for all  $h \in H_v^\infty$ , since by definition  $h\nu \in L^\infty$ . Although  $T_f h$  of (1.8) is a well-defined holomorphic function it might not be an element of  $H_v^\infty$  and  $T_f$  might not be a bounded operator  $H_v^\infty \rightarrow H_v^\infty$ . Actually it is an elementary consequence of the closed graph theorem that  $T_f$  is a bounded operator  $H_v^\infty \rightarrow H_v^\infty$  if and only if  $T_f(H_v^\infty) \subset H_v^\infty$ . (We remark that the a priori assumption  $f \in L^1$  is usual also in the theory of Toeplitz operators in the reflexive Bergman spaces, but in that case this assumption does not guarantee that the defining integral (1.8) converges for all  $h \in A_v^p$ . From this point of view, the case  $p = \infty$  is more simple.)

If  $h \in H_v^\infty$  is such that  $f \cdot h \in L_v^2$ , we also have

$$(1.9) \quad \begin{aligned} (T_f h)(z) &= \sum_{n=0}^{\infty} \langle f \cdot h, e_n \rangle e_n(z) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_{\mathbb{D}} f(w) h(w) \overline{w^n} \nu(w) dA, \end{aligned}$$

where the series converges in  $L_v^2$ . However, the formula also holds for all  $h \in H_v^\infty$  (since we are assuming  $f \in L^1$ ) and the product  $f h \nu$  thus belongs to  $L^1$ , and one can commute the summation and integration in (1.8), due to (1.6). In the latter case, the sum (1.9) converges uniformly for  $z$  in compact subsets of the disc.

**Remark 1.1.** Let us make some general comments on the definition of Toeplitz operators. In the case of Bergman spaces with weighted  $L^2$ -norms, the way of defining Toeplitz operators is clear, namely, by using the uniquely defined orthogonal projection from  $L_v^2$  onto the closed subspace of analytic functions. Concerning the definition of Toeplitz operators in Banach spaces which are not Hilbert spaces, one naturally proceeds in the same way, if the orthogonal projection is still a bounded operator in the space in question; see also the above remark on the use of the closed graph theorem.

This is however not the case here: in [5] it was shown that for the exponentially decreasing weight  $v(z) = \exp(-1/(1 - |z|))$ , the orthogonal projection  $L_v^2 \rightarrow A_v^2$  is bounded in  $L_v^p$  if and only if  $p = 2$ . We also extend this result here in Corollary 2.4. Moreover, in [14] the second named author proved that the mere existence of a bounded projection from  $L_v^\infty$  onto  $H_v^\infty$  is equivalent to  $H_v^\infty$  being isomorphic to the Banach space  $\ell^\infty$  which in turn is equivalent to  $v$  satisfying a so called condition (B); see Definition 3.1, below. For example, the exponentially decreasing weight mentioned above satisfies (B), but there also exist natural weights which do not, like  $v(z) = (1 - \log(1 - |z|))^{-1}$  (see the statement after Theorem 1.2. of [14] and Example 2.4. of the same paper for other examples). Even if condition (B) is satisfied, there does not exist a natural choice of a bounded projection from  $L_v^\infty$  onto  $H_v^\infty$ . The result in [14] is basically an existence proof for the bounded projection and does not contain an explicit formula. Formulas for these projections were presented in [15], but apparently there is no obvious choice for a canonical one, which could replace the orthogonal projection. In view of these remarks it is natural to consider Toeplitz operators defined via the orthogonal projection of  $L_v^2$ , since this can be done in all  $H_v^\infty$ -spaces and the definition is in accordance with the tradition in the simple cases like the standard weights  $v(r) = (1 - r)^\alpha$ ,  $\alpha > 0$ ; see [28]. The symbol needs to satisfy non-trivial, relatively strong conditions in order to make the Toeplitz operator bounded, cf. Theorem 2.3.

As for the contents of this paper, we recall in Section 2 the known fact that a Toeplitz operator  $T_f$  on  $H_v^\infty$  with holomorphic symbol is a bounded operator  $H_v^\infty \rightarrow H_v^\infty$  if and only if  $f$  is an element of  $H^\infty$ . The situation is completely different if harmonic symbols are considered instead of holomorphic ones. In particular we construct in Theorem 2.3 a bounded harmonic function  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that  $T_f$  is unbounded on  $H_v^\infty$  for every weight  $v$ . This result also has the consequence, Corollary 2.4, that the Bergman projection  $P_v$  is never bounded on  $L_v^\infty$ .

Moreover in Section 3 we give our characterization of the continuity and compactness of Toeplitz operators with radial symbols in  $H_v^\infty$ , see Theorem 3.6. Toeplitz operators with radial symbols are nothing but Taylor coefficient multipliers. They are studied at first in Section 3. Our main result for multipliers, Theorem 3.4, is a generalization of a result in [13].

The negative result of Theorem 2.3 motivates the further studies of Toeplitz operators with bounded symbols in Section 4. We show, among other things that any Toeplitz operator with a trigonometric polynomial as the symbol is bounded, at least if the weight is normal. In Section 5 we put the condition (3.7) of Theorem 3.6 into a form which is natural for the Bergman spaces  $A_v^p$ ,  $1 < p < \infty$ , and show that the condition is sufficient for the boundedness of  $T_f$  in that case, see Proposition 5.1.

As for the notation on analytic function spaces and operators in them, we refer to [28]. All function spaces are defined over the domain  $\mathbb{D}$  unless otherwise stated. In addition we only remark that  $c, C, C_r$  etc. denote generic positive constants the exact value may change from place to place. We sometimes denote the pointwise multiplication by  $f \cdot h$  for clarity.

## 2. TOEPLITZ OPERATORS WITH HOLOMORPHIC AND HARMONIC SYMBOLS

Let us consider a symbol  $f : \mathbb{D} \rightarrow \mathbb{C}$  which is holomorphic and integrable over the disc, i.e.  $f \in A^1$ . If  $h \in H_v^\infty$ , then  $f \cdot h$  is holomorphic on  $\mathbb{D}$  so that there are numbers  $a_n \in \mathbb{C}$  with

$$(f \cdot h)(re^{i\varphi}) = \sum_{n=0}^{\infty} a_n r^n e^{in\varphi}.$$

This implies in view of (1.8), (1.4) that  $T_f(h) = f \cdot h$ . Hence  $T_f$  is just the pointwise multiplier with symbol  $f$ ; we denote this operator by  $S_f$  as the notation  $M_f$  will be reserved for the coefficient multiplier, see Section 2.

The following result for multiplication operators is known, see [1], and by the above explanation it can also be interpreted as a result for  $T_f$ ,  $f \in A^1$ . This seemingly simple result should be compared with Theorem 2.3, below.

**Proposition 2.1.** *Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be holomorphic. Then  $S_f$  is a bounded operator  $H_v^\infty \rightarrow H_v^\infty$  if and only if  $f \in H^\infty$ . Assuming in addition  $f \in A^1$ , the operator  $T_f$  is bounded  $H_v^\infty \rightarrow H_v^\infty$ , if and only if  $f \in H^\infty$ .*

Since the reference does not contain a proof and since our weights are pretty general, we prove the necessity statement for the multiplier; the other parts are quite trivial. Indeed, if  $S_f$  is continuous on  $H_v^\infty$ , then its transpose map  $S_f^* : (H_v^\infty)^* \rightarrow (H_v^\infty)^*$  is continuous in the dual space  $((H_v^\infty)^*, \|\cdot\|_*)$ . Clearly, given  $z \in \mathbb{D}$ , the point evaluation functional  $\delta_z : f \mapsto f(z)$  belongs to the dual, and we have  $S_f^*(\delta_z) = f(z)\delta_z$  for each  $z \in \mathbb{D}$ . Therefore

$$|f(z)| = \|S_f^*(\delta_z)\|_* / \|\delta_z\|_* \leq \|S_f^*\|_{\text{op}} = \|S_f\|_{\text{op}}$$

for all  $z \in \mathbb{D}$ , where we denoted by  $\|\cdot\|_{\text{op}}$  the operator norm in the relevant spaces. We get  $f \in H^\infty$ .

We have the following corollary.

**Corollary 2.2.** *For any weight  $v$  there is an element  $f \in H_v^\infty \cap L^1$  such that  $T_f$  is unbounded on  $H_v^\infty$ .*

For the same reason as above, let us sketch the proof that the set  $(H_v^\infty \cap L^1) \setminus H^\infty$  is non-empty. First, the usual argument based on Montel's theorem and the assumption on the vanishing of the weight  $v$  on the boundary imply that the embedding  $H^\infty \hookrightarrow H_v^\infty$  is compact. The sequence of monomials  $(z^n)_{n=1}^\infty$  is bounded in  $H^\infty$  and converges to 0 uniformly on compact subsets of  $\mathbb{D}$ , hence  $\|z^n\|_v \rightarrow 0$  as  $n \rightarrow \infty$  (see [20], Section 2.4). Also  $\|z^n\|_1 \rightarrow 0$  and  $\|z^n\|_\infty = 1$  for all  $n$ , by direct calculations.

If the space  $H^\infty$  were equal to  $H_v^\infty \cap L^1$ , the closed graph theorem would yield a constant  $C > 0$  such that

$$(2.1) \quad \|h\|_\infty \leq C \max(\|h\|_v, \|h\|_1)$$

for all  $h \in H^\infty$  (since the converse of the inequality (2.1) holds trivially). We get a contradiction from the above norm estimates for the monomials  $z^n$ .

We proceed to study the case of harmonic symbols, which is much more complicated. Since the Bergman projection is known to be unbounded with respect to the norm  $\|\cdot\|_v$  for many weights  $v$ , one may expect that there are examples of bounded symbols  $f \in L^\infty$  so that  $T_f$  is not a bounded operator from  $H_v^\infty$  into itself. While we do not exactly know such examples in the literature, let us mention Section 5 of [18], where possible pathologies of Toeplitz operators with bounded symbols were considered in the case of reflexive Bergman spaces on polygonal domains. In the following theorem we find a very strong negative example; cf. also Proposition 2.1.

**Theorem 2.3.** *There is a bounded harmonic function  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that  $T_f$  is not a bounded operator  $H_v^\infty \rightarrow H_v^\infty$  for any weight  $v$  on  $\mathbb{D}$ .*

Since the pointwise multiplication with a bounded function  $f$  is always a bounded operator  $H_v^\infty \rightarrow L_v^\infty$ , this result immediately implies the following conclusion.

**Corollary 2.4.** *The Bergman projection  $P_v$  is never (for any weight under consideration) a bounded mapping  $L_v^\infty \rightarrow L_v^\infty$ .*

Namely, if  $P_v$  were bounded, this would imply  $T_f : H_v^\infty \rightarrow H_v^\infty$  is bounded for every  $f \in L^\infty$ , which would contradict Theorem 2.3.

As was already explained in Remark 1.1, in spite of Corollary 2.4 the Banach space  $H_v^\infty$  is quite often complemented in  $L_v^\infty$ , which means that there exists a bounded projection  $P : L_v^\infty \rightarrow H_v^\infty$  different from  $P_v$ . This happens, if and only if  $H_v^\infty$  is isomorphic to  $\ell^\infty$ , and if and only if the weight  $v$  satisfies condition (B) of Definition 3.1. This is true for example for all normal weights, see [14] for details.

To prove Theorem 2.3 we need some lemmas. Fix a weight  $v$  on  $\mathbb{D}$ . Let  $\tilde{f}_0 : \partial\mathbb{D} \rightarrow \mathbb{C}$  be the map with

$$\tilde{f}_0(e^{i\varphi}) = \begin{cases} 1, & \text{if } -\pi/2 \leq \varphi \leq \pi/2 \\ 0 & \text{else.} \end{cases}$$

Then, the following is true.

**Lemma 2.5.** *Let  $f_0$  be the harmonic extension of  $\tilde{f}_0$  on  $\mathbb{D}$ . We have*

$$f_0(z) = \frac{1}{2} + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (z^{2k+1} + \bar{z}^{2k+1}), \quad z \in \mathbb{D}.$$

Clearly,  $f_0$  is bounded on the disc due to the maximum principle.

**Proof.** Let  $a_k, k \in \mathbb{Z}$ , be the Fourier coefficients of  $\tilde{f}_0$ . Then we have

$$\begin{aligned} a_k &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ikt} dt = \frac{e^{ik\pi/2} - e^{-ik\pi/2}}{2k\pi i} = \frac{e^{i|k|\pi/2} - e^{-i|k|\pi/2}}{2|k|\pi i} \\ &= \begin{cases} \frac{(-1)^j}{(2j+1)\pi}, & \text{if } |k| = 2j+1 \\ 0 & \text{else,} \end{cases} \end{aligned}$$

provided that  $k \neq 0$ . Moreover,  $a_0 = 1/2$ . This proves the lemma.  $\square$

**Lemma 2.6.** *Let*

$$f(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}, \quad z \in \mathbb{D},$$

for some  $a_k$ . Put  $(Sf)(z) = (f(z) - if(iz))/2$ . Then

$$(Sf)(z) = \sum_{k=0}^{\infty} a_{4k+1} z^{4k+1} \quad \text{and} \quad \sup_{|z|=r} |(Sf)(z)| \leq \sup_{|z|=r} |f(z)|$$

for all  $r$ .

**Proof.** The first assertion follows from

$$1 - i \cdot i^{2k+1} = 1 + (-1)^k = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

The second assertion is trivial.  $\square$

Consider  $m > 0$  and let  $r_m$  be a point where the function  $r^m v(r)$  attains its absolute maximum on  $[0, 1]$ . We easily see that  $r_n \geq r_m$  if  $n \geq m$  and  $\lim_{m \rightarrow \infty} r_m = 1$ ; see for example [12] for details.

Let us set for all  $m \in \mathbb{N}_0$

$$g_m(re^{i\varphi}) = \frac{r^m e^{im\varphi}}{r_m^m v(r_m)}, \quad re^{i\varphi} \in \mathbb{D}.$$

Then  $\|g_m\|_v = 1$ . Recalling the notation (1.2) for  $\Gamma_k$  we state the following result.

**Lemma 2.7.** *Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be harmonic, say  $f(re^{i\varphi}) = \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\varphi}$ . For all  $m \in \mathbb{N}_0$  we have*

$$(2.2) \quad T_f(g_m)(re^{i\varphi}) = \sum_{k=0}^m b_{k-m} \frac{\Gamma_{2m}}{\Gamma_{2k}} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)} + \sum_{k=m+1}^{\infty} b_{k-m} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)}$$

**Proof.** This follows from

$$\begin{aligned} f(re^{i\varphi}) \cdot g_m(re^{i\varphi}) &= \sum_{j \in \mathbb{Z}} b_j \frac{r^{m+|j|} e^{i(j+m)\varphi}}{r_m^m v(r_m)} \\ &= \sum_{k=m+1}^{\infty} b_{k-m} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)} + \sum_{k=-\infty}^m b_{k-m} \frac{r^{2m-k} e^{ik\varphi}}{r_m^m v(r_m)} \end{aligned}$$

and (1.9).  $\square$

**Proof of Theorem 2.3.** We take  $f_0$  of Lemma 2.5 and show that  $T_{f_0}$  is unbounded on  $H_v^\infty$ . Put

$$f_1(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} (z^{2j+1} + \bar{z}^{2j+1}).$$

It suffices to show that  $T_{f_1}$  is unbounded since  $T_{f_0} = T_{1/2} + \pi^{-1} T_{f_1}$  and  $T_{1/2}$  is bounded. Fix a positive integer  $m$ , say  $m = 4m_0$  for  $m_0 \in \mathbb{N}$ . Then

$$k - m \text{ is } \begin{cases} \text{odd} & \text{if } k \text{ is odd} \\ \text{even} & \text{if } k \text{ is even} \end{cases} \quad \text{and} \quad j - 2m_0 \text{ is } \begin{cases} \text{odd} & \text{if } j \text{ is odd} \\ \text{even} & \text{if } j \text{ is even.} \end{cases}$$

Lemma 2.7 yields with  $b_k = 0$ , if  $k$  is even, and with  $b_k = (-1)^k/|2k + 1|$  if  $k$  is odd

$$T_{f_1}(g_m)(re^{i\varphi}) = \sum_{\substack{k=0, \\ k \text{ odd}}}^m b_{k-m} \frac{\Gamma_{2m}}{\Gamma_{2k}} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)} + \sum_{\substack{k=m+1, \\ k \text{ odd}}}^{\infty} b_{k-m} \frac{r^k e^{ik\varphi}}{r_m^m v(r_m)}.$$

Using Lemma 2.6 we obtain

$$\begin{aligned} S(T_{f_1}(g_m))(re^{i\varphi}) &= \sum_{0 \leq 4j+1 \leq m} b_{4j+1-m} \frac{\Gamma_{2m}}{\Gamma_{8j+2}} \frac{r^{4j+1} e^{i(4j+1)\varphi}}{r_m^m v(r_m)} \\ &+ \sum_{m+1 \leq 4j+1 < \infty} b_{4j+1-m} \frac{r^{4j+1} e^{i(4j+1)\varphi}}{r_m^m v(r_m)}. \end{aligned}$$

Recall that  $b_{4j+1-m} = 1/|4(j - m_0) + 1|$ . So if we take  $\varphi = 0$  then all summands in the preceding sum are non-negative. Hence

$$\begin{aligned} \frac{r_m}{5} \log \left( \frac{1}{1 - r_m^4} \right) &= \frac{r_m}{5} \sum_{j=1}^{\infty} \frac{(r_m^4)^j}{j} \leq \sum_{j=0}^{\infty} \frac{r_m^{4j+1}}{4j+1} \\ &= \sum_{m+1 \leq 4j+1 < \infty} b_{4j+1-m} \frac{r_m^{4j+1} v(r_m)}{r_m^m v(r_m)} \leq S(T_{f_1}(g_m))(r_m) v(r_m) \\ &\leq \|S(T_{f_1}(g_m))\|_v \leq \|T_{f_1}(g_m)\|_v. \end{aligned}$$

Since  $\lim_{m \rightarrow \infty} r_m = 1$ , the left-hand side of the preceding estimate grows to the infinity, when  $m \rightarrow \infty$ . Hence  $T_{f_1}$  and also  $T_{f_0}$  cannot be bounded.  $\square$

### 3. MULTIPLIERS FROM $h_v^\infty$ INTO $H_v^\infty$ AND TOEPLITZ OPERATORS

Toeplitz operators with radial (thus in general non-harmonic) symbols on the disc correspond to Taylor coefficient multipliers so we proceed to study them. At first we mention some general results concerning the Banach space  $h_v^\infty$ . These are collected from the references [12], [14] and [15]. We recall that the numbers  $r_m \in ]0, 1[$  were defined above Lemma 2.7.

**Definition 3.1.** (i) The weight  $v$  satisfies the condition (B), if

$$\begin{aligned} \forall b_1 > 1 \exists b_2 > 1 \exists c > 0 \forall m, n > 0 \\ \left( \frac{r_m}{r_n} \right)^m \frac{v(r_m)}{v(r_n)} \leq b_1 \quad \text{and} \quad m, n, |m - n| \geq c \Rightarrow \left( \frac{r_n}{r_m} \right)^n \frac{v(r_n)}{v(r_m)} \leq b_2. \end{aligned}$$

(ii) Also,  $v$  is called *normal* if

$$\sup_{n \in \mathbb{N}} \frac{v(1 - 2^{-n})}{v(1 - 2^{-n-1})} < \infty \quad \text{and} \quad \inf_{k \in \mathbb{N}} \limsup_{n \rightarrow \infty} \frac{v(1 - 2^{-n-k})}{v(1 - 2^{-n})} < 1.$$

Note that in (i),  $m$  and  $n$  need not be integers. Condition (B) is crucial for the structure, in particular for the isomorphic character of  $H_v^\infty$ . Actually it is equivalent to the fact that  $H_v^\infty$  is isomorphic to the Banach space  $\ell^\infty$  of bounded sequences (Theorem 1.1 of [14]). Examples of weights satisfying (B) are all normal weights, in particular the standard weights  $v(r) = (1 - r)^\alpha$  (or  $v(r) = (1 - r^2)^\alpha$ ) where  $\alpha > 0$ . Moreover, for  $\beta > 0$  and  $\gamma > 0$  the weight  $v(r) = \exp(-\gamma/(1 - r)^\beta)$  satisfies (B) but is not normal; see [14].



Fix  $b > 2$ . We define by induction the indices  $0 \leq m_1 < m_2 < \dots$  such that

$$b = \min \left( \left( \frac{r_{m_n}}{r_{m_{n+1}}} \right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \right).$$

This is always possible according to Lemma 5.1. of [14]. (Actually it suffices to choose the indices such that the preceding minimum lies between  $b$  and some constant  $b_1 > b$ .)

Now let the numbers  $b_k \in \mathbb{C}$ ,  $k \in \mathbb{Z}$ , be given and denote by  $h(\varphi) = \sum_{k \in \mathbb{Z}} b_k e^{ik\varphi}$  a series which may or may not converge. We take the preceding numbers  $m_n$  and put for all  $n \in \mathbb{N}$

$$\begin{aligned} (W_n h)(\varphi) &= \sum_{m_{n-1} < |k| \leq m_n} \frac{|k| - [m_{n-1}]}{[m_n] - [m_{n-1}]} b_k e^{ik\varphi} + \sum_{m_n < |k| \leq m_{n+1}} \frac{[m_{n+1}] - |k|}{[m_{n+1}] - [m_n]} b_k e^{ik\varphi} \\ &=: \sum_{k \in \mathbb{Z}} w_{nk} b_k e^{ik\varphi} \end{aligned}$$

where  $m_0 = 0$ . Here  $[r]$  is the largest integer not greater than  $r$ . The operators  $W_n$  are also considered as acting on the harmonic functions by

$$W_n : \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\varphi} \mapsto \sum_{k=-\infty}^{\infty} w_{nk} b_k r^{|k|} e^{ik\varphi}$$

For any function  $g : \mathbb{D} \rightarrow \mathbb{C}$  and radius  $0 \leq r \leq 1$  we denote

$$M_\infty(g, r) = \sup_{|z|=r} |g(z)|.$$

The Riesz projection  $P$  is defined by

$$(3.1) \quad P \left( \sum_{k \in \mathbb{Z}} a_k r^{|k|} e^{ik\varphi} \right) = \sum_{k=0}^{\infty} a_k r^k e^{ik\varphi}.$$

**Theorem 3.2.** *Let  $v$  satisfy (B). Then there are constants  $c_1, c_2 > 0$  such that, for all  $g \in h_v^\infty$ ,*

$$(3.2) \quad c_1 \sup_{n \in \mathbb{N}} M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \leq \|g\|_v \leq c_2 \sup_{n \in \mathbb{N}} M_\infty(W_n g, r_{m_n}) v(r_{m_n})$$

and

$$(3.3) \quad c_1 M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \leq \|W_n g\|_v \leq c_2 M_\infty(W_n g, r_{m_n}) v(r_{m_n})$$

for all  $n$ . Moreover,

$$(3.4) \quad \sup_n (m_{n+1} - m_n) / (m_n - m_{n-1}) < \infty.$$

Finally, the Riesz projection  $P : h_v^\infty \rightarrow H_v^\infty$  is bounded.

This is Theorem 1 of [15]. See also Propositions 4.1. and 5.2. of [14]. One can even show that the boundedness of the Riesz projection in  $h_v^\infty$  is equivalent to (B) (for details, see [14]).

**Remark 3.3.** If a sequence  $(b_k)_{k=-\infty}^\infty$  of complex numbers is given such that

$$\sup_{n \in \mathbb{N}} M_\infty \left( \sum_{k=-\infty}^{\infty} w_{nk} b_k r^{|k|} e^{ik\varphi}, r_{m_n} \right) v(r_{m_n})$$

$$(3.5) \quad = \sup_{n \in \mathbb{N}} \sup_{\varphi \in [0, 2\pi]} \left| \sum_{m_{n-1} < |k| \leq m_{n+1}} w_{nk} b_k r_{m_n}^k e^{ik\varphi} \right| v(r_{m_n}) < \infty,$$

then the series defining the harmonic function  $g(re^{i\varphi}) = \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\varphi}$  converges in the compact-open topology, and  $g$  belongs to  $h_v^\infty$  and  $\|g\|_v$  is bounded by a constant times the expression in (3.5). For this statement, see Remark 1, (iii) of [15].

**Examples.** If  $v$  is normal then one can take  $m_n = 2^{kn}$  for suitable fixed  $k > 0$  (see [14], Example 2.4., and [12]). For  $v(r) = \exp(-\gamma/(1-r)^\beta)$  one can take  $m_n = \beta^2(\beta/\gamma)^{1/\beta} n^{2+2/\beta} - \beta^2 n^2$ , see [2].

We next turn to a theorem which was proven for a more restricted class of weights in Theorem 4.1 of [13]. In the theorem we assume that a sequence  $(\gamma_k)_{k=0}^\infty$  of complex numbers is given, and consider the formal series  $f(\varphi) = \sum_{k=0}^\infty \gamma_k e^{ik\varphi}$  and the multiplier  $M_f$  with

$$(3.6) \quad (M_f h)(re^{i\varphi}) = \sum_{k=0}^\infty \gamma_k b_k r^k e^{ik\varphi}$$

for harmonic functions  $h(re^{i\varphi}) = \sum_{k=-\infty}^\infty b_k r^{|k|} e^{ik\varphi}$ . By definition,  $M_f h$  is holomorphic, if the series (3.6) converges.

**Theorem 3.4.** *Let the weight  $v$  satisfy condition (B). Then  $M_f$  maps  $h_v^\infty$  into  $H_v^\infty$  and is bounded, if and only if*

$$(3.7) \quad \sup_{n \in \mathbb{N}} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi < \infty.$$

Moreover, assume (3.7) holds. Then  $M_f : h_v^\infty \rightarrow H_v^\infty$  is compact, if and only if

$$(3.8) \quad \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

**Proof.** Assume (3.7) holds. We first remark that  $M_f$  is a convolution operator, i.e. at least in the case of only finitely many non-zero entries  $\gamma_k$ , the expression (3.6) can be written as

$$(M_f h)(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi - \psi) h(re^{i\psi}) d\psi.$$

So, if  $h \in h_v^\infty$ , then we have for all  $re^{i\varphi} \in \mathbb{D}$

$$(3.9) \quad |(M_{W_n f} h)(re^{i\varphi})| v(r) \leq \frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi \|h\|_v.$$

Hence,

$$M_\infty(M_{W_n f} h, r)v(r) \leq C \|h\|_v$$

for all  $n$  and  $r$ , where the constant  $C > 0$  is the supremum on the left-hand side of (3.7). This bound and Remark 3.3 imply that the series on the right-hand side

of (3.6) converges in the compact-open topology, defines an element of  $H_v^\infty$  and is bounded by  $\|h\|_v$ . In other words,  $M_f$  maps  $h_v^\infty$  continuously into  $H_v^\infty$ .

As for compactness of the operator  $M_f$ , let  $(h_j)_{j=1}^\infty$  be a sequence which converges to 0 uniformly on compact subsets of  $\mathbb{D}$  and which is contained in the closed unit ball of  $h_v^\infty$ . It suffices to show that  $M_f$  maps such a sequence into a one converging to 0 with respect to the norm; see for example [20], Section 2.4. Let  $\varepsilon > 0$ . If (3.8) is assumed, we can fix  $N \in \mathbb{N}$  such that

$$(3.10) \quad \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi < \varepsilon$$

for  $n > N$ . Moreover, we note that for every  $n \in \mathbb{N}$ , the operator

$$W_n M_f : \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\varphi} = \sum_{k=0}^{\infty} w_{nk} \gamma_k b_k r^k e^{ik\varphi}$$

is bounded in the space  $h_v^\infty$  when this space is endowed with the norm

$$(3.11) \quad \sup_{|z| \leq r_{m_n}} |h(z)|;$$

to see this notice that every functional

$$g \mapsto r_{m_n}^{-k} \int_0^{2\pi} g(r_{m_n} e^{ik\varphi}) e^{-ik\varphi} d\varphi, \quad g \in h_v^\infty,$$

is bounded with respect to the norm (3.11) on  $h_v^\infty$ , and  $W_n M_f$  is a finite linear combination of these functionals. Consequently, due to the uniform convergence on compact sets, we can choose a large enough  $J = J(N) \in \mathbb{N}$  such that

$$(3.12) \quad \sup_{|z| \leq r_{m_n}} |W_n M_f h_j(z)| v(z) < \varepsilon$$

for all  $n \leq N$ , all  $j \geq J$ . For such  $j$  we obtain by Theorem 3.2

$$(3.13) \quad c_2^{-1} \|M_f h_j\|_v \leq \sup_{n \leq N} M_\infty(W_n M_f h_j, r_{m_n}) v(r_{m_n}) + \sup_{n > N} M_\infty(W_n M_f h_j, r_{m_n}) v(r_{m_n})$$

The first term on the right-hand side of (3.13) is bounded by  $\varepsilon$  due to (3.12), and the second one can be estimated in the same way as in (3.9), and (3.10) implies that this term is bounded by  $\varepsilon$ . Thus,  $M_f$  is compact.

To prove the necessity of (3.7) for the boundedness, we fix an arbitrary  $0 < \varepsilon < 1$ , and  $n \in \mathbb{N}$  and  $\psi \in [0, 2\pi]$  and find, by, for example, the Fejer approximation theorem, a trigonometric polynomial  $h$ , depending on  $n, \psi$  and  $\varepsilon$ ,

$$h(re^{i\varphi}) = \sum_{k \in \mathbb{Z}} h_k r^{|k|} e^{ik\varphi}$$

such that

$$(3.14) \quad \left| h(r_{m_n} e^{i\varphi}) - \frac{\overline{W_n f(\psi - \varphi)}}{|W_n(\psi - \varphi)| v(r_{m_n})} \right| < \frac{\varepsilon}{v(r_{m_n})}$$

for all  $\varphi \in [0, 2\pi]$ , in particular

$$(3.15) \quad M_\infty(h, r_{m_n}) v(r_{m_n}) \leq 2.$$

As a consequence,

$$\begin{aligned}
& \frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi = \frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\psi - \varphi)| d\varphi \\
& \leq \frac{1}{2\pi} \left| \int_0^{2\pi} (W_n f)(\psi - \varphi) h(r_{m_n} e^{i\varphi}) d\varphi \right| v(r_{m_n}) + \varepsilon \\
& = \frac{1}{2\pi} \left| \int_0^{2\pi} f(\psi - \varphi) (W_n h)(r_{m_n} e^{i\varphi}) d\varphi \right| v(r_{m_n}) + \varepsilon \\
(3.16) \quad & = |M_f W_n h(r_{m_n} e^{i\psi})| v(r_{m_n}) + \varepsilon.
\end{aligned}$$

We obtain

$$(3.17) \quad \frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi \leq \|M_f\| \cdot \|W_n h\|_v + \varepsilon.$$

For any  $r > 0$ , Lemma 3.3. of [14] implies

$$(3.18) \quad M_\infty(W_n h, r) \leq 4 \left( \frac{[m_{n+1}] - [m_{n-1}]}{[m_n] - [m_{n-1}]} \right) \left( 3 + 4 \frac{[m_{n+1}] - [m_{n-1}]}{[m_{n+1}] - [m_n]} \right) M_\infty(h, r).$$

Due to Theorem 3.2 (in particular (3.4)) and (3.15) we find a universal constant  $d > 0$  such that

$$\|W_n h\|_v \leq c_2 M_\infty(W_n h, r_{m_n}) v(r_{m_n}) \leq c_2 d M_\infty(h, r_{m_n}) v(r_{m_n}) \leq 2c_2 d.$$

Hence  $\sup_n \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi < \infty$ .

Finally, to prove the necessity of the condition (3.8) for the compactness of  $M_f$ , we first observe that given any  $k \in \mathbb{N}$  we have, for all  $r \leq r_k$ ,

$$(3.19) \quad \left( \frac{r}{r_{m_n}} \right)^{m_n} \frac{v(r)}{v(r_{m_n})} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

To see this, fix  $k$  for a moment and denote for all  $m \in \mathbb{N}$  and  $r \in [0, 1[$

$$G_m(r) = \left( \frac{r}{r_m} \right)^m \frac{v(r)}{v(r_m)} \quad \text{and} \quad c_k = \sup_{r \leq r_k} G_k(r) < \infty.$$

For all  $r \leq r_k$  we get

$$\begin{aligned}
\frac{G_{m_n}(r)}{G_{k+1}(r)} &= \left( \frac{r}{r_{m_n}} \right)^{m_n} \left( \frac{r_{k+1}}{r} \right)^{k+1} \frac{v(r)}{v(r_{m_n})} \frac{v(r_{k+1})}{v(r)} \\
&= \left( \frac{r}{r_{k+1}} \right)^{m_n - (k+1)} \frac{r_{k+1}^{m_n} v(r_{k+1})}{r_{m_n}^{m_n} v(r_{m_n})} \leq \left( \frac{r}{r_{k+1}} \right)^{m_n - k - 1},
\end{aligned}$$

where the last inequality follows from the definition that  $r_{m_n}$  is the maximum point of the function  $r^{m_n} v(r)$ . We see that (3.19) holds, since

$$\sup_{r \leq r_k} G_{m_n}(r) \leq c_k \sup_{r \leq r_k} \frac{G_{m_n}(r)}{G_{k+1}(r)} \leq c_k \sup_{r \leq r_k} \left( \frac{r}{r_{k+1}} \right)^{m_n - k} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We next choose for every  $n$  and  $\psi$  the trigonometric polynomial  $h =: h_{n,\psi}$  with  $\varepsilon = 1/n$ , as in (3.14). As a consequence of Theorem 3.2, (3.15) and Lemma 3.3. of [14] (cf. (3.18)),

$$(3.20) \quad \begin{aligned} M_\infty(W_n h_{n,\psi}, r_{m_{n-1}})v(r_{m_{n-1}}) &\leq \|W_n h_{n,\psi}\|_v \\ &\leq c_2 M_\infty(W_n h_{n,\psi}, r_{m_n})v(r_{m_n}) \leq C M_\infty(h_{n,\psi}, r_{m_n})v(r_{m_n}) \leq C'. \end{aligned}$$

Let us again fix  $k \in \mathbb{N}$ . We claim that for every  $\delta > 0$  there exists  $N$  such that

$$(3.21) \quad \sup_{|z| \leq r_k} |W_n h_{n,\psi}|v(z) < \delta$$

for all  $n \geq N$  and  $\psi$ . To see this, notice that the smallest power of  $r$  in the trigonometric polynomial  $W_n h_{n,\psi}$  is  $m_{n-1}$ , hence, assuming  $n$  is so large that  $m_{n-1} > k$ , Lemma 3.1.(b) of [14] yields for all  $|z| \leq r_k$

$$(3.22) \quad |W_n h_{n,\psi,\varepsilon}(z)| \leq 2 \left( \frac{|z|}{r_{m_{n-1}}} \right)^{m_{n-1}} M_\infty(W_n h_{n,\psi}, r_{m_{n-1}}).$$

We obtain (3.21) for large enough  $N$  by using (3.22), (3.19), (3.20), since

$$\begin{aligned} &\sup_{|z| \leq r_k} |W_n h_{n,\psi,\varepsilon}(z)|v(z) \\ &\leq 2 M_\infty(W_n h_{n,\psi}, r_{m_{n-1}})v(r_{m_{n-1}}) \sup_{|z| \leq r_k} \left( \frac{|z|}{r_{m_{n-1}}} \right)^{m_{n-1}} \frac{v(z)}{v(r_{m_{n-1}})}. \end{aligned}$$

In other words, the functions  $W_n h_{n,\psi}$  form a sequence converging to zero uniformly on compact subset of the open disc (and also uniformly with respect to  $\psi$ ). Fixing  $\psi \in [0, 2\pi]$ , the compact operator  $M_f$  maps the sequence  $(W_n h_{n,\psi})_{n=1}^\infty$  into a sequence converging to 0 in the norm. Taking this into account in the estimate (3.16)–(3.17), we get (3.8).  $\square$

**Corollary 3.5.** *Let the weight  $v$  satisfy condition (B). Then  $M_f$  maps  $H_v^\infty$  continuously into  $H_v^\infty$  if and only if*

$$\sup_n \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi < \infty.$$

Moreover, a bounded  $M_f : H_v^\infty \rightarrow H_v^\infty$  is compact, if and only if

$$\sup_{n \rightarrow \infty} \int_0^{2\pi} |(W_n f)(\varphi)| d\varphi = 0.$$

**Proof.** The sufficiency follows directly from Theorem 3.4. As for necessity, note that the Riesz projection  $P$  is bounded, by the assumption and Theorem 3.2. Thus, if  $M_f|_{H_v^\infty}$  is bounded then  $M_f$  is also bounded on  $h_v^\infty$ , and the necessary condition follows from Theorem 3.4. The statements concerning compactness can be proven by analogous arguments.  $\square$

Now we go back to Toeplitz operators. Let  $T_a$  be a Toeplitz operator on  $H_v^\infty$  with a given radial symbol  $a \in L^1$ , i.e. where  $a(z) = a(|z|)$  for all (almost every)  $z$ . Then,

with  $h(z) = \sum_{n=0}^{\infty} h_n z^n \in H_v^\infty$ , (1.9) reduces to

$$\begin{aligned} T_a h(z) &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_0^1 \int_0^{2\pi} a(r) h(re^{i\varphi}) r^{n+1} e^{-in\varphi} v(r) d\varphi dr \\ (3.23) \quad &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_0^1 a(r) r^{2n+1} v(r) h_n dr = \sum_{n=0}^{\infty} \gamma_n h_n z^n = M_{f_a} h(z) \end{aligned}$$

where

$$(3.24) \quad \gamma_n = \frac{1}{\Gamma_{2n}} \int_0^1 r^{2n+1} v(r) a(r) dr \quad \text{and} \quad f_a(\varphi) = \sum_{k=0}^{\infty} \gamma_k e^{ik\varphi}.$$

We obtain by Corollary 3.5

**Theorem 3.6.** *Let the weight satisfy (B). If  $a \in L^1$  is radial then  $T_a$  is bounded as operator  $H_v^\infty \rightarrow H_v^\infty$  if and only if*

$$\sup_n \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi < \infty,$$

and  $T_a$  is a compact operator  $H_v^\infty \rightarrow H_v^\infty$ , if and only if

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi = 0.$$

#### 4. MORE ON TOEPLITZ OPERATORS.

Since it was observed above that the boundedness of a symbol is not enough to guarantee the boundedness of the Toeplitz operator, we present in this section some complementary results and examples on this topic; see also the remark at the end of this section. In the following we denote by  $Q_m$ ,  $m \in \mathbb{N}_0$ , the projection

$$(4.1) \quad Q_m \left( \sum_{k=0}^{\infty} b_k e^{ik\varphi} \right) = \sum_{k=0}^m b_k e^{ik\varphi} \quad \text{or} \quad Q_m \left( \sum_{k=0}^{\infty} b_k z^k \right) = \sum_{k=0}^m b_k z^k$$

It is well-known that

$$\left| Q_m \left( \sum_{k=0}^{\infty} c_k e^{ik\varphi} \right) \right| \leq d \log m \sup_{0 \leq \psi \leq 2\pi} \left| \sum_{k=0}^{\infty} c_k e^{ik\psi} \right|$$

where  $d > 0$  is a universal constant independent of  $m$  and  $c_k$ .

At first we show

**Theorem 4.1.** *Let  $a_j \in L^1$ ,  $j = -n, \dots, n$ , be radial functions and define*

$$f(z) = \sum_{j=-n}^n a_j(|z|) z^j, \quad z \in \mathbb{D} \setminus \{0\},$$

and  $f_j(z) = a_j(|z|) z^j$ . Then the following are equivalent:

- (i)  $T_f$  is bounded on  $H_v^\infty$ .
- (ii)  $T_{f_j}$  are bounded on  $H_v^\infty$  for all  $j$ .

(iii)  $T_{a_j}$  are bounded on  $H_v^\infty$  for  $j = -n, \dots, 0$  and  $T_{a_j} \circ (id - Q_{j-1})$  are bounded on  $H_v^\infty$  for  $j = 1, \dots, n$ .

(iv) The multipliers  $M_{g_j}$  are bounded on  $H_v^\infty$  for all  $j$  where

$$g_j(\varphi) = \sum_{k=\max(j,0)}^{\infty} \frac{1}{\Gamma_{2k}} \int_0^1 r^{2k+1} a_j(r) v(r) dr e^{ik\varphi}.$$

We prove Theorem 4.1 below. Notice that any  $f \in L^1$  can be expanded as follows:

$$f(z) \sim \sum_{j=-\infty}^{\infty} a_j(|z|) z^j,$$

for some radial functions  $a_j$ . (Expand  $f(re^{i\varphi})$  into a Fourier series for each fixed  $r \in [0, 1[$ .)

**Example.** Let  $f(z) = 1/z$ . Then  $f \in L^1$  and  $T_f$  is bounded according to Theorem 4.1. since  $f(z) = 1 \cdot z^{-1}$  and  $T_1 = id$  is bounded but  $f$  is unbounded. This is no contradiction to Theorem 2.1 since  $f$  is not holomorphic in 0.

Let  $h(z) = \sum_{k=0}^{\infty} b_k z^k$  for given constant coefficients  $b_k$ . Notice that we have

$$(4.2) \quad |b_k| r^k v(r) = \frac{1}{2\pi} \left| \int_0^{2\pi} e^{-ik\varphi} h(re^{i\varphi+i\psi}) d\psi \right| v(r) \leq |h(re^{i\varphi})| v(r) \leq \|h\|_v$$

for each  $r$ . For  $j \in \mathbb{Z}$  we introduce the shift

$$S_j(h)(z) = \sum_{k=-\min(j,0)}^{\infty} b_k z^{k+j}.$$

For  $\psi \in \mathbb{R}$  let  $R_\psi$  be the translation

$$R_\psi(h)(z) = h(e^{i\psi} z) = \sum_{k=0}^{\infty} b_k z^k e^{ik\psi}.$$

**Lemma 4.2.**  $R_\psi$  and  $S_j$  are bounded operators on  $H_v^\infty$ . Moreover, we have

$$S_j S_{-j} = \begin{cases} \text{id}_{H_v^\infty}, & \text{if } j \leq 0 \\ (\text{id}_{H_v^\infty} - Q_{j-1}), & \text{if } j > 0. \end{cases}$$

**Proof.** The boundedness of  $R_\psi$  is a direct consequence of the definition. If  $j \geq 0$  then  $S_j(h)(z) = z^j h(z)$  and hence  $S_j$  is bounded.

Now let  $j < 0$ . Put  $h(z) = \sum_{k=0}^{\infty} b_k z^k$ . Then

$$S_j(h)(z) = \sum_{k=|j|}^{\infty} b_k z^{k-|j|} = z^j ((\text{id}_{H_v^\infty} - Q_{|j|-1})h)(z).$$

Hence, if  $|z| > 1/2$  then  $|S_j(h)(z)| v(|z|) \leq 2d_1(1 + \log |j|) \|h\|_v$  for some universal constant  $d_1$ . By the preceding and (4.2), where  $r = 3/4$ , we have for  $|z| \leq 1/2$ ,

$$|S_j(h)(z)| v(|z|) \leq \sum_{k=|j|}^{\infty} |b_k| \frac{1}{2^{k-|j|}} v(0)$$

$$\leq \sum_{k=|j|}^{\infty} \frac{\|h\|_v}{v(3/4)} \left(\frac{4}{3}\right)^k \left(\frac{1}{2}\right)^{k-|j|} v(0) = 3 \left(\frac{4}{3}\right)^{|j|} \frac{v(0)}{v(3/4)} \|h\|_v.$$

Thus  $S_j$  is bounded. The last identities of Lemma 4.2 follow from the definition.  $\square$

**Proof of Theorem 4.1** The implication (ii)  $\Rightarrow$  (i) follows from the fact that  $T_f = \sum_{j=-n}^n T_{f_j}$ .

Using (3.23) we see that, with  $h(z) = \sum_{k=0}^{\infty} b_k z^k$ , we have

$$(4.3) \quad T_{f_j}(h)(z) = \sum_{k=-\min(j,0)}^{\infty} \frac{1}{\Gamma_{k+j}} \int_0^1 a_j(r) r^{k+j+1} v(r) dr b_k z^{k+j} = T_{a_j} S_j(h)(z)$$

so that (iii)  $\Rightarrow$  (ii) follows from (4.3) and Lemma 4.2.

For (i)  $\Rightarrow$  (iii) we note that (1.9) implies

$$\begin{aligned} T_f(R_\psi h)(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi\Gamma_{2n}} \int_0^{2\pi} \int_0^1 f(re^{i\varphi}) h(re^{i\varphi+i\psi}) r^{n+1} e^{-in\varphi} v(r) dr d\varphi z^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi\Gamma_{2n}} \int_0^{2\pi} \int_0^1 f(re^{i\varphi-i\psi}) h(re^{i\varphi}) r^{n+1} e^{-in+i\psi\varphi} v(r) dr d\varphi z^n \\ &= T_{R_{-\psi}f}(h)(e^{i\psi}z) = \sum_{j=-n}^n e^{-ij\psi} T_{f_j}(h)(e^{i\psi}z). \end{aligned}$$

This yields

$$T_{f_j}(h)(z) = \frac{1}{2\pi} \int_0^{2\pi} R_{-\psi} T_f(R_\psi h)(z) e^{ij\psi} d\psi$$

and hence

$$\|T_{f_j}(h)\|_v \leq \|T_f\| \cdot \|R_\psi h\|_v = \|T_f\| \cdot \|h\|_v.$$

Therefore  $T_{f_j}$  is bounded for all  $j$ . Now (4.3) and Lemma 4.2 imply

$$T_{f_j} \circ S_{-j} = \begin{cases} T_{a_j} & \text{if } j \leq 0 \\ T_{a_j} \circ (id - Q_{j-1}) & \text{if } j > 0. \end{cases}$$

Finally, (iii)  $\Leftrightarrow$  (iv) follows from  $T_{a_j} = M_{g_j}$ , if  $j \leq 0$ , and  $T_{a_j} \circ (id - Q_{j-1}) = M_{g_j}$ , if  $j > 0$ .  $\square$

**Theorem 4.3.** *Assume that  $v$  is normal. Let  $a_j$  be polynomials in  $r$ , hence  $f_j \in L^1$ , where  $f_j(z) = a_j(|z|)z^j$ ,  $j = -n, \dots, n$ . Put*

$$f(z) = \sum_{k=-n}^n a_k(|z|)z^k.$$

Then  $T_f$  is bounded on  $H_v^\infty$ .

We prove Theorem 4.3 at the end of this section. We immediately get, in contrast to Theorem 2.3,

**Corollary 4.4.** *Let  $v$  be normal. Then, for any trigonometric polynomial  $f$ , the Toeplitz operator  $T_f$  is bounded on  $H_v^\infty$ .*



**Proof.** Let  $f_j(re^{i\varphi}) = \alpha_j r^{|j|} e^{ij\varphi}$  and  $f = \sum_{j=-n}^n f_j$ . Then all  $f_j \in L^1$ . Put  $a_j(r) = \alpha_j$  if  $j \geq 0$  and  $a_j(r) = \alpha_j r^{2|j|}$  if  $j < 0$ . Then  $f_j(z) = a_j(|z|)z^j$  for all  $j$  and the corollary follows from Theorem 4.3  $\square$

To prove Theorem 4.3 we need the following

**Lemma 4.5.** *Let  $v$  be normal. Then there is a universal constant  $c > 0$  such that, for any  $k, m$  with  $0 < k \leq m \leq 2k$ , we have*

$$\frac{\Gamma_{k-1}}{\Gamma_{m-1}} \leq c.$$

**Proof.** It follows from the definition of normal weight that there is a constant  $d > 0$  with  $\sup_{0 \leq r < 1} v(r^2)/v(r) < d$ . With the substitution  $s^{(m+1)/(k+1)} = r$  we see that

$$\Gamma_{k-1} = \int_0^1 r^k v(r) dr = \frac{m+1}{k+1} \int_0^1 s^m v(s^{(m+1)/(k+1)}) ds \leq 2d \int_0^1 s^m v(s) ds = 2d\Gamma_{m-1}.$$

Here we used  $s^{(m+1)/(k+1)} \geq s^2$  and hence  $v(s^{(m+1)/(k+1)}) \leq v(s^2) \leq dv(s)$ . Hence the lemma follows with  $c = 2d$ .  $\square$

**Proposition 4.6.** *Assume that  $v$  is normal. Let  $a \in L^1$  be radial such there is a constant  $d > 0$  with*

$$(4.4) \quad r^k |a(r)| \leq \frac{d}{k} \quad \text{for all } k \in \mathbb{N}.$$

or

$$(4.5) \quad r^k |a(r) - 1| \leq \frac{d}{k} \quad \text{for all } k \in \mathbb{N}.$$

Then  $T_a$  is bounded on  $H_v^\infty$ .

**Proof.** Since  $v$  is normal it satisfies condition (B). Let  $m_n$  be the indices of Theorem 3.2 and let  $\gamma_n$  and  $f_a$  be as in (3.24). We have to study the boundedness of the multiplier  $M_{f_a} = T_a$ . At first assume that  $a$  satisfies (4.4). We obtain

$$|\gamma_k| \leq \frac{1}{\Gamma_{2k}} \int_0^1 r^{2k+1} v(r) |a(r)| dr \leq \frac{d}{k\Gamma_{2k}} \int_0^1 r^{2k+1} v(r) \frac{1}{r^k} dr = \frac{d}{k} \frac{\Gamma_k}{2\pi\Gamma_{2k}}.$$

Let  $D > 0$  be the supremum in (3.4). We have

$$\begin{aligned} & \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi \\ & \leq d \left( \sum_{m_{n-1} < k \leq m_n} \frac{k - [m_{n-1}]}{[m_n] - [m_{n-1}]} \frac{\Gamma_k}{k\Gamma_{2k}} + \sum_{m_n < k \leq m_{n+1}} \frac{[m_{n+1}] - k}{[m_{n+1}] - [m_n]} \frac{\Gamma_k}{k\Gamma_{2k}} \right) \\ & \leq Dd \sum_{m_{n-1} < k \leq m_{n+1}} \frac{\Gamma_k}{k\Gamma_{2k}} \leq cDd \frac{m_{n+1} - m_{n-1}}{m_{n-1}} \end{aligned}$$

where  $c$  is the constant of Lemma 4.5. We can apply (3.4) again to conclude

$$\sup_n \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi < \infty.$$

According to Theorem 3.6  $T_a$  is bounded.

If (4.5) holds then  $\tilde{a} = a - 1$  satisfies (4.4). Hence  $T_{\tilde{a}}$  is bounded. But  $T_a = T_{\tilde{a}} + T_1$  and  $T_1 = id$  which implies  $T_a$  is bounded.  $\square$

The idea of the proof of the last statement can clearly be generalized: if  $b \in L^1$  is a symbol such that  $T_b$  is bounded in  $H_v^\infty$  and  $\tilde{a} \in L^1$  is another symbol such that  $a := b - \tilde{a}$  is a radial function satisfying (4.4), then  $T_{\tilde{a}}$  is bounded in  $H_v^\infty$ .

**Proof of Theorem 4.3.** In view of Theorem 4.2. it suffices to show that  $T_a$  is bounded when  $a(r) = r^\ell$  for some  $\ell > 0$ . But this follows from Proposition 4.6 since  $a$  satisfies (4.5). Indeed, fix  $k$  and consider the polynomial  $g(r) = r^k - r^{k+\ell}$ ,  $0 \leq r \leq 1$ . Clearly,  $g$  attains its supremum at  $(k/(k+\ell))^{1/\ell}$  and we have

$$0 \leq g(r) \leq \left(\frac{k}{k+\ell}\right)^{k/\ell} \frac{\ell}{k+\ell} \leq \frac{\ell}{k+\ell} \leq \frac{\ell}{k} \text{ for all } r. \quad \square$$

We finally remark that condition (4.4) holds for symbols  $a(r) = (1-r)^\alpha$ , if and only if  $\alpha \geq 1$ . This is not a precise condition for the boundedness of  $T_a$ , since for the normal weights  $v(r) = (1-r)^\delta$ ,  $0 < \delta < 1$ , any symbol  $a$  with  $|a(z)| \leq C(1-r)^\delta$ , produces a bounded Toeplitz operator  $T_a$  in  $H_v^\infty$ . This so since the pointwise multiplier  $S_a : h \mapsto a \cdot h$  maps  $H_v^\infty$  into the space  $L^\infty$ , and the Bergman projection only causes at most logarithmic singularity on the boundary of the disc, i.e. it maps  $L^\infty$  into  $H_w^\infty$  with the weight  $w(r) = 1/(|\log(1-r)|+1)$ , and this space is of course continuously embedded into  $H_v^\infty$ .

A more careful study of these growth estimates is postponed to a planned future work.

## 5. REMARKS ON OPERATORS ON REFLEXIVE BERGMAN SPACES.

For radial symbols, the boundedness of  $T_a$  as an operator from the Bergman-Hilbert space  $A_v^2$  into itself is characterized by the condition

$$(5.1) \quad \sup_{n \in \mathbb{N}} |\gamma_n| < \infty,$$

where the numbers  $\gamma_n$  are as in (3.24). However, the conditions (3.7) and (5.1) seem not to "interpolate" easily in a way, which would characterize the boundedness and compactness of  $T_a : A_v^p \rightarrow A_v^p$  for  $2 < p < \infty$  (or  $1 < p < 2$ ). Nevertheless we will still show that a condition analogous to (3.7) is sufficient for the boundedness of  $T_a$  in  $A_v^p$ . Let us remark that in [16] the authors used somewhat similar methods to show the connection of the boundedness problem for  $T_a : A_v^p \rightarrow A_v^p$  to the boundedness problem for multipliers in Hardy spaces.

We need to introduce some more notation and definitions: for details of these, see [16]. For a holomorphic  $g(z) = \sum_{k=0}^{\infty} g_k z^k$  and  $0 < r < 1$  we define

$$M_p(g, r) = \left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\varphi})|^p d\varphi \right)^{1/p}$$

and recall the notation  $Q_n g(z) = \sum_{k=0}^n g_k z^k$ , see (4.1). It is well-known that, for  $1 < p < \infty$ , there are universal constants  $c_p > 0$  with  $M_p(Q_n g, r) \leq c_p M_p(g, r)$  where  $c_p$  does not depend on  $g$ ,  $n$  or  $r$ . Moreover, we fix a number  $\beta > 16 \cdot 3^{p-1} (1+2^p) c_p^p + 2$

and use induction to obtain the increasing numerical sequences  $0 = \ell_1 < \ell_2 < \ell_3 \dots$  and  $0 \leq s_1 < s_2 \dots < R$  such that

$$(5.2) \quad \int_0^{s_n} r^{\ell_n p} d\mu = \beta \int_{s_n}^R r^{\ell_n p} d\mu \quad \text{and} \quad \int_0^{s_n} r^{\ell_{n+1} p} d\mu = \frac{1}{\beta} \int_{s_n}^R r^{\ell_{n+1} p} d\mu.$$

(These numbers were calculated in some examples in the paper [3].) We define for all  $n \in \mathbb{N}$

$$Z_n f = (Q_{[\ell_{n+1}]} - Q_{[\ell_n]})f,$$

and

$$\omega_n = \left( \int_0^{s_n} \left( \frac{r}{s_n} \right)^{\ell_n p} d\mu + \int_{s_n}^R \left( \frac{r}{s_n} \right)^{\ell_{n+1} p} d\mu \right)^{1/p}.$$

We get for the norm of  $A_v^p$  a representation analogous to (3.2): there are constants  $d_1, d_2 > 0$  such that, for every  $f \in A_\mu^p$ ,

$$(5.3) \quad d_1 \|f\|_{p,v} \leq \left( \sum_{n=1}^{\infty} \omega_n^p M_p^p(Z_n f, s_n) \right)^{1/p} \leq d_2 \|f\|_{p,v}.$$

This was shown in [9] for  $p = 1$  and in [15] for  $1 < p < \infty$  and  $R = 1$ ,

**Proposition 5.1.** *Let the weight satisfy (B), let  $a \in L^1$  be a radial function and let  $f_a(\varphi) = \sum_{k=0}^{\infty} \gamma_k e^{ik\varphi}$  be as in (3.24). Then the Toeplitz operator  $T_a$  is a well-defined, bounded operator from  $A_v^p$  into itself, if*

$$(5.4) \quad \sup_{n \in \mathbb{N}} \int_0^{2\pi} |(Z_n f_a)(\varphi)| d\varphi < \infty.$$

Moreover,  $T_a : A_v^p \rightarrow A_v^p$  is compact, if

$$(5.5) \quad \int_0^{2\pi} |(Z_n f_a)(\varphi)| d\varphi \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

**Proof.** Let us denote by  $M_f$  the convolution operator, or the sequence space multiplier, corresponding to  $T_a$ , see (3.24). So, if  $h \in A_v^p$  then for all  $re^{i\varphi} \in \mathbb{D}$  we get by the usual orthogonality relations of functions  $e^{ik\varphi}$ ,

$$\begin{aligned} (Z_n M_f h)(re^{i\varphi}) &= (M_{Z_n f} h)(re^{i\varphi}) = \int_0^{2\pi} Z_n f(\varphi - \psi) h(re^{id\psi}) d\psi \\ &= \int_0^{2\pi} Z_n f(\varphi - \psi) Z_n h(re^{id\psi}) d\psi. \end{aligned}$$

We apply the Young inequality

$$\|a * b\|_{L^p(\partial\mathbb{D})} \leq \|a\|_{L^1(\partial\mathbb{D})} \|b\|_{L^p(\partial\mathbb{D})}$$

to get

$$(5.6) \quad M_p(Z_n M_f h, r) \leq \int_0^{2\pi} |(Z_n f)(\varphi)| d\varphi M_p(Z_n h, r)$$

The inequality  $\|M_f h\|_{p,v} \leq C\|h\|_{p,v}$  thus follows by applying (5.4) and (5.3) to both  $\|M_f h\|_{p,v}$  and  $\|h\|_{p,v}$ . This shows that (5.4) is sufficient for  $T_a$  to map  $A_v^p$  continuously into itself.

If (5.5) holds, the proof for the compactness of  $T_a$  is similar to the corresponding proof in Theorem 3.4. We again let  $(h_j)_{j=1}^\infty$  be a sequence which is contained in the unit ball of  $A_v^p$  and which converges to 0 uniformly on compact subsets of  $\mathbb{D}$ , and assume  $\varepsilon > 0$  is given. We choose  $N \in \mathbb{N}$  such that  $\int_0^{2\pi} |(Z_n f)(\varphi)| d\varphi < \varepsilon$ . Then, we use the convergence of our sequence in the compact-open topology and the argument in the proof of Theorem 3.4 to find a large enough  $J \in \mathbb{N}$  such that

$$\sup_{|z| \leq r_{m_n}} |Z_n M_f h_j(z)| v(z) < \frac{\varepsilon}{2\pi N \omega_n} \Rightarrow M_p(Z_n M_f h_j, r_{m_n}) < \frac{\varepsilon}{N \omega_n}$$

for all  $n \leq N$ , all  $j \geq J$ . In view of (5.6) and (5.3) this implies

$$\begin{aligned} \|M_f h_j\|_{p,v}^p &\leq \sum_{n=1}^N \omega_n^p M_p(Z_n M_f h_j, r_{m_n})^p + \sum_{n=N+1}^\infty \omega_n^p M_p(Z_n M_f h_j, r_{m_n})^p \\ &\leq \varepsilon + \varepsilon \sum_{n=N+1}^\infty \omega_n^p M_p(Z_n h_j, r_{m_n})^p \leq 2\varepsilon \|h_j\|_{p,v}^p \leq 2\varepsilon. \quad \square \end{aligned}$$

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INSTITUTO UNIVERSITARIO DE MATEMÁTICA PURA Y APLICADA IUMPA, UNIVERSITAT POLITÈCNICA DE VALÈNCIA, E-46071 VALENCIA, SPAIN

*E-mail address:* jbonet@mat.upv.es

INSTITUTE OF MATHEMATICS, UNIVERSITY OF PADERBORN, WARBURGER STR. 100, D-33098 PADERBORN, GERMANY

*E-mail address:* lusky@uni-paderborn.de

DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68, UNIVERSITY OF HELSINKI, 00014 HELSINKI, FINLAND

*E-mail address:* jari.taskinen@helsinki.fi