

# SOLID HULLS AND CORES OF WEIGHTED $H^\infty$ -SPACES.

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**ABSTRACT.** We determine the solid hull and solid core of weighted Banach spaces  $H_v^\infty$  of analytic functions  $f$  such that  $v|f|$  is bounded, both in the case of the holomorphic functions on the disc and on the whole complex plane, for a very general class of radial weights  $v$ . Precise results are presented for concrete weights on the disc that could not be treated before. It is also shown that if  $H_v^\infty$  is solid, then the monomials are an (unconditional) basis of the closure of the polynomials in  $H_v^\infty$ . As a consequence  $H_v^\infty$  does not coincide with its solid hull and core in the case of the disc. An example shows that this does not hold for weighted spaces of entire functions.

## 1. INTRODUCTION AND PRELIMINARIES

The solid hulls of weighted  $H^\infty$ -type Banach spaces  $H_v^\infty$  of analytic functions on the open unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  were characterized in [8] for a large class of weight functions  $v$ , and a similar study for entire functions was made in [7]. In Theorem 3.1 we extend the results of [8] by means of the calculations of certain numerical constants, which yields many novel, concrete examples of solid hulls. At the same time, we also describe in Theorem 2.4 the solid cores of these Banach spaces in a similar way as we did in Theorem 2.3 (Theorem 2.1 in [7]) for the solid hull. Moreover, we prove that in the case of analytic functions on the disc the Banach space  $H_v^\infty$  is always different from both its solid hull and core; see Corollary 5.3. However, an example is given in the case of entire functions to show that  $H_v^\infty$  may coincide with both its solid hull and core, in particular it is a solid space. We also prove in Theorem 5.2 that if  $H_v^\infty$  coincides with its solid hull, then the monomials are an (unconditional) basis of the closure of the polynomials in  $H_v^\infty$ .

To describe the results in detail, let us introduce some notation and terminology. We set  $R = 1$  (for the case of holomorphic functions on the unit disc) and  $R = +\infty$  (for the case of entire functions). A *weight*  $v$  is a continuous function  $v : [0, R[ \rightarrow ]0, \infty[$ , which is non-increasing on  $[0, R[$  and satisfies  $\lim_{r \rightarrow R} r^n v(r) = 0$  for each  $n \in \mathbb{N}$ . We extend  $v$  to  $\mathbb{D}$  if  $R = 1$  and to  $\mathbb{C}$  if  $R = +\infty$  by  $v(z) := v(|z|)$ . For such a weight  $v$ , we study the Banach space  $H_v^\infty$  of analytic functions  $f$  on the disc  $\mathbb{D}$  (if  $R = 1$ ) or on the whole complex plane  $\mathbb{C}$  (if  $R = +\infty$ ) such that  $\|f\|_v := \sup_{|z| < R} v(z)|f(z)| < \infty$ . For an analytic function  $f \in H(\{z \in \mathbb{C}; |z| < R\})$  and  $r < R$ , we denote  $M(f, r) := \max\{|f(z)|; |z| = r\}$ . Using the notation  $O$  and  $o$  of Landau,  $f \in H_v^\infty$  if and only if  $M(f, r) = O(1/v(r))$ ,  $r \rightarrow R$ . It is known that the closure of the polynomials in  $H_v^\infty$  coincides with the Banach space  $H_v^0$  of all those analytic functions on  $\{z \in \mathbb{C}; |z| < R\}$  such that  $M(f, r) = o(1/v(r))$ ,  $r \rightarrow R$ . see e.g. [3]. It will be clear from the context in the rest of the article when we refer to analytic functions on the disc or entire functions. Anyway, if it is necessary to distinguish at some point, we will use the notations  $H_v^\infty(\mathbb{D})$  and  $H_v^\infty(\mathbb{C})$ .

We shall identify an analytic function  $f(z) = \sum_{n=0}^\infty a_n z^n$  with the sequence of its Taylor coefficients  $(a_n)_{n=0}^\infty$ . Let  $A$ ,  $B$  and  $H$  be vector spaces of complex sequences

containing the space of all the sequences with finitely many non-zero coordinates. The space  $A$  is *solid* if  $a = (a_n) \in A$  and  $|b_n| \leq |a_n|$  for each  $n$  implies  $b = (b_n) \in A$ . The *solid hull* of  $A$  is

$$S(A) := \{(c_n) : \exists (a_n) \in A \text{ such that } |c_n| \leq |a_n| \forall n \in \mathbb{N}\}.$$

It coincides with the smallest solid space containing  $A$ .

The *solid core* of  $A$  is

$$s(A) := \{(c_n) : (c_n a_n) \in A \forall (a_n) \in \ell_\infty\}.$$

The *set of multipliers from  $A$  into  $B$*  is

$$(A, B) := \{c = (c_n) : (c_n a_n) \in B \forall (a_n) \in A\}.$$

The following facts are also known, see [1]: 1.  $A$  is solid if and only if  $\ell_\infty \subset (A, A)$ ; 2.  $A \subset (B, H)$  if and only if  $B \subset (A, H)$ ; 3. the solid core  $s(A)$  of  $A$  is the largest solid space contained in  $A$ , and moreover  $s(A) = (\ell_\infty, A)$ ; 4. the solid hull  $S(A)$  of  $A$  is the smallest solid space containing  $A$ ; 5. If  $X$  is solid,  $(A, X) = (S(A), X)$  and  $(X, A) = (X, s(A))$ .

The results on [7] and [8] contain a characterization of the solid hull of  $H_v^\infty$ , if  $v$  satisfies a condition (b), see (2.1) below. This general characterization is in terms of a numerical sequence  $(m_n)_{n=0}^\infty$ , which depends on the weight and was studied by the second named author in [16]. However, given a concrete weight on the disc like

$$(1.1) \quad v(r) = \exp(-a/(1-r)^b)$$

the calculation of the numbers  $m_n$  is not an easy matter, and in [8], this was only done in the case  $0 < b \leq 2$ . In Theorem 3.1 we calculate these numbers and thus determine the solid hull for weights

$$(1.2) \quad v(r) = w(r) \exp(-a/(1-r)^b)$$

with any  $a, b > 0$ , where  $w$  is a differentiable positive function with some growth restriction. Moreover, the same theorem also contains the analogous characterization of the solid core. In Section 4 we show how these results can be used also in the case of the variant

$$(1.3) \quad v(r) = \exp(-a/(1-r^2)^b)$$

of the weight. The general characterization of solid cores for weights satisfying condition (b) is given in Theorem 2.4. As explained above, further interesting related results are presented in Section 5.

Bennet, Stegenga and Timoney in their paper [2] determined the solid hull and the solid core of the weighted spaces  $H_v^\infty(\mathbb{D})$  in the case the weight  $v$  is doubling. Exponential weights  $v(r) = \exp(-a/(1-r)^b)$  with  $a, b > 0$  are not doubling. Not much seems to be known about multipliers and solid hulls of weighted spaces of analytic functions on the unit disc in the case of exponential weights. Hadamard multipliers of certain weighted space  $H_a^1(\alpha)$ ,  $\alpha > 0$ , were completely described by Dostanić in [10] (see also Chapter 13 in [13]). Other aspects of weighted spaces of analytic functions on the unit disc with exponential weights, like integration operators or Bergman projections, have been investigated recently by Constantin, Dostanić, Pau, Pavlović, Peláez and Rättyä, among others; see [9], [11], [17], [18] and [20]. The solid hull and multipliers on spaces of analytic functions on the disc

has been investigated by many authors. In addition to [2], we mention for example [1], [5], [6], [12], the books [13] and [19] and the many references therein.

Spaces of type  $H_v^\infty(\mathbb{C})$  and  $H_v^\infty(\mathbb{D})$  appear in the study of growth conditions of analytic functions and have been investigated in various articles since the work of Shields and Williams, see *e.g.* [3],[4], [15], [16], [21] and the references therein.

In the case of a "standard" weight  $v_\alpha(z) = (1 - |z|^2)^\alpha$ , where  $\alpha \geq 0$ , we denote for every  $H_\alpha^\infty := H_\alpha^\infty(\mathbb{D}) := H_{v_\alpha}^\infty$ . The solid hull  $S(H_\alpha^\infty)$  of  $H_\alpha^\infty$  is known: it equals

$$S(H_\alpha^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_{n \in \mathbb{N}_0} \left( \sum_{m=2^n}^{2^{n+1}-1} |b_m|^2 (m+1)^{-2\alpha} \right)^{1/2} < \infty \right\}.$$

This is Theorem 8.2.1 of [13]. Moreover, the solid core  $s(H_\alpha^\infty)$  can also be characterized, see Theorem 8.3.4 of [13]:

$$s(H_\alpha^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_{n \in \mathbb{N}_0} \left( \sum_{m=2^n}^{2^{n+1}-1} |b_m| (m+1)^{-\alpha} \right) < \infty \right\}.$$

## 2. SOLID HULL AND CORE FOR WEIGHTS WITH CONDITION (b).

In this section we consider the quite large class of weights on  $\mathbb{D}$  or  $\mathbb{C}$  satisfying the regularity condition (b). For such a weight, the solid hull was found in the paper [8], and we determine the solid core here.

**Definition 2.1.** Let  $r_n \in ]0, R[$  be a global maximum point of the function  $r^m v(r)$  for any  $m > 0$ . The weight  $v$  satisfies the *condition (b)* if there exist numbers  $b > 2$ ,  $K > b$  and  $0 < m_1 < m_2 < \dots$  with  $\lim_{n \rightarrow \infty} m_n = \infty$  such that

$$(2.1) \quad b \leq \left( \frac{r_{m_n}}{r_{m_{n+1}}} \right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq K.$$

**Remark 2.2.** (1) The second named author introduced the following *condition (B)* on the weight  $v$  in [16]:

$$\forall b_1 > 0 \exists b_2 > 1 \exists c > 0 \forall m, n :$$

$$\left( \frac{r_m}{r_n} \right)^m \frac{v(r_m)}{v(r_n)} \leq b_1 \quad \text{and} \quad |m - n| \geq c \Rightarrow \left( \frac{r_n}{r_m} \right)^n \frac{v(r_n)}{v(r_m)} \leq b_2.$$

It was observed in Remark 2.7 of [7] that if a weight  $v$  satisfies condition (B), then it also satisfies condition (b) for some  $b > 2$ ,  $K > b$  and  $0 < m_1 < m_2 < \dots$ .

(2) As a consequence of this observation and Section 2 in [16], the following weights satisfy condition (b): For  $R = 1$ ,

- (i)  $v(r) = (1 - r)^\alpha$  with  $\alpha > 0$ , which are the standard weights on the disc, and
- (ii)  $v(r) = \exp(-(1 - r)^{-1})$ .

More examples can be seen in Example 3.3. For  $R = +\infty$ ,

- (i)  $v(r) = \exp(-r^p)$  with  $p > 0$ ,
- (ii)  $v(r) = \exp(-\exp r)$ , and
- (iii)  $v(r) = \exp(-(\log^+ r)^p)$ , where  $p \geq 2$  and  $\log^+ r = \max(\log r, 0)$ .

We recall the result [8], Theorem 2.1 for  $\mathbb{D}$ , or [7], Theorem 2.5 for entire functions:

**Theorem 2.3.** *If the weight  $v$  satisfies (b), we have*

$$(2.2) \quad S(H_v^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_n v(r_{m_n}) \left( \sum_{m_n < m \leq m_{n+1}} |b_m|^2 r_{m_n}^{2m} \right)^{1/2} < \infty \right\}.$$

Now let us prove the following statement.

**Theorem 2.4.** *For a weight  $v$  satisfying (b), we have*

$$(2.3) \quad s(H_v^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_n v(r_{m_n}) \left( \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m \right) < \infty \right\}.$$

Proof. For a holomorphic function  $f$  with  $f(z) = \sum_{n=0}^\infty a_n z^n$ , we let  $h_f$  denote the function defined by

$$h_f(z) = \sum_{n=0}^\infty |a_n| z^n \quad \text{for all } z.$$

It is easy to see that the solid core  $s(H_v^\infty)$  of  $H_v^\infty$  coincides with the set

$$\{f : f \text{ holomorphic}, h_f \in H_v^\infty\}.$$

Now, let  $f \in H_v^\infty$ ,  $f(z) = \sum_{n=0}^\infty b_n z^n$ . If

$$(2.4) \quad h_f(z) = \sum_{n=0}^\infty |b_n| z^n,$$

belongs to  $H_v^\infty$ , it is easily seen that

$$\|h_f\|_v = \sup_{0 < r < R} v(r) \sum_{n=0}^\infty |b_n| r^n.$$

This implies

$$(2.5) \quad \sup_n v(r_{m_n}) \left( \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m \right) \leq \|h_f\|_v.$$

Thus the solid core is contained in the right-hand side of (2.3). Now we proceed with the reverse inclusion.

According to [16], Proposition 5.2., there are numbers  $\beta_m \in [0, 1]$  and a constant  $c > 0$  (independent of  $f$ ) such that

$$\|h_f\|_v \leq c \sup_n \sup_{r_{m_{n-1}} \leq |z| \leq r_{m_n+1}} v(z) \left| \sum_{m_{n-1} < m \leq m_{n+1}} \beta_m |b_m| z^m \right|.$$

This implies

$$(2.6) \quad \|h_f\|_v \leq c \sup_n \sup_{r_{m_{n-1}} \leq r \leq r_{m_n+1}} v(r) \left( \sum_{m_{n-1} < m \leq m_{n+1}} |b_m| r^m \right).$$

For  $r_{m_{n-1}} \leq r \leq r_{m_n+1}$  we have

$$\begin{aligned} & v(r) \sum_{m_{n-1} < m \leq m_n} |b_m| r^m \\ &= \frac{v(r)}{v(r_{m_{n-1}})} v(r_{m_{n-1}}) \sum_{m_{n-1} < m \leq m_n} |b_m| r_{m_{n-1}}^m \left( \frac{r}{r_{m_{n-1}}} \right)^m \\ &\leq \left( \frac{r}{r_{m_{n-1}}} \right)^{m_n} \frac{v(r)}{v(r_{m_{n-1}})} v(r_{m_{n-1}}) \sum_{m_{n-1} < m \leq m_n} |b_m| r_{m_{n-1}}^m \\ &\leq \left( \frac{r_{m_n}}{r_{m_{n-1}}} \right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n-1}})} v(r_{m_{n-1}}) \sum_{m_{n-1} < m \leq m_n} |b_m| r_{m_{n-1}}^m \end{aligned}$$

$$(2.7) \quad \leq K v(r_{m_{n-1}}) \sum_{m_{n-1} < m \leq m_n} |b_m| r_{m_{n-1}}^m.$$

Here we used that  $r_{m_n}$  is a global maximum point for  $r^{m_n} v(r)$ .

Similarly, for  $r_{m_{n-1}} \leq r \leq r_{m_n}$  we have

$$(2.8) \quad \begin{aligned} & v(r) \sum_{m_n < m \leq m_{n+1}} |b_m| r^m \\ &= \frac{v(r)}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m \left( \frac{r}{r_{m_n}} \right)^m \\ &\leq \left( \frac{r}{r_{m_n}} \right)^{m_n} \frac{v(r)}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m \\ &\leq v(r_{m_n}) \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m. \end{aligned}$$

Finally, if  $r_{m_n} \leq r \leq r_{m_{n+1}}$  then

$$(2.9) \quad \begin{aligned} & v(r) \sum_{m_n < m \leq m_{n+1}} |b_m| r^m \\ &= \frac{v(r)}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m \left( \frac{r}{r_{m_n}} \right)^m \\ &\leq \left( \frac{r}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m \\ &\leq \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m \\ &\leq K v(r_{m_n}) \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m. \end{aligned}$$

Hence, according to (2.6), with (2.7), (2.8) and (2.9),

$$\|h_f\|_v \leq c2K \sup_n v(r_{m_n}) \left( \sum_{m_n < m \leq m_{n+1}} |b_m| r_{m_n}^m \right).$$

This together with (2.1) yields (2.3).  $\square$

### 3. A WEIGHT OF THE FORM $w(r) \exp(-a/(1-r)^b)$ .

In this section we only deal with weights defined on the unit disc  $\mathbb{D}$ . Our purpose is to improve the results of [8] by calculating the solid hulls and cores for a larger class of concrete examples, namely, for weights of the form

$$(3.1) \quad v(z) = w(r) \exp \left( - \frac{a}{(1-r)^b} \right), \quad z \in \mathbb{D},$$

where  $a, b > 0$  are given constants and  $w : [0, 1[ \rightarrow ]0, \infty[$  is a differentiable function, extended to  $\mathbb{D}$  by  $w(z) = w(|z|)$ . We remark the examples in [8] only contain the case  $b \leq 2$ ,  $w \equiv 1$ .

We will prove

**Theorem 3.1.** *Let  $w'(r)/w(r)$  be a decreasing function and assume that there are  $n_0 > 0$  and  $\alpha \in ]0, 1 + b/2[$  such that*

$$(3.2) \quad (1-r)^\alpha \frac{w'(r)}{w(r)} \text{ is bounded on } [0, 1[$$

$$(3.3) \quad \frac{1}{e} \leq \frac{w(1 - (\frac{a}{bn^2})^{1/b})}{w(1 - (\frac{a}{b(n+1)^2})^{1/b})} \leq e \text{ for } n \geq n_0.$$

*Then, the solid hull of  $H_v^\infty$  is equal to*

$$\left\{ (b_m)_{m=0}^\infty : \sup_n w \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right) e^{-bn^2} \left( \sum_{\substack{m \in \mathbb{N} \\ m_n < m \leq m_{n+1}}} |b_m|^2 \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right)^{2m} \right)^{1/2} < \infty \right\},$$

*where*

$$m_n = b \left( \frac{b}{a} \right)^{1/b} n^{2+2/b} - bn^2 - \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right) \frac{w'(1 - (\frac{a}{bn^2})^{1/b})}{w(1 - (\frac{a}{bn^2})^{1/b})}$$

*(which reduces to  $m_n = b^{1+1/b} a^{-1/b} n^{2+2/b} - bn^2$ , if  $w \equiv 1$ ).*

*Moreover, the solid core of  $H_v^\infty$  is equal to*

$$\left\{ (b_m)_{m=0}^\infty : \sup_n w \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right) e^{-bn^2} \sum_{\substack{m \in \mathbb{N} \\ m_n < m \leq m_{n+1}}} |b_m| \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right)^m < \infty \right\}.$$

We postpone the proof a bit and consider some remarks and examples. Let us start with the following quite trivial observation which, however, is very useful to simplify the presentations of the solid hulls and cores.

**Lemma 3.2.** *Let  $1 \leq p < \infty$ , and let  $(K_n)_{n \in \mathbb{N}_0}$ ,  $(\tilde{K}_n)_{n \in \mathbb{N}_0}$ ,  $(L_m)_{m \in \mathbb{N}_0}$  and  $(\tilde{L}_m)_{m \in \mathbb{N}_0}$  be sequences of positive numbers. Assume that there are given two increasing, unbounded sequences  $(m_n)_{n \in \mathbb{N}_0}$  and  $(\tilde{m}_n)_{n \in \mathbb{N}_0}$  of positive real numbers such that*

$$(3.4) \quad m_n < \tilde{m}_n < m_{n+1} \quad \forall n \in \mathbb{N}_0$$

*and such that for some constants  $C > c > 0$ , for all  $n \in \mathbb{N}$ ,*

$$(3.5) \quad \begin{aligned} cK_n^p L_m &\leq \tilde{K}_{n-1}^p \tilde{L}_m \leq CK_n^p L_m & \forall m \text{ with } m_n < m \leq \tilde{m}_n \\ cK_n^p L_m &\leq \tilde{K}_n^p \tilde{L}_m \leq CK_n^p L_m & \forall m \text{ with } \tilde{m}_n < m \leq m_{n+1}. \end{aligned}$$

*Then, we have*

$$(3.6) \quad \begin{aligned} &\left\{ (b_m)_{m=0}^\infty : \sup_{n \in \mathbb{N}_0} K_n \left( \sum_{m_n < m \leq m_{n+1}} |b_m|^p L_m \right)^{1/p} < \infty \right\} \\ &= \left\{ (b_m)_{m=0}^\infty : \sup_{n \in \mathbb{N}_0} \tilde{K}_n \left( \sum_{\tilde{m}_n < m \leq \tilde{m}_{n+1}} |b_m|^p \tilde{L}_m \right)^{1/p} < \infty \right\} \end{aligned}$$

*Proof.* We have, by (3.5),

$$\begin{aligned} &\sup_{n \in \mathbb{N}_0} \left( \sum_{m_n < m \leq m_{n+1}} K_n^p |b_m|^p L_m \right)^{1/p} \\ &\leq \sup_{n \in \mathbb{N}_0} \left( \left( \sum_{m_n < m \leq \tilde{m}_n} K_n^p |b_m|^p L_m \right)^{1/p} + \left( \sum_{\tilde{m}_n < m \leq m_{n+1}} K_n^p |b_m|^p L_m \right)^{1/p} \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{c} \sup_{n \in \mathbb{N}_0} \left( \left( \sum_{m_n < m \leq \tilde{m}_n} \tilde{K}_{n-1}^p |b_m|^p \tilde{L}_m \right)^{1/p} + \left( \sum_{\tilde{m}_n < m \leq m_{n+1}} \tilde{K}_n^p |b_m|^p \tilde{L}_m \right)^{1/p} \right) \\
&\leq \frac{2}{c} \sup_{n \in \mathbb{N}_0} \left( \sum_{\tilde{m}_n < m \leq \tilde{m}_{n+1}} \tilde{K}_n^p |b_m|^p \tilde{L}_m \right)^{1/p}.
\end{aligned}$$

The converse inequality can be shown in the same way, using the other inequalities in (3.5).  $\square$

It is obvious that the representations of the solid hull and core of a weighted space are by no means unique: the sequence  $m_n$  and even the coefficients and exponents can be chosen in many ways. We discuss this in the first example.

**Example 3.3.** (i):  $a = b = 1$ ,  $w = 1$ . Here  $m_n = n^4 - n^2$ . However, in [8] the representation of this solid hull was found with the more simple numbers  $\tilde{m}_n = n^4$  instead:

$$\begin{aligned}
S(H_v^\infty) &= \left\{ (b_m)_{m=0}^\infty : \sup_n e^{-n^2} \left( \sum_{m=n^4+1}^{(n+1)^4} |b_m|^2 (1 - n^{-2})^{2m} \right)^{1/2} < \infty \right\}, \\
s(H_v^\infty) &= \left\{ (b_m)_{m=0}^\infty : \sup_n e^{-n^2} \sum_{m=n^4+1}^{(n+1)^4} |b_m| (1 - n^{-2})^m < \infty \right\}.
\end{aligned}$$

Let us verify that the condition (3.5) of Lemma 3.2 is satisfied with  $p = 2$ ; the proof for the case  $p = 1$  follows by taking square roots. We may obviously assume  $n \geq 2$ . For  $m$  with  $m_n < m \leq \tilde{m}_n$ , i.e.,

$$(3.7) \quad n^4 - n^2 < m \leq n^4,$$

we have  $K_n = e^{-n^2} = \tilde{K}_n$ ,  $L_m = (1 - n^{-2})^{2m}$  and  $\tilde{L}_m = (1 - (n-1)^{-2})^{2m}$ , hence,

$$\begin{aligned}
K_n^2 L_m &= e^{-2n^2} \left( 1 - \frac{1}{n^2} \right)^{2m} \geq e^{-2n^2} \left( 1 - \frac{1}{n^2} \right)^{n^2 2n^2} \\
(3.8) \quad &\geq C e^{-2n^2} e^{-2n^2} = C e^{-4n^2},
\end{aligned}$$

$$\begin{aligned}
K_n^2 L_m &\leq e^{-2n^2} \left( 1 - \frac{1}{n^2} \right)^{2n^4 - 2n^2} \\
&\leq e^{-2n^2} \left( 1 - \frac{1}{n^2} \right)^{n^2 2n^2} \left( 1 - \frac{1}{n^2} \right)^{-2n^2} \\
(3.9) \quad &\leq C e^{-4n^2} \left( 1 - \frac{1}{n^2} \right)^{-2n^2} \leq C' e^{-4n^2},
\end{aligned}$$

Furthermore, we write

$$n^4 = (n-1)^4 + 4(n-1)^3 + \rho(n) = (n-1)^2(n-1)^2 + 4(n-1)^3 + \rho(n),$$

where  $\rho(n) = 6n^2 - 8n + 3$ . Using the trivial estimates  $\rho(n) - n^2 \geq -60(n-1)^2$  and  $\rho(n) \leq 50(n-1)^2$  for all  $n \geq 2$ , we obtain

$$\begin{aligned}
\tilde{K}_{n-1}^2 \tilde{L}_m &= e^{-2(n-1)^2} \left( 1 - \frac{1}{(n-1)^2} \right)^{2m} \leq e^{-2(n-1)^2} \left( 1 - \frac{1}{(n-1)^2} \right)^{2n^4 - 2n^2} \\
&= e^{-2(n-1)^2} \left( 1 - \frac{1}{(n-1)^2} \right)^{2(n-1)^2(n-1)^2 + 8(n-1)^3 + 2(\rho(n) - n^2)}
\end{aligned}$$

$$\begin{aligned}
&\leq e^{-2(n-1)^2} \left( \left( 1 - \frac{1}{(n-1)^2} \right)^{(n-1)^2} \right)^{2(n-1)^2+8(n-1)-120} \\
(3.10) \quad &\leq C e^{-4(n-1)^2} e^{-8(n-1)} = C e^{-4n^2+8n-4} e^{-8(n-1)} \leq C' e^{-4n^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{K}_{n-1}^2 \tilde{L}_m &= e^{-2(n-1)^2} \left( 1 - \frac{1}{(n-1)^2} \right)^{2m} \geq e^{-2(n-1)^2} \left( 1 - \frac{1}{(n-1)^2} \right)^{2n^4} \\
&= e^{-2(n-1)^2} \left( 1 - \frac{1}{(n-1)^2} \right)^{2(n-1)^2(n-1)^2+8(n-1)^3+2\rho(n)} \\
&\geq e^{-2(n-1)^2} \left( \left( 1 - \frac{1}{(n-1)^2} \right)^{(n-1)^2} \right)^{2(n-1)^2+8(n-1)+100} \\
(3.11) \quad &\geq C e^{-4(n-1)^2} e^{-8(n-1)} \geq C' e^{-4n^2}.
\end{aligned}$$

We thus see that the first pair of inequalities (3.5) holds. The second one is trivial since for  $\tilde{m}_n < m \leq m_{n+1}$ , i.e.,

$$n^4 < m \leq (n+1)^4 - (n+1)^2,$$

we have  $L_m = (1 - n^{-2})^{2m} = \tilde{L}_m$ .

(ii):  $a = 1$ ,  $b = 2$ ,  $w(r) = 1 - r$ . Here

$$m_n = 2^{3/2}n^3 - 2n^2 + 2^{1/2}n - 1$$

and

$$\begin{aligned}
S(H_v^\infty) &= \left\{ (b_m)_{m=0}^\infty : \sup_n \frac{e^{-2n^2}}{\sqrt{2}n} \left( \sum_{\substack{m \in \mathbb{N} \\ m_n < m \leq m_{n+1}}} |b_m|^2 (1 - (\sqrt{2}n)^{-1})^{2m} \right)^{1/2} < \infty \right\}, \\
s(H_v^\infty) &= \left\{ (b_m)_{m=0}^\infty : \sup_n \frac{e^{-2n^2}}{\sqrt{2}n} \sum_{\substack{m \in \mathbb{N} \\ m_n < m \leq m_{n+1}}} |b_m| (1 - (\sqrt{2}n)^{-1})^m < \infty \right\}.
\end{aligned}$$

(iii):  $a = b = 1$ ,  $w(r) = (1 - \log(1 - r))^{-1}$ . Here, a direct calculation yields

$$(3.12) \quad m_n = n^4 - n^2 + \frac{n^2 - 1}{1 + \log(n^2)},$$

but we can again use Lemma 3.2 with  $\tilde{m}_n = n^4$ ,  $K_n = e^{-n^2} (1 + \log(n^2))^{-1} = \tilde{K}_n$ ,  $L_m = (1 - n^{-2})^{2m}$  and  $\tilde{L}_m = (1 - (n-1)^{-2})^{2m}$ , since the calculation (3.8)–(3.11) shows that both the expressions  $K_n^2 L_m$  and  $\tilde{K}_{n-1}^2 L_m$  are proportional to

$$(3.13) \quad \frac{e^{-4n^2}}{(1 + \log(n^2))^2}$$

for all  $m$  with  $m_n < m \leq \tilde{m}_n$ . (To see this, we observe that in comparison with (3.8)–(3.11),  $\tilde{K}_n$  only has the new factor  $(1 + \log(n^2))^{-1} =: g_n$  for which  $g_n$  and  $g_{n-1}$  are proportional, and that  $m_n$  of (3.12) satisfies  $n^4 - n^2 < m_n < n^4$  so that  $m$



in (3.13) falls into the interval considered also in (3.8)–(3.11)). Moreover, of course  $K_n^2 L_m = \tilde{K}_n^2 L_m$  for all  $m$  with  $\tilde{m}_n < m \leq m_{n+1}$ . Thus, we have

$$S(H_v^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_n \frac{e^{-n^2}}{1 + \log(n^2)} \left( \sum_{m=n^4+1}^{(n+1)^4} |b_m|^2 (1 - n^{-2})^{2m} \right)^{1/2} < \infty \right\},$$

$$s(H_v^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_n \frac{e^{-n^2}}{1 + \log(n^2)} \sum_{m=n^4+1}^{(n+1)^4} |b_m| (1 - n^{-2})^m < \infty \right\}.$$

(iv):  $a = b = 1$ ,  $w(r) = \exp(-\log^2(1 - r))$ . Here we have  $w'(r)/w(r) = 2(1 - r)^{-1} \log(1 - r)$ . It is easily seen that (3.2) and (3.3) are satisfied. We obtain  $m_n = n^4 - n^2 + 4(n^2 - 1) \log(n)$  and

$$S(H_v^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_n \exp(-4 \log^2(n) - n^2) \left( \sum_{\substack{m \in \mathbb{N} \\ m_n < m \leq m_{n+1}}} |b_m|^2 (1 - n^{-2})^{2m} \right)^{1/2} < \infty \right\},$$

$$s(H_v^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_n \exp(-4 \log^2(n) - n^2) \sum_{\substack{m \in \mathbb{N} \\ m_n < m \leq m_{n+1}}} |b_m| (1 - n^{-2})^m < \infty \right\}.$$

**Remark 3.4.** Fix  $m > 1$  and put

$$f(r) = r^m v(r) = r^m w(r) \exp\left(-\frac{a}{(1-r)^b}\right).$$

Due to the continuity of  $f$  and the fact that  $f(0) = f(1) = 0 \leq f(r)$ ,  $r \in ]0, 1[$ , the function  $f$  has a global maximum on  $]0, 1[$ . It is easily seen that  $r \in ]0, 1[$  is a zero of  $f'$  if and only if

$$(3.14) \quad m = ab \frac{r}{(1-r)^{b+1}} - r \frac{w'(r)}{w(r)}.$$

Since  $-rw'(r)/w(r)$  is assumed to be increasing, the right-hand side of (3.14) is strictly increasing in  $r$ . Hence (3.14) has exactly one solution, denoted by  $r_m$ , which is the unique global maximum of  $f$ . In particular, if

$$(3.15) \quad M = abm^{1+1/b} \left(1 - \left(\frac{1}{m}\right)^{1/b}\right) - \left(1 - \left(\frac{1}{m}\right)^{1/b}\right) \frac{w'\left(1 - \left(\frac{1}{m}\right)^{1/b}\right)}{w\left(1 - \left(\frac{1}{m}\right)^{1/b}\right)}$$

for some  $m > 1$ , then

$$(3.16) \quad r_M = 1 - \left(\frac{1}{m}\right)^{1/b}.$$

Proof of Theorem 3.1. 1°. We first consider that case  $b \geq 1$ .

a) Some estimates. If  $1 \leq x \leq y$  and  $0 < \beta < 1$  then the mean value theorem yields

$$(3.17) \quad y^\beta - x^\beta \leq \beta x^{\beta-1}(y - x) \quad \text{and} \quad x^\beta - y^\beta \leq \beta y^{\beta-1}(x - y).$$

Moreover we use

$$(3.18) \quad 1 + x \leq e^x \quad \text{for all } x \in \mathbb{R}.$$

Now let  $1 < m \leq k$  and define  $M$  as in (3.15) and  $K$  in the same way with  $k$  replacing  $m$ . Then

$$r_M = 1 - \frac{1}{m^{1/b}} \quad \text{and} \quad r_K = 1 - \frac{1}{k^{1/b}}$$

and we can rewrite (3.15) as

$$(3.19) \quad M = abm^{1+1/b}r_M - r_M \frac{w'(r_M)}{w(r_M)}, \quad K = abk^{1+1/b}r_K - r_K \frac{w'(r_K)}{w(r_K)}.$$

Then, with (3.17), (3.18) we obtain

$$(3.20) \quad \begin{aligned} \frac{r_K}{r_M} &= \frac{1 - \left(\frac{1}{k}\right)^{1/b}}{1 - \left(\frac{1}{m}\right)^{1/b}} = 1 + \frac{\left(\frac{1}{m}\right)^{1/b} - \left(\frac{1}{k}\right)^{1/b}}{1 - \left(\frac{1}{m}\right)^{1/b}} \\ &\leq \exp\left(\frac{1}{b} \left(\frac{1}{k}\right)^{1/b-1} \left(\frac{1}{m} - \frac{1}{k}\right) \frac{1}{r_M}\right) = \exp\left(\frac{1}{b} \frac{k-m}{k^{1/b}m} \frac{1}{r_M}\right) \end{aligned}$$

and

$$(3.21) \quad \begin{aligned} \frac{r_M}{r_K} &= \frac{1 - \left(\frac{1}{m}\right)^{1/b}}{1 - \left(\frac{1}{k}\right)^{1/b}} = 1 - \frac{\left(\frac{1}{m}\right)^{1/b} - \left(\frac{1}{k}\right)^{1/b}}{1 - \left(\frac{1}{k}\right)^{1/b}} \\ &\leq \exp\left(-\frac{1}{b} \frac{k-m}{m^{1+1/b}} \frac{1}{r_K}\right). \end{aligned}$$

Now we write

$$\left(\frac{r_K}{r_M}\right)^K \frac{v(r_K)}{v(r_M)} = \exp\left(K \log\left(\frac{r_K}{r_M}\right)\right) \frac{w(r_K)}{w(r_M)} \exp(-a(k-m))$$

which in view of (3.19), (3.20) is bounded by

$$(3.22) \quad \begin{aligned} &\frac{w(r_K)}{w(r_M)} \exp\left(a \frac{k(k-m)}{m} \frac{r_K}{r_M} - \frac{1}{b} \frac{r_K}{r_M} \frac{w'(r_K)}{w(r_M)} \frac{k-m}{k^{1/b}m} - a(k-m)\right) \\ &= \frac{w(r_K)}{w(r_M)} \exp\left(-a(k-m) + a\left(\frac{k-k^{1-1/b}}{m-m^{1-1/b}}\right)(k-m) + c_1(k,m)\right) \\ &= \frac{w(r_K)}{w(r_M)} \exp\left(a \frac{(k-m)^2}{m-m^{1-1/b}} \left(1 - \frac{k^{1-1/b}-m^{1-1/b}}{k-m}\right) + c_1(k,m)\right) \end{aligned}$$

where

$$(3.23) \quad c_1(k,m) = -\frac{1}{b} \frac{r_K}{r_M} \frac{w'(r_K)}{w(r_K)} (1-r_K) \frac{k-m}{m}.$$

Using (3.21) instead of (3.20) we get with the help of (3.17)

$$(3.24) \quad \begin{aligned} &\left(\frac{r_M}{r_K}\right)^K \frac{v(r_M)}{v(r_K)} \\ &\leq \frac{w(r_M)}{w(r_K)} \exp\left(a(k-m) - a(k-m) \left(\frac{k}{m}\right)^{1/b} + c_2(k,m)\right) \\ &= \frac{w(r_M)}{w(r_K)} \exp\left(a(k-m) \left(\frac{m^{1/b}-k^{1/b}}{m^{1/b}}\right) + c_2(k,m)\right) \\ &\leq \frac{w(r_M)}{w(r_K)} \exp\left(-\frac{a}{b} (k-m)^2 \frac{1}{k^{1-1/b}m^{1/b}} + c_2(k,m)\right) \end{aligned}$$

where

$$(3.25) \quad c_2(k, m) = \frac{1}{b} \left( \frac{1}{m} \right)^{1/b} \frac{w'(r_K)}{w(r_K)} \frac{k-m}{k}.$$

Similarly, (3.20) implies

$$(3.26) \quad \begin{aligned} & \left( \frac{r_K}{r_M} \right)^M \frac{v(r_K)}{v(r_M)} \\ & \leq \frac{w(r_K)}{w(r_M)} \exp \left( -a(k-m) + a \left( \frac{m}{k} \right)^{1/b} (k-m) + c_3(k, m) \right) \\ & = \frac{w(r_K)}{w(r_M)} \exp \left( -a(k-m) \left( \frac{k^{1/b} - m^{1/b}}{k^{1/b}} \right) + c_3(k, m) \right) \\ & \leq \frac{w(r_K)}{w(r_M)} \exp \left( -\frac{a}{b} (k-m)^2 \frac{1}{k} + c_3(k, m) \right) \end{aligned}$$

with

$$(3.27) \quad c_3(k, m) = -\frac{1}{b} \frac{w'(r_M)}{w(r_M)} \frac{k-m}{m} (1-r_K).$$

Finally, (3.21) implies

$$(3.28) \quad \begin{aligned} & \left( \frac{r_M}{r_K} \right)^M \frac{v(r_M)}{v(r_K)} \\ & = \frac{w(r_M)}{w(r_K)} \exp \left( a(k-m) - a \frac{m}{k} (k-m) \frac{r_M}{r_K} + c_4(k, m) \right) \\ & = \frac{w(r_M)}{w(r_K)} \exp \left( a \frac{(k-m)^2}{k - k^{1-1/b}} \left( 1 - \frac{k^{1-1/b} - m^{1-1/b}}{k-m} \right) + c_4(k, m) \right) \end{aligned}$$

with

$$(3.29) \quad c_4(k, m) = \frac{1}{b} \frac{r_M}{r_K} \frac{w'(r_M)}{w(r_M)} \frac{k-m}{m^{1/b}k}.$$

b) The parameters  $m_n$ . Now put for every  $n \in \mathbb{N}$  large enough

$$(3.30) \quad j_n = \frac{b}{a} n^2$$

and in the above calculations choose

$$m = j_n \quad , \quad k = j_{n+1}.$$

We denote the numbers in (3.19) by

$$m_n = M \quad , \quad m_{n+1} = K$$

so that the following relations hold, by (3.16):

$$r_{m_n} = 1 - \frac{1}{j_n^{1/b}} = 1 - \left( \frac{a}{bn^2} \right)^{1/b} \quad , \quad r_{m_{n+1}} = 1 - \left( \frac{a}{b(n+1)^2} \right)^{1/b}.$$

c) Final estimates. With (3.23) we obtain

$$c_1(j_{n+1}, j_n) = -\frac{1}{b} \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right) \frac{w'(r_{m_{n+1}})}{w(r_{m_{n+1}})} (1 - r_{m_{n+1}}) \frac{2n+1}{n^2}$$

By assumption (3.2) there is  $d > 0$  and  $0 < \alpha < 1 + b/2$  with

$$\frac{w'(r_{m_{n+1}})}{w(r_{m_{n+1}})} \leq \frac{d}{(1 - r_{m_{n+1}})^\alpha}.$$

Hence

$$\begin{aligned} |c_1(j_{n+1}, j_n)| &\leq \frac{d}{b} \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right) (1 - r_{m_{n+1}})^{1-\alpha} \left( \frac{2n+1}{n^2} \right) \\ &= \frac{d}{b} \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right) \left( \frac{a}{b} \right)^{(1-\alpha)/b} (n+1)^{2(\alpha-1)/b} \left( \frac{2n+1}{n^2} \right). \end{aligned}$$

Since  $2(\alpha - 1)/b < 1$  we obtain

$$\lim_{n \rightarrow \infty} c_1(j_{n+1}, j_n) = 0$$

By assumption (3.3) we have  $w(r_{m_{n+1}})/w(r_{m_n}) \leq e$ . So, using (3.22) and (3.30) we see that there is a constant  $K_0$  with

$$(3.31) \quad \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq K_0 \quad \text{for all } n.$$

This follows from the fact that, for  $k = j_{n+1}$  and  $m = j_n$ , the expression in (3.22),

$$\frac{(k-m)^2}{m-m^{1/b}} \frac{1-k^{1-1/b}-m^{1-1/b}}{k-m},$$

remains uniformly bounded for all  $n$ .

To obtain a lower estimate of

$$\left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}$$

consider (3.24) and (3.25). Exactly as before we see that

$$(3.32) \quad \lim_{n \rightarrow \infty} c_2(j_{n+1}, j_n) = 0$$

For  $k = j_{n+1}$  and  $m = j_n$  we obtain

$$\left( \frac{a}{b} \right) \frac{(k-m)^2}{k^{1-1/b}m^{1/b}} = \frac{(2n+1)^2}{(n+1)^{2-2/b}n^{2/b}}$$

which tends to 4 as  $n \rightarrow \infty$ . Together with (3.32) we find  $n_0$  such that

$$-\left( \frac{a}{b} \right) \frac{(j_{n+1} - j_n)^2}{j_{n+1}^{1-1/b} j_n^{1/b}} + c_2(j_{n+1}, j_n) \leq -2 \quad \text{for } n \geq n_0.$$

Since by assumption  $w(r_{m_n})/w(r_{m_{n+1}}) \leq e$  the estimate (3.24) implies

$$\left( \frac{r_{m_n}}{r_{m_{n+1}}} \right)^{m_{n+1}} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} \leq \frac{1}{e},$$

hence

$$(3.33) \quad 2 < e \leq \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \quad \text{for } n \geq n_0.$$

Repeating the preceding arguments using (3.26), (3.27), (3.28), (3.29) instead of (3.22), (3.23), (3.24), (3.25) we see that

$$\lim_{n \rightarrow \infty} c_3(j_{n+1}, j_n) = \lim_{n \rightarrow \infty} c_4(j_{n+1}, j_n) = 0$$

and there are  $n_1, N_1$  with

$$(3.34) \quad 2 < e \leq \left( \frac{r_{m_n}}{r_{m_{n+1}}} \right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} \leq K_1 \quad \text{for } n \geq n_1.$$

Then the assertion of the theorem in the case  $b \geq 1$  follows from (3.31), (3.33), (3.34) and [8], Theorem 2.1.

2°. We prove Theorem 3.1 in the case  $0 < b < 1$ . Here we use, for  $\gamma > 1$ ,  $0 \leq x \leq y$ ,

$$y^\gamma - x^\gamma \leq \gamma y^{\gamma-1}(y-x) \quad \text{and} \quad x^\gamma - y^\gamma \leq \gamma x^{\gamma-1}(x-y).$$

We obtain, instead of (3.20) and (3.20),

$$\begin{aligned} \frac{1 - \left(\frac{1}{k}\right)^{1/b}}{1 - \left(\frac{1}{m}\right)^{1/b}} &= 1 + \frac{\left(\frac{1}{m}\right)^{1/b} - \left(\frac{1}{k}\right)^{1/b}}{1 - \left(\frac{1}{m}\right)^{1/b}} \\ &\leq \exp \left( \frac{1}{b} \left(\frac{1}{m}\right)^{1/b} \frac{k-m}{\left(1 - \left(\frac{1}{m}\right)^{1/b}\right)k} \right) \end{aligned}$$

and

$$\begin{aligned} \frac{1 - \left(\frac{1}{m}\right)^{1/b}}{1 - \left(\frac{1}{k}\right)^{1/b}} &= 1 - \frac{\left(\frac{1}{m}\right)^{1/b} - \left(\frac{1}{k}\right)^{1/b}}{1 - \left(\frac{1}{k}\right)^{1/b}} \\ &\leq \exp \left( -\frac{1}{b} \left(\frac{1}{k}\right)^{1/b} \frac{k-m}{\left(1 - \left(\frac{1}{k}\right)^{1/b}\right)m} \right). \end{aligned}$$

Then the theorem follows by repeating the same arguments as in the preceding section.  $\square$

#### 4. A WEIGHT OF THE FORM $v_2(z) = \exp(-a/(1-r^2)^b)$ .

As a consequence of the preceding discussion we consider here the weight

$$v_2(z) = \exp \left( \frac{-a}{(1-r^2)^b} \right)$$

for given constants  $a, b > 0$ . We compare  $v_2$  with the weight  $v_1(z) = \exp(-a/(1-r)^b)$  of the preceding section (with  $w \equiv 1$ ).

Put

$$A = \left\{ f \in H_{v_2}^\infty : f(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k} \text{ for some } a_{2k} \right\}$$

and

$$B = z \cdot A = \left\{ g \in H_{v_2}^\infty : g(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1} \text{ for some } a_{2k+1} \right\}.$$

Moreover, let  $T_1, T_2 : H_{v_1}^\infty \rightarrow H_{v_2}^\infty$  be the maps with

$$(T_1 h)(z) = h(z^2) \quad \text{and} \quad (T_2 h)(z) = zh(z^2), \quad h \in H_{v_1}^\infty, z \in \mathbb{D}.$$

**Proposition 4.1.** *The operator  $T_1$  maps  $H_{v_1}^\infty$  isometrically onto  $A$ . The map  $T_2$  is a contractive operator from  $H_{v_1}^\infty$  onto  $B$ . Moreover, we have*

$$H_{v_2}^\infty = A \oplus B.$$

Proof. The map  $T_1$  is certainly an isometry into  $A$  and  $T_2$  is a contractive operator into  $B$ . To show the surjectivity, let  $f \in B$ , say

$$(4.1) \quad f(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}.$$

Then put  $h(z) = \sum_{k=0}^{\infty} a_{2k+1} z^k$ . In view of (4.1), since the series representing  $f(z)/z$  converges uniformly on compact subsets of  $\mathbb{D}$ , we have

$$c := \sup_{|z| \leq 1/2} \frac{|f(z)|}{|z|} < \infty.$$

Hence,

$$(4.2) \quad \begin{aligned} \|h\|_{v_1} &= \sup_{|z| < 1} |h(z)| v_1(z) = \sup_{|z| < 1} |h(z^2)| v_1(z^2) = \sup_{|z| < 1} \frac{|f(z)|}{|z|} v_2(z) \\ &= \max \left( \sup_{|z| \leq 1/2} \frac{|f(z)|}{|z|} v_2(z), \sup_{1/2 < |z| < 1} \frac{|f(z)|}{|z|} v_2(z) \right) \end{aligned}$$

We obtain  $h \in H_{v_1}^{\infty}$  and clearly  $T_2 h = f$ . This shows that  $T_2$  maps  $H_{v_1}^{\infty}$  onto  $B$ . Similarly we see that  $T_1$  maps  $H_{v_1}^{\infty}$  onto  $A$ .

Now consider the operator  $P$  with  $(Pf)(z) = (f(z) + f(-z))/2$  for  $f \in H_{v_2}^{\infty}$  and  $z \in \mathbb{D}$ .  $P$  is a contractive projection from  $H_{v_2}^{\infty}$  onto  $A$ . We clearly get  $(id - P)(H_{v_2}^{\infty}) = B$ . Hence  $H_{v_2}^{\infty} = A \oplus B$ .  $\square$

Now we take the numbers  $m_n$  of Theorem 3.1 for  $w \equiv 1$ , i.e.

$$m_n = \frac{b^{1+1/b}}{a^{1/b}} n^{2+2/b} - bn^2.$$

Let  $[s]$  denote the largest integer which is smaller than or equal to  $s$ .

**Theorem 4.2.** *The solid hull of  $H_{v_2}^{\infty}$  is equal to*

$$\left\{ (b_m) : \sup_n e^{-bn^2} \left( \sum_{\substack{m \in \mathbb{N} \\ 2[m_n]+1 < m \leq 2[m_{n+1}]+1}} |b_m|^2 \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right)^{2[m/2]} \right)^{1/2} < \infty \right\}.$$

Moreover, the solid core of  $H_{v_2}^{\infty}$  is equal to

$$\left\{ (b_m) : \sup_n e^{-bn^2} \sum_{\substack{m \in \mathbb{N} \\ 2[m_n]+1 < m \leq 2[m_{n+1}]+1}} |b_m| \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right)^{[m/2]} < \infty \right\}.$$

Proof. Using Proposition 4.1 and Theorem 3.1 we see that  $(b_m) \in S(H_{v_2}^{\infty})$  if and only if

$$(4.3) \quad \sup_n e^{-bn^2} \left( \sum_{\substack{m \in \mathbb{N} \\ m_n < m \leq m_{n+1}}} |b_{2m}|^2 \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right)^{2m} \right)^{1/2} < \infty$$

and

$$(4.4) \quad \sup_n e^{-bn^2} \left( \sum_{\substack{m \in \mathbb{N} \\ m_n < m \leq m_{n+1}}} |b_{2m+1}|^2 \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right)^{2m} \right)^{1/2} < \infty.$$

If  $m \in \mathbb{N}$  and  $m_n < m \leq m_{n+1}$  then  $[m_n] + 1 \leq m \leq [m_{n+1}]$ . We obtain  $2[m_n] + 2 \leq 2m \leq 2[m_{n+1}]$  and  $2[m_n] + 3 \leq 2m + 1 \leq 2[m_{n+1}] + 1$ . Hence (4.3) and (4.4) are equivalent to

$$\sup_n e^{-bn^2} \left( \sum_{\substack{m \in \mathbb{N} \\ 2[m_n]+1 < m \leq 2[m_{n+1}]+1}} |b_m|^2 \left( 1 - \left( \frac{a}{bn^2} \right)^{1/b} \right)^{2[m/2]} \right)^{1/2} < \infty.$$

The proof for the solid core is the same.  $\square$

## 5. MAXIMAL SOLID CORES AND MINIMAL SOLID HULLS.

In this section we prove some general results on the relations of Schauder bases and solid hulls and cores for  $H_v^\infty$ -spaces. We refer the reader to [14] for terminology about bases in Banach spaces. If the monomials form a Schauder basic sequence in  $H_v^\infty$ , then obviously the condition of being solid is related with the property of  $\{z^m\}$  being an unconditional basis. Another, related fact will be proven in Theorem 5.2. We also show the unexpected fact that for some special weights,  $H_v^\infty$  is solid.

**Proposition 5.1.** *We have  $S(H_v^\infty) = H_v^\infty$  if and only if  $s(H_v^\infty) = H_v^\infty$ .*

Proof. If  $S(H_v^\infty) = H_v^\infty$  then  $h_f \in H_v^\infty$  (see (2.4)) for all  $f \in H_v^\infty$ . Hence  $s(H_v^\infty) = H_v^\infty$ .

Now assume  $s(H_v^\infty) = H_v^\infty$  and take  $g \in S(H_v^\infty)$  with  $g(z) = \sum_{k=0}^\infty b_k z^k$ . There is  $f \in H_v^\infty$  with  $f(z) = \sum_{k=0}^\infty a_k z^k$  and  $|b_k| \leq |a_k|$  for all  $k$ . Since by assumption  $h_f \in H_v^\infty$  we obtain

$$\|g\|_v \leq \sup_r v(r) \sum_{k=0}^\infty |a_k| r^k = \|h_f\|_v < \infty$$

which implies  $g \in H_v^\infty$ . Hence  $S(H_v^\infty) = H_v^\infty$ .  $\square$

**Example.** Consider the weight  $v(r) = \exp(-\log^2(r))$  on the complex plane  $\mathbb{C}$ . According to [15], Theorem 2.5., there is a constant  $d > 0$  such that for every  $f \in H_v^\infty$  with  $f(z) = \sum_{k=0}^\infty a_k z^k$  we have

$$\sup_k (|a_k| \exp(k^2/4)) \leq \|f\|_v \leq d \sup_k (|a_k| \exp(k^2/4)).$$

Then clearly  $h_f \in H_v^\infty$ . Indeed, let  $(h_f)_n$  be the partial sums of  $h_f$ , i.e.  $(h_f)_n(z) = \sum_{k=0}^n |a_k| z^k$ . Then  $(h_f)_n \in H_v^\infty$ ,  $(h_f)_n \rightarrow h_f$  pointwise on  $\mathbb{C}$  and

$$\|h_f\|_v \leq \sup_n \|(h_f)_n\|_v \leq d \sup_k (|a_k| \exp(k^2/4)) \leq d \|f\|_v < \infty.$$

Hence  $S(H_v^\infty) = H_v^\infty = s(H_v^\infty)$ .

Recall that we denote by  $H_v^0$  the closure of the polynomials in  $H_v^\infty$ . We put  $\Lambda = \{z^k : k = 0, 1, 2, \dots\}$ .

**Theorem 5.2.** *If  $S(H_v^\infty) = H_v^\infty$  then  $\Lambda$  is a Schauder basis of  $H_v^0$ .*

We prove Theorem 5.2 at the end of this section. At first we state

**Corollary 5.3.** *In the case of analytic functions on the disc  $\mathbb{D}$ , one always has  $S(H_v^\infty(\mathbb{D})) \neq H_v^\infty(\mathbb{D})$  and  $s(H_v^\infty(\mathbb{D})) \neq H_v^\infty(\mathbb{D})$ .*

**Proof.** According to [15], Theorem 2.2.,  $\Lambda$  is never a basis for  $H_v^0(\mathbb{D})$ . This proves Corollary 5.3 in view of Theorem 5.2  $\square$

For the proof of Theorem 5.2 we need two lemmas.

**Lemma 5.4.** (i) Fix  $m \in \mathbb{N}$  and  $\epsilon > 0$ . Then there is  $r_0 < R$  such that, for every  $f$  with  $f(z) = \sum_{k=0}^m a_k z^k$ , we have

$$\sup_{r_0 \leq |z| < R} |f(z)|v(z) \leq \epsilon \|f\|_v.$$

(ii) Fix  $0 \leq r_1 < R$  and  $\epsilon > 0$ . Then there is  $n \in \mathbb{N}$  such that, for any  $g \in H_v^\infty$  with  $g(z) = \sum_{k=n}^\infty a_k z^k$ , we have

$$\sup_{0 \leq |z| \leq r_1} |g(z)|v(z) \leq \epsilon \|g\|_v.$$

**Proof.** (i) Fix  $r < R$ . Then we clearly have

$$|a_k| \leq \frac{\|f\|_v}{r^k v(r)} \quad \text{for all } k = 0, 1, \dots, m.$$

We obtain

$$|f(z)|v(z) \leq \sum_{k=0}^m \left(\frac{|z|}{r}\right)^k \frac{v(|z|)}{v(r)} \|f\|_v.$$

Find  $r_0 > 0$  such that

$$\left(\frac{|z|}{r}\right)^k \frac{v(|z|)}{v(r)} \leq \frac{\epsilon}{m+1}$$

whenever  $|z| > r_0$ . This is possible since, by assumption,  $\lim_{|z| \rightarrow R} |z|^k v(|z|) = 0$  for all  $k$ . This implies (i).

(ii) Fix  $r > r_1$  and consider  $g(z) = \sum_{k=n}^\infty a_k z^k$ . We have  $|a_k| \leq \|g\|_v / (r^k v(r))$  for all  $k$ . This implies, if  $|z| \leq r_1$ ,

$$|g(z)|v(z) \leq \sum_{k=n}^\infty \frac{|z|^k v(|z|)}{r^k v(r)} \|g\|_v \leq \sum_{k=n}^\infty \left(\frac{r_1}{r}\right)^k \frac{v(0)}{v(r)} \|g\|_v.$$

We find  $n$  so large that

$$\sum_{k=n}^\infty \left(\frac{r_1}{r}\right)^k \frac{v(0)}{v(r)} \leq \epsilon$$

which proves the lemma.  $\square$

**Lemma 5.5.** Let  $f = \sum_{j=0}^\infty a_j z^j$  be an analytic function on the disc, let  $m_1 < m_2 < \dots$  be indices and  $f_n(z) = \sum_{j=m_n+1}^{m_{n+1}} a_j z^j$ . Then there is a subsequence  $(f_{n_k})_{k=0}^\infty$  such that

$$(5.1) \quad \sup_{k \in \mathbb{N}} \|f_{n_k}\|_v \leq 2 \left\| \sum_{k=0}^\infty f_{n_k} \right\|_v.$$

We remark that for any subsequence  $(f_{n_k})_{k=0}^\infty$ , the sum  $\sum_k f_{n_k}$  on the right-hand side of (5.1) is the Taylor series of an analytic function on the disc, so the sum converges at least uniformly on compact subsets of  $\mathbb{D}$ ; if the sum does not belong to  $H_v^\infty$ , its norm is infinity and the inequality (5.1) becomes a triviality.



**Proof.** Use Lemma 5.4 and induction to find a subsequence  $(f_{n_k})$  and radii  $r_1 < r_2 < \dots$  such that  $|f_{n_k}(z)|v(|z|) \leq 3^{-k}\|f_{n_k}\|_v$  whenever  $|z| \leq r_k$  or  $|z| \geq r_{k+1}$ . Hence

$$(5.2) \quad \|f_{n_k}\|_v = \sup_{r_k \leq |z| \leq r_{k+1}} |f_{n_k}(z)|v(|z|).$$

By the remark above, we may assume that  $\sum_k f_{n_k} \in H_v^\infty$ . Fix  $j$ . If  $r_j \leq |z| \leq r_{j+1}$  we obtain

$$\begin{aligned} \left\| \sum_k f_{n_k} \right\|_v &\geq \left| \sum_k f_{n_k}(z)v(z) \right| \\ &\geq |f_{n_j}(z)|v(z) - \sum_{k \neq j} |f_{n_k}(z)|v(z) \\ &\geq |f_{n_j}(z)|v(z) - \sum_{k \neq j} \frac{1}{3^k} \|f_{n_k}\|_v \\ &\geq |f_{n_j}(z)|v(z) - \frac{1}{2} \sup_k \|f_{n_k}\|_v. \end{aligned}$$

In view of (5.2) this implies

$$\left\| \sum_k f_{n_k} \right\|_v \geq \|f_{n_j}\|_v - \frac{1}{2} \sup_k \|f_{n_k}\|_v \quad \text{for all } j$$

and hence

$$\left\| \sum_k f_{n_k} \right\|_v \geq \frac{1}{2} \sup_k \|f_{n_k}\|_v$$

which proves the lemma.  $\square$

**Proof of Theorem 5.2.** For any subset  $N$  of  $\mathbb{N}$ , let  $T_N$  be the operator with  $T_N(\sum_{k=0}^\infty a_k z^k) = \sum_{k \in N} a_k z^k$ . If  $S(H_v^\infty) = H_v^\infty = s(H_v^\infty)$  then  $T_N(H_v^\infty) \subset H_v^\infty$ . The closed graph theorem implies that  $T_N$  is bounded.

Now let  $P_n$  be the Dirichlet projections, i.e.  $P_n(\sum_{k=0}^\infty a_k z^k) = \sum_{k=0}^n a_k z^k$ . Assume that  $\Lambda$  is not a basis for  $H_v^0$ . Then the  $P_n$  are not uniformly bounded. By the uniform boundedness theorem we obtain a function  $f \in H_v^0$  such that  $\sup_n \|P_n(f)\|_v = \infty$ . Hence we can find a subsequence  $P_{n_m}$  with  $\lim_{m \rightarrow \infty} \|(P_{n_{m+1}} - P_{n_m})(f)\|_v = \infty$ . Put  $f_m = (P_{n_{m+1}} - P_{n_m})(f)$ . Then,  $\sum_m f_m \in H_v^\infty$ , since this sum is of the form  $T_N f$  for some subset  $N$  of  $\mathbb{N}$ . We apply Lemma 5.5 to find a subsequence  $f_{m_k}$  such that

$$\sup_k \|f_{m_k}\|_v \leq 2 \left\| \sum_k f_{m_k} \right\|_v.$$

The left hand side of this inequality is infinite while the function on the right-hand side is again of the form  $T_{\tilde{N}} f$  for some  $\tilde{N} \subset \mathbb{N}$  and thus has finite norm as an element  $H_v^\infty$ . So we arrive at a contradiction. Therefore  $\Lambda$  is a basis of  $H_v^0$ .  $\square$

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## REFERENCES

- [1] J.M. Anderson, A.L. Shields, Coefficient multipliers of Bloch functions, *Trans. Amer. Math. Soc.* 224 (1976), 255-265.
- [2] G. Bennet, D.A. Stegenga, R.M. Timoney, Coefficients of Bloch and Lipschitz functions, *Illinois J. Math.* 25 (1981), 520-531.
- [3] K.D. Bierstedt, J. Bonet, A. Galbis, Weighted spaces of holomorphic functions on bounded domains, *Michigan Math. J.* 40 (1993), 271-297.
- [4] K.D. Bierstedt, J. Bonet, J. Taskinen, Associated weights and spaces of holomorphic functions, *Studia Math.* 127 (1998), 137-168.
- [5] O. Blasco, A. Galbis, On Taylor coefficients of entire functions integrable against exponential weights, *Math. Nachr.* 223 (2001), 5-21.
- [6] O. Blasco, M. Pavlovic, Coefficient multipliers on Banach spaces of analytic functions, *Rev. Mat. Iberoam.* 27 (2011), 415-447.
- [7] J. Bonet, J. Taskinen, Solid hulls of weighted Banach spaces of entire functions, to appear in *Rev. Mat. Iberoamericana*. arXiv: 1607.02237v1
- [8] J. Bonet, J. Taskinen, Solid hulls of weighted Banach spaces of analytic functions on the unit disc with exponential weights, to appear in *Ann. Acad. Sci. Fenn.* arXiv:1702.00145v1.
- [9] O. Constantin, J.A. Peláez, Boundedness of the Bergman projection on  $L_p$ -spaces with exponential weights, *Bull. Sci. Math.* 139 (2015), no. 3, 245-268.
- [10] M.R. Dostanić, Multipliers in the space of analytic functions with exponential mean growth, *Asymptot. Anal.* 65 (2009), no. 3-4, 191-201.
- [11] M-R. Dostanić, Integration operators on Bergman spaces with exponential weight, *Rev. Mat. Iberoam.* 23 (2007), no. 2, 421-436.
- [12] M. Jevtić, M. Pavlović, On the solid hull of the Hardy-Lorentz space, *Publ. Inst. Math. (Beograd) (N.S.)* 85(99) (2009), 55-61.
- [13] M. Jevtić, D. Vukotić, M. Arsenović, Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-Type Spaces, *RSME Springer Series*, Volume 2. Springer 2016.
- [14] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces I*, Springer, Berlin, 1977.
- [15] W. Lusky, On the Fourier series of unbounded harmonic functions, *J. of the Lond. Math. Soc.* (2) 61, 568-580, (2000)
- [16] W. Lusky, On the isomorphism classes of weighted spaces of harmonic and holomorphic functions, *Studia Math.* 175, 19-45, (2006)
- [17] J. Pau, J.A. Peláez, Volterra type operators on Bergman spaces with exponential weights. *Topics in complex analysis and operator theory*, 239-252, *Contemp. Math.*, 561, Amer. Math. Soc., Providence, RI, 2012.
- [18] M. Pavlović, On harmonic conjugates with exponential mean growth, *Czech. Math. J.* 49 (1999), 733-742.
- [19] M. Pavlović, *Function classes on the unit disc. An introduction*, De Gruyter Studies in Mathematics, 52. De Gruyter, Berlin, 2014. xiv+449 pp.
- [20] J.A. Peláez, J. Rättyä, Weighted Bergman spaces induced by rapidly increasing weights, *Mem. Amer. Math. Soc.* 227 (2014), no. 1066, vi+124 pp.
- [21] A.L. Shields, D.L. Williams, Bounded projections, duality and multipliers in spaces of analytic functions, *Trans. Amer. Math. Soc.* 162 (1971), 287-302.

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