SCHAUDER BASES AND THE DECAY RATE OF THE HEAT EQUATION

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Abstract. We consider the classical Cauchy problem for the linear heat equation and integrable initial data in the Euclidean space $\mathbb{R}^N$. In the case $N = 1$ we show that given a weighted L^p-space $L^p_w(\mathbb{R})$ with $1 \le p < \infty$ and a fast growing weight w, there is a Schauder basis $(e_n)_{n=1}^\infty$ in $L^p_w(\mathbb{R})$ with the following property: given a positive integer m there exists $n_m > 0$ such that, if the initial data f belongs to the closed linear space of e_n with $n \ge n_m$, then the decay rate of the solution of the heat equation is at least t^{-m}. The result is also generalized to the case $N > 1$ with a slightly weaker formulation. The proof is based on a construction of a Schauder basis of $L^p_w(\mathbb{R}^N)$, which annihilates an infinite sequence of bounded functionals.

1. Introduction.

Given an integrable function $f \in L^1(\mathbb{R}^N)$ in the Euclidean space $\mathbb{R}^N$, $N \in \mathbb{N}$, the unique solution of the classical Cauchy problem for the linear heat (or diffusion) equation

$$\partial_t u(x,t) = \Delta u(x,t) \quad \text{for } x \in \mathbb{R}^N,\ t > 0 \tag{1.1}$$

$$u(x,0) = f(x) \quad \text{for } x \in \mathbb{R}^N, \tag{1.2}$$

has the decay rate $t^{-N/2}$ for large "times" t. This follows directly from the well-known solution formula

$$u(x,t) = e^{t\Delta} f(x) := \frac{1}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{1}{4t}(x-y)^2} f(y)dy, \tag{1.3}$$

where we write $x^2 := |x|^2 = \sum_{j=1}^N x_j^2$ for vectors $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\Delta = \sum_{j=1}^N \partial_j^2 = \sum_{j=1}^N (\partial/\partial x_j)^2$ for the Laplacian; if $N = 1$ we denote $e^{t\partial_x^2} f$ instead of $e^{t\Delta} f$. Indeed, (1.3) implies the bound

$$\|u(\cdot,t)\|_p := \Big(\int\limits_{\mathbb{R}^N} |u(x,t)|^p dx \Big)^{1/p} \le C_p t^{-N/2} \tag{1.4}$$

for large t, for any $p \in [1,\infty)$ and also the same estimate for the sup-norm $\|u(\cdot,t)\|_\infty$ with the usual definition.

For general initial data $f \in L^1(\mathbb{R}^N)$, which is not necessarily positive, cancellation phenomena may cause faster decay rates. For example in the case $N = 1$, if f is such that $\int_{-\infty}^{\infty} f(x)dx = 0$, then a simple argument shows that $e^{t\partial_x^2} f$ decays at least with the speed t^{-1}; see Proposition 2.2 for an exact, more general formulation of this phenomenon.

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To describe our main result on decay rates we fix a continuous weight function $w : \mathbb{R}^N \to \mathbb{R}^+$ with symmetry $w(x) := w(-x)$ for all $x \in \mathbb{R}^N$. We assume w is fast growing which means that

$$\sup_{x\in\mathbb{R}^N} \frac{1}{w(x)}(1+|x|)^m < \infty \ \ \forall\, m \in \mathbb{N}. \tag{1.5}$$

Given $p \in [1,\infty)$ we denote by $L^p_w(\mathbb{R}^N)$ the weighted L^p-space on $\mathbb{R}^N$ endowed with the norm

$$\|f\|_{p,w} := \Big(\int\limits_{\mathbb{R}^N} |f(x)|^p w(x) dx \Big)^{1/p}. \tag{1.6}$$

Our main result, in addition to Theorem 2.4 on Schauder bases which annihilate linear functionals, reads as follows:

Theorem 1.1. *Let $1 \le p < \infty$ and let the weight w satisfy the conditions above.*
1°. *Let $N = 1$. There exists a Schauder basis $(e_n)_{n=1}^\infty$ of the Banach-space $L^p_w(\mathbb{R})$ with the following property: given $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that any initial data f*

$$f = \sum_{n=1}^\infty f_n e_n \in L^p_w(\mathbb{R}), \tag{1.7}$$

with the property $f_n = 0$ for all $n = 1, \ldots, n_m$, has the fast decay property

$$\|e^{t\Delta} f\|_\infty \le \frac{C}{t^m} \|f\|_{p,w} \tag{1.8}$$

for all $t \ge 1$.
2°. *If $N > 1$, then there exists a weight $\tilde{w} : \mathbb{R}^N \to \mathbb{R}^+$ satisfying the assumptions around* (1.5) *such that $L^p_w(\mathbb{R}^N) \subset L^1_{\tilde{w}}(\mathbb{R}^N)$ and such that the space $L^1_{\tilde{w}}(\mathbb{R}^N)$ has a Schauder basis $(e_n)_{n=1}^\infty$ with the same property as in 1° ($\|f\|_{1,\tilde{w}}$ replacing $\|f\|_{p,w}$ in* (1.8)*) .*

In other words, if initial data is included in the finite co-dimensional subspace $G_m = \overline{\mathrm{sp}(e_n : n \ge n_m)}$, then the corresponding solution decays at least at the speed t^{-m}; leaving out finitely many coordinates in the Banach-space of initial data makes the solution decay fast. The subspace G_m thus has an explicit description in terms of the Schauder basis, although in general we are not able to determine the precise decay rate, if the initial data is in the complement space.

If $N > 1$ and $p = 1$ and the weight w has a special symmetric form, then we may still take $\tilde{w} = w$. See Section 4 for details.

Remark 1.2. a) We emphasize the following general aspect of our construction in the case $N = 1$, $p > 1$: to find the basis we split the space $L^p_w(\mathbb{R}) = L^{p;-}_w(\mathbb{R}) \oplus L^{p;+}_w(\mathbb{R})$, where the two subspaces consist of functions vanishing on the positive or negative real line, respectively. Then, the basis in Theorem 1.1, is constructed as small perturbations of any given Schauder bases of $L^{p;\pm}_w(\mathbb{R})$. Due this general nature of the result, we only obtain the existence of the desired basis, but not explicit information on the magnitude of the numbers n_m. See the end of Section 4.

b) By classical arguments, the heat kernel in (1.3) can be expanded as the series

$$e^{\frac{1}{4t}(x-y)^2} = \sum_{n\in\mathbb{N}_0} \frac{1}{t^{n/2}} H_n(x) y^n \tag{1.9}$$

where H_n are suitably normalized Hermite functions. If $m \in \mathbb{N}$, one can write a given f, say, belonging to $L^2_w(\mathbb{R})$ with $w(x) = e^{-x^2/2}$, as

$$f = \sum_{n=1}^{m} f_n H_n + g,$$

where the coefficients f_n are chosen such that $\int_{\mathbb{R}} y^n g(y) dy = 0$ for $n = 1, \ldots, m$. Then, the solution with initial data g has the decay rate $t^{-(m+1)/2}$. This known observation gives information resembling our result, although it does not give such a general decomposition of the initial data space as Theorem 1.1. We also mention [5], Appendix A, where analogous results in the form of spectral decompositions are derived for more general equations.

There is an extensive literature dealing with the decay rate of the solution to the Cauchy problem of the heat equation. For example, precise decay rates in the linear case have been considered in [2], although most of the recent research is concentrated on semilinear or other nonlinear generalizations of (1.1)—(1.2). As a slightly random sample we mention the papers [3, 4, 6, 7, 9, 13, 14, 16, 17, 23]; see also the monograph [18] for an exposition. We especially mention the papers [1, 9, 10, 11, 19, 20], where the asymptotic large time behavior of the semilinear problem is considered by separating the faster decay of terms with vanishing integrals. The paper [10] contains the state of art in this direction and in fact has partially been a source of inspiration for the present work.

We organize our paper as follows. Section 2 is devoted to the case $N = 1$. We discuss the known phenomenon that for special initial data with certain vanishing iterated integrals the decay rate can be made arbitrarily fast. This leads to the definition of special continuous linear functionals in the space $L^p_w(\mathbb{R})$, $1 \le p < \infty$, and to the formulation of Theorem 2.4 concerning the existence of Schauder basis annihilating given functionals. We show how Theorem 1.1 follows from this result, although the proof of Theorem 2.4 is only presented in Section 3. Theorem 2.4 uses the concept of a shrinking Schauder basis: since an arbitrary basis of the non-reflexive space $L^1_w(\mathbb{R})$ is not necessarily shrinking, this case requires a separate treatment, which is contained in Lemma 2.5.

The case $N > 1$ of Theorem 1.1 will be considered in Section 4. The proof is based on decomposing a given initial data of several variables into a convergent sum of products of functions in one variable and using the already proven one-dimensional case. Here, our method requires the use of L^1-norms and a little abstract tensor product techniques. At the end of the Section 4 we discuss some interesting open problems.

We will use the following general notation. By C, C' etc. we denote generic positive constants, the exact value of which may change from place to place. The possible dependence, say, on a parameter p is indicated as C_p. By $\operatorname{supp} f$ we denote the support of a function f and by $\operatorname{sp}(A)$ the linear span of a subset A of a vector space. Its closure is denoted by $\overline{\operatorname{sp}(A)}$. We write $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $\mathbb{R}^{\pm} = \{x \in \mathbb{R} : \pm x \ge 0\}$. The characteristic or indicator function of a set A is denoted by $\mathbf{1}_A$. We use standard notation $L^p(\mathbb{R}^N)$, $L^p(0,1)$ etc. for unweighted Lebesgue spaces. Moreover, X^* stands for the dual of a Banach space X, i.e. the space of bounded linear functionals on X. The norm of X^* is denoted $\| \cdot \|_{X^*}$.

The identity operator $X \to X$ is denoted by id_X. For a linear operator T between Banach spaces, $\|T\|$ denotes the operator norm.

If X denotes a separable Banach space over the scalar field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$), we recall that a sequence $(e_n)_{n=1}^{\infty} \subset X$ is a Schauder basis, if every element $f \in X$ can be presented as a convergent sum

$$f = \sum_{n=1}^{\infty} f_n e_n \tag{1.10}$$

where the numbers $f_n \in \mathbb{K}$ are unique for f. For example in a separable Hilbert space, every orthonormal basis is a Schauder basis, but the converse is of course not true. There are many well-known constructions of Schauder bases in classical Banach spaces; among them, the wavelet bases are most studied in the recent years. We refer to [15], [22] for this topic.

2. Proof of Theorem 1.1 in the case $N = 1$

In this section we show how Theorem 1.1 follows from an abstract result concerning bases which annihilate linear functionals, Theorem 2.4. First, we recursively define the linear operators

$$I^0 f := f \ , \quad I^m f(x) = \int_{-\infty}^{x} I^{m-1} f(y) dy \quad \text{for } m \in \mathbb{N} \ ,$$

$$J^m f = \int_{-\infty}^{0} I^{m-1} f(y) dy \text{ for } m \in \mathbb{N}. \tag{2.1}$$

By the Cauchy formula for repeated integrations these can be written as

$$I^m f(x) = \frac{1}{(m-1)!} \int_{-\infty}^{x} (x-y)^{m-1} f(y) dy \ , \tag{2.2}$$

$$J^m f = \frac{1}{(m-1)!} \int_{-\infty}^{0} (-y)^{m-1} f(y) dy \ , \quad m \in \mathbb{N}. \tag{2.3}$$

The operators I^m do not map $L^p_w(\mathbb{R})$ even into $L^p(\mathbb{R})$ (since $If := I^1 f$ may be bounded from below by a positive constant for large x). However, we have the following simple observations. We denote $L^{p,-}_w(\mathbb{R}) := \{f \in L^p_w(\mathbb{R}) : \mathrm{supp} f \subset \mathbb{R}^-\}$.

Lemma 2.1. *(i) If $m \in \mathbb{N}$ and $f \in L^p_w(\mathbb{R})$, then the restriction of $I^m f$ to $\mathbb{R}^-$ is rapidly decreasing, as $x \to -\infty$: we have*

$$\sup_{x \in \mathbb{R}^-} (1 + |x|)^k |I^m f(x)| \le C_{k,m,p} \|f\|_{p,w} < \infty \quad \forall\, k \in \mathbb{N}. \tag{2.4}$$

(ii) If $m \in \mathbb{N}$ is given and $f \in L^{p,-}_w(\mathbb{R})$ has the property that $J^k f = 0$ for all $k \in \mathbb{N}$ with $k \le m$, then

$$\mathrm{supp}\, I^k f \subset \mathbb{R}^- \tag{2.5}$$

for all $k \le m$. In particular $I^k f \in L^1(\mathbb{R})$ and

$$\|I^k f\|_1 \le C_{k,p,w} \|f\|_{p,w} \tag{2.6}$$

for every $k \le m$.
(iii) J^m *is a bounded linear functional on* $L^p_w(\mathbb{R})$.

Proof. As for (i), we consider $p > 1$ with the dual exponent $p' = p/(p-1)$. Then, (2.2), (1.5) and the Hölder inequality imply for $x \le 0$

$$(1+|x|)^k |I^m f(x)| \le C_m (1+|x|)^k \int_{-\infty}^{x} |x-y|^{m-1} |f(y)| dy$$

$$\le C_m (1+|x|)^k \int_{-\infty}^{x} |y|^{m-1} (1+|y|)^{-k-m+1-\frac{2}{p'}} (1+|y|)^{k+m-1+\frac{2}{p'}} |f(y)| dy$$

$$\le C_m \int_{-\infty}^{x} (1+|y|)^{-\frac{2}{p'}} (1+|y|)^{k+m-1+\frac{2}{p'}} |f(y)| dy$$

$$\le C_m \Big(\int_{-\infty}^{0} (1+|y|)^{-2} dy \Big)^{1/p'} \Big(\int_{-\infty}^{0} (1+|y|)^{p(k+m-1+\frac{2}{p'})} |f(y)|^p dy \Big)^{1/p}$$

$$\le C_{k,m,p} \Big(\int_{-\infty}^{0} w(y) |f(y)|^p dy \Big)^{1/p} \le C_{k,m,p} \|f\|_{p,w}.$$

The proof for the case $p = 1$ is simpler, as the exponents $2/p'$ are omitted and the Hölder inequality is not needed.

Concerning (ii), a simple induction argument yields (2.5): assume that $J^k f = 0$ for all $k \le m$ and that $\tilde{m} < m$ and (2.5) holds for all $k \le \tilde{m}$. Then, by the definition of $I^{\tilde{m}+1}$, for $x \ge 0$,

$$I^{\tilde{m}+1} f(x) = \int_{-\infty}^{0} I^{\tilde{m}} f(y) dy + \int_{0}^{x} I^{\tilde{m}} f(y) dy$$

Here, the first term equals $J^{\tilde{m}+1} f$ and is thus 0, and the second term also vanishes by the induction assumption. The bound (2.6) follows from (2.5), (2.4) and an application of the Hölder inequality.

The statement (iii) follows from (2.3), (1.5), and the Hölder inequality. □

The following fact about faster convergence rates for special initial data is known, but we need to present and prove a formulation, which precisely fits to our arguments.

Proposition 2.2. *Let* $N = 1$ *and let* $f \in L^{p,-}_w(\mathbb{R})$ *be such that for some* $m \in \mathbb{N}$*, it satisfies* $J^k f = 0$ *for all* $k \in \mathbb{N}$ *with* $k \le m$*. Then, there holds the bound*

$$\|e^{t\partial_x^2} f\|_\infty \le \frac{C_{p,w,m} \|f\|_{p,w}}{t^{(1+m)/2}} \tag{2.7}$$

for the solution of (1.1)–(1.2) *with the initial data* f.

Proof. Let m and f be as in the assumption. We employ repeated integration by parts with respect to y in order to evaluate (1.3). In this process there appear the expressions $I^k f$, which according to our assumptions and Lemma 2.1 belong to $L^1(\mathbb{R})$. At the first step we write

$$u(x,t) = -\frac{1}{\sqrt{2\pi t}} \int\limits_{\mathbb{R}} \big(\partial_y e^{-\frac{1}{4t}(x-y)^2}\big)(If)(y)dy = -\frac{1}{2\sqrt{2\pi t}} \int\limits_{\mathbb{R}} \frac{x-y}{t} e^{-\frac{1}{4t}(x-y)^2}(If)(y)dy,$$

where obviously the replacement term vanishes since both the Gaussian kernel and If are rapidly decreasing functions. Repeating integration by parts k times, an induction proof shows that

$$u(x,t) = \frac{C}{\sqrt{t}} \int\limits_{\mathbb{R}} P_k(x,y,t) e^{-\frac{1}{4t}(x-y)^2}(I^k f)(y)dy \tag{2.8}$$

where the function P_k is a finite sum of terms of the form

$$C_a \frac{1}{t^{k/2}} \frac{(x-y)^a}{t^{a/2}} \tag{2.9}$$

where $a \in \mathbb{N}_0$ and C_a are some constants. Indeed, given $k \in \mathbb{N}$ and $a \geq 1$,

$$\begin{aligned} &\partial_y\Big(\frac{1}{t^{k/2}} \frac{(x-y)^a}{t^{a/2}} e^{-\frac{1}{4t}(x-y)^2}\Big) \\ &= \Big(\frac{-1}{t^{k/2}} \frac{a(x-y)^{a-1}}{t^{a/2}} + \frac{1}{t^{k/2}} \frac{2(x-y)^{a+1}}{4t^{a/2+1}}\Big) e^{-\frac{1}{4t}(x-y)^2} \\ &= \Big(\frac{-1}{t^{(k+1)/2}} \frac{a(x-y)^{a-1}}{t^{(a-1)/2}} + \frac{1}{t^{(k+1)/2}} \frac{(x-y)^{a+1}}{2t^{(a+1)/2}}\Big) e^{-\frac{1}{4t}(x-y)^2} \end{aligned}$$

where we have two terms of the form (2.9) for $k+1$.

We evaluate

$$\begin{aligned} &\frac{1}{t^{1/2}} \Big| \int\limits_{\mathbb{R}} \frac{1}{t^{k/2}} \frac{(x-y)^a}{t^{a/2}} e^{-\frac{1}{4t}(x-y)^2}(I^k f)(y)dy \Big| \\ &\leq \frac{C_{p,w,k,a}}{t^{(1+k)/2}} \sup_{x,y\in\mathbb{R}} \Big(\Big(\frac{|x-y|^2}{4t}\Big)^{a/2} e^{-\frac{|x-y|^2}{4t}}\Big) \int\limits_{\mathbb{R}} \big|(I^k f)(y)\big| dy \\ &\leq \frac{C'_{p,w,k,a}}{t^{(1+k)/2}} \|f\|_{p,w}, \end{aligned} \tag{2.10}$$

since the supremum is bounded by a constant independent of t and the integral is bounded according to (2.6). We get the bound (2.7) for (2.8), by the remark on P_k around (2.9). □

Given a Schauder basis $(e_n)_{n=1}^\infty$ of X we denote for every $n \in \mathbb{N}$ by P_n the basis projection

$$P_n f = P_n\Big(\sum_{k=1}^\infty f_k e_k\Big) = \sum_{k=1}^n f_k e_k\ , \quad \text{where } f = \sum_{k=1}^\infty f_k e_k \in X. \tag{2.11}$$

The number $K = \sup_n \|P_n\|$ is called the basis constant of $(e_n)_{n=1}^\infty$; the supremum defining K is always finite, see [15].

Definition 2.3. Let $x^* \in X^*$. We say that a Schauder basis $(e_n)_{n=1}^\infty$ of X is *shrinking* with respect to x^* if

$$\lim_{n\to\infty} \|x^* \circ (\mathrm{id}_X - P_n)\|_{X^*} = 0. \tag{2.12}$$

For a basis $(e_n)_{n=1}^\infty$ of X consider the biorthogonal functionals $e_n^* \in X^*$, where $e_n^*(e_m) = \delta_{mn}$ (Kronecker delta); let $W = \overline{\mathrm{sp}\{e_n^* : n \in \mathbb{N}\}} \subset X^*$. It is easily seen that $(e_n^*)_{n=1}^\infty$ is a Schauder basis of W with the basis projections P_n^*, where $P_n^*(x^*) = x^* \circ P_n$ for $x^* \in X^*$. However we have $W \neq X^*$ in general. We obtain that $(e_n)_{n=1}^\infty$ is shrinking with respect to $x^* \in X^*$, if and only if $x^* \in W$.

Definition 2.3 extends slightly the classical notion of a shrinking basis, see [15]. A basis $(e_n)_{n=1}^\infty$ of X is shrinking, if it is shrinking with respect to all elements in X^* in the sense of the preceding definition, i.e. if $W = X^*$. In this case X^* must be separable. It is well-known that every basis of X is shrinking, if X is reflexive. Again, see [15] for more details.

Theorem 1.1 is a consequence of the following result, the proof of which is postponed to Section 3.

Theorem 2.4. *Let $x_m^* \in X^*$ for all $m \in \mathbb{N}$, and let $\epsilon > 0$. Assume that $(\tilde{e}_n)_{n=1}^\infty$ is a Schauder basis of X which is shrinking with respect to all x_m^*. Then, there exists an increasing sequence $(n_m)_{m=1}^\infty \subset \mathbb{N}$ and a basis $(e_n)_{n=1}^\infty$ of X such that*

$$x_m^*(e_n) = 0 \quad \text{for all} \quad n \geq n_m. \tag{2.13}$$

If $T : X \to X$ is the linear operator with $T\tilde{e}_n = e_n$ for all n, then we have

$$\|\mathrm{id}_X - T\| < \epsilon. \tag{2.14}$$

Obviously, condition (2.14) means that T is a bijection and the new basis $(e_n)_{n=1}^\infty$ can be considered as perturbation of the given basis $(\tilde{e}_n)_{n=1}^\infty$.

We repeat that every Schauder basis of a Banach space X is shrinking, if X is reflexive. This is in particular true for any orthonormal basis in a Hilbert space. However, in order to treat the case $p = 1$ we state the following result, which also will proven only in Section 3.

Lemma 2.5. *There exists a Schauder basis $(\tilde{e}_n^-)_{n=1}^\infty$ of $L_w^{1;-}(\mathbb{R})$ which is shrinking for all functionals J^m defined in (2.3).*

Proof of Theorem 1.1. Let p and w be as in the assumption and first consider the Banach space $X = L_w^{p;-}(\mathbb{R})$. The functionals $J^m =: x_m^*$ of (2.1) are well defined and bounded on X, by Lemma 2.1, (iii). We fix a basis $(\tilde{e}_n^-)_{n=1}^\infty$, which is shrinking with respect to all x_m^*; in the case $p = 1$ we use Lemma 2.5 to find this. Then, Theorem 2.4 yields the desired basis $(e_n^-)_{n=1}^\infty$ of $L_w^{p;-}(\mathbb{R})$ and the sequence of indices $(n_m)_{m=1}^\infty$; in particular, given $m \in \mathbb{N}$ we have

$$J^k(e_n^-) = 0 \tag{2.15}$$

for every $k \leq m$, $n \geq n_m$. To see that (1.8) holds for a given m and for any initial data $f^- \in G_{n_m}^- := \overline{\mathrm{sp}\,\{e_n^- : n \geq n_m\}} \subset L_w^{p;-}(\mathbb{R})$ we remark that such a f^- has a representation

$$f^- = \sum_{n=n_m}^{\infty} f_n^- e_n^-. \tag{2.16}$$

Since this series converges in $L_w^p(\mathbb{R})$ and every J^k is a continuous mapping, (2.15) implies $J^k g = 0$ for all $k \leq m$. Hence, (1.8) follows from Proposition 2.2.

To complete the proof we remark that the space $L_w^p(\mathbb{R})$ equals in a natural way the direct sum $L_w^{p;-}(\mathbb{R}) \oplus L_w^{p;+}(\mathbb{R})$, where the second component is defined as the closed subspace of $L^p_w(\mathbb{R})$ consisting of functions with supports in $\mathbb{R}^+$. The functions

$$e_n^+ := e_n^- \circ \psi \quad , \quad \text{where } \psi(x) := -x \ \forall x \in \mathbb{R} \tag{2.17}$$

form a Schauder basis of $L^{p,+}_w(\mathbb{R})$, which plays the same role as the basis $(e_n^-)_{n=1}^\infty$ has in $L^{p,-}_w(\mathbb{R})$. This follows from the formal commutation relations

$$\partial_x^2(f \circ \psi) = (\partial_x^2 f) \circ \psi \quad , \quad e^{t\partial_x^2}(f \circ \psi) = \big(e^{t\partial_x^2} f\big) \circ \psi. \tag{2.18}$$

Consequently, the union of the sequences $(e_n^-)_{n=1}^\infty$ and $(e_n^+)_{n=1}^\infty$ is the desired Schauder basis. $\square$

3. Proofs of Theorem 2.4 and Lemma 2.5.

We need the following elementary

Lemma 3.1. *Let $(h_n)_{n=1}^\infty$ be a basis of the Banach space Y with basis projections Q_n, $n \in \mathbb{N}$, and basis constant K. Moreover, let $T : Y \to Y$ be a linear operator with $c := \|\mathrm{id}_Y - T\| < 1$. Then $(Th_n)_{n=1}^\infty$ is a basis of Y with basis constant at most $K(1+c)/(1-c)$.*

Proof. By the assumption and the Neumann series, T is an isomorphism (linear homeomorphism), and we have $T^{-1} = \sum_{k=0}^\infty (\mathrm{id}_Y - T)^k$, hence $\|T^{-1}\| \le (1-c)^{-1}$. Moreover, $\|T\| \le 1 + \|\mathrm{id}_Y - T\| \le 1 + c$. Hence, $(Te_n)_{n=1}^\infty$ is a basis of Y with basis projections TP_nT^{-1} and basis constant at most $K\|T\|\,\|T^{-1}\| \le K(1+c)/(1-c)$. $\square$

Proposition 3.2. *Let $(\tilde{h}_n)_{n=1}^\infty$ be a basis of the Banach space Y with basis projections Q_n, $n \in \mathbb{N}$, and basis constant K. Moreover, let $L, M \in \mathbb{N}$ and assume that $y_m^* \in Y^*$, $m \in \mathbb{N}$, satisfy*

$$y_1^*|_{(\mathrm{id}_Y - Q_L)Y}, \ldots, y_M^*|_{(\mathrm{id}_Y - Q_L)Y} = 0$$

and

$$\lim_{n\to\infty} \|y_m^*|_{(\mathrm{id}_Y - Q_n)Y}\| = 0 \quad \text{for all } m. \tag{3.1}$$

Then for any $\delta > 0$ there is a basis $(h_n)_{n=1}^\infty$ of Y and an index $N > L$ with

$$h_n = \tilde{h}_n, \quad n = 1, \ldots, N,$$

$$y_{M+1}^*(h_n) = 0 \quad \text{if } n > N, \qquad y_k^*(h_n) = 0 \quad \text{for } k = 1, \ldots, M \text{ and } n \ge L+1, \tag{3.2}$$

and

$$\|\mathrm{id}_Y - S\| \le K\delta \tag{3.3}$$

for the linear operator $S : Y \to Y$ with $S\tilde{h}_n = h_n$ for all $n \in \mathbb{N}$. The basis constant of $(h_n)_{n=1}^\infty$ is at most $K(1+K\delta)/(1-K\delta)$.

Proof. If $y_{M+1}^*|_{(\mathrm{id}_Y - Q_L)Y} = 0$ then we can take $h_n = \tilde{h}_n$ for all n. Otherwise let $N > L$ be large enough and put

$$\rho = \frac{\|y_{M+1}^*|_{(\mathrm{id}_Y - Q_N)Y}\|}{\|y_{M+1}^*|_{(Q_N - Q_L)Y}\|}.$$

According to (3.1) we can choose N so large that $\rho < \delta$ and

$$K\rho < 1. \tag{3.4}$$

In fact ρ can be made arbitrarily small since the denominator in the definition of ρ goes to $\|y_{M+1}^*|(\mathrm{id}_Y - Q_L)Y\| > 0$ if N tends to ∞ while the numerator tends to 0 in view of (3.1). We find $x \in (Q_N - Q_L)Y$ with $\|x\| = 1$ and $y^*_{M+1}(x) = \|y^*_{M+1}|_{(Q_N-Q_L)Y}\|$. (Take into account that $(Q_N - Q_L)Y$ is finite dimensional.)

Put $Sf = f$ if $f \in Q_N Y$ and

$$Sg = g - \frac{y^*_{M+1}(g)}{\|y^*_{M+1}|_{(Q_N-Q_L)Y}\|}x \quad \text{if } g \in (\mathrm{id}_Y - Q_N)Y. \tag{3.5}$$

Then we have

$$\|f + g - S(f + g)\| = \|g - Sg\| \leq \rho\|g\| \leq \rho K\|f + g\|. \tag{3.6}$$

Let $h_n = S\tilde{h}_n$ for all n. According to Lemma 3.1 and in view of (3.4), $(h_n)_{n=1}^\infty$ is a basis of Y with basis constant smaller than or equal to

$$K\left(\frac{1 + K\rho}{1 - K\rho}\right) \leq K\left(\frac{1 + K\delta}{1 - K\delta}\right).$$

Formula (3.5) yields $y^*_{M+1}(h_j) = 0$ if $j > N$. Moreover, since $x \in (\mathrm{id}_Y - Q_L)Y$ we have $y^*_k(h_l) = 0$ for $k \leq M$, $l \geq L + 1$. Together with (3.6) this proves the proposition. □

Conclusion of the proof of Theorem 2.4. Consider $\delta_n > 0$ such that

$$\sum_{n=1}^\infty K2^{n-1}\delta_n \leq \epsilon, \quad \frac{1 + K2^{n-1}\delta_n}{1 - K2^{n-1}\delta_n} \leq 2 \quad \text{and} \quad \prod_{n=1}^\infty \left(\frac{1 + K2^{n-1}\delta_n}{1 - K2^{n-1}\delta_n}\right) \quad \text{converges.}$$

Then, we use induction and apply Proposition 3.2 as follows.

We start with the basis $(\tilde{e}_n)_{n=1}^\infty =: (e_n^{(1)})_{n=1}^\infty$ and $n_1 := 0$. If we are in the step m, and we already have the indices n_k, $k \leq m$, and a basis $(e_n^{(m)})_{n=1}^\infty$ with basis constant at most

$$K\prod_{k=1}^{m-1}\left(\frac{1 + K2^{k-1}\delta_k}{1 - K2^{k-1}\delta_k}\right) \leq 2^{m-1}K,$$

such that $x^*_k(e_n^{(m)}) = 0$ for all $n \geq n_k$ and all $k \leq m$, then we apply Proposition 3.2 with $\tilde{h}_n = e_n^{(m)}$, $L = n_m$, $M = m$ and $\delta = \delta_m$. This yields an index $N > n_m$ and a basis $(e_n^{(m+1)})_{n=1}^\infty$ with basis constant not larger than

$$K\prod_{k=1}^{m}\left(\frac{1 + K2^{k-1}\delta_k}{1 - K2^{k-1}\delta_k}\right)$$

such that, in view of (3.2), $e_n^{(m+1)} = e_n^{(m)}$ for $n \leq N$ and $x^*(e_n^{(m+1)}) = 0$ for all $n > N$. Put $n_{m+1} = N$ and continue the induction.

At the mth step of the process, the first n_m elements of the basis remain unchanged so that we end up with a basic sequence $(e_n)_{n=1}^\infty$ with basis constant at most

$$K\prod_{k=1}^{\infty}\left(\frac{1 + K2^{k-1}\delta_k}{1 - K2^{k-1}\delta_k}\right)$$

and such that (2.13) holds. In view of (3.3) the linear operator $T : X \to X$ with $T\tilde{e}_n = e_n$ for all n satisfies

$$\|\mathrm{id}_Y - T\| \leq \sum_{m=1}^\infty K\prod_{k=1}^{m-1}\left(\frac{1 + K2^{k-1}\delta_k}{1 - K2^{k-1}\delta_k}\right)\delta_n \leq \sum_{m=1}^\infty K2^{m-1}\delta_m \leq \epsilon.$$

If we choose $\epsilon < 1$ then T is surjective and $(e_n)_{n=1}^{\infty}$ is a basis of X with the required properties. $\square$

Proof of Lemma 2.5. We consider the Haar system $(e_n)_{n=1}^{\infty}$ in $L^1(0,1)$, where $e_1 \equiv 1$ and

$$e_{2^k+j}(t) = \begin{cases} 1, & \text{if } t \in [(2j-2)2^{-k-1}, (2j-1)2^{-k-1}], \\ -1, & \text{if } t \in [(2j-1)2^{-k-1}, (2j)2^{-k-1}], \\ 0, & \text{otherwise,} \end{cases}$$

for $k = 0, 1, 2, \dots$ and $j = 1, \dots, 2^k$. It is well-known that the Haar system is a Schauder basis for $L^1(0,1)$ with basis constant 1 (see [15]). Put

$$A_{2^k+j-1} = \Big[\frac{j-1}{2^k}, \frac{j}{2^k}\Big].$$

Then we have $\mathbf{1}_{A_1} = e_1$, $\mathbf{1}_{A_2} = (e_1+e_2)/2$ and $\mathbf{1}_{A_3} = (e_1 - e_2)/2$. By induction we see that any element $\mathbf{1}_{A_m}$ is a linear combination of the Haar functions e_n.

For $h \in L^{\infty}(0,1)$ let Φ_h be the linear functional on $L^1(0,1)$ defined by

$$\Phi_h(f) = \int_0^1 f(s)h(s)ds \quad \text{for all } f \in L^1(0,1).$$

Recall that the map $h \mapsto \Phi_h$ is an isometric isomorphism between $L^{\infty}(0,1)$ and $L^1(0,1)^*$. It is easily seen that the biorthogonal functionals e_n^* of the Haar elements e_n are, up to constant factors, the functionals Φ_{e_n}. Let $W = \overline{\mathrm{sp}\,\{\Phi_{e_n}\}} \subset L^1(0,1)^*$. Then $\Phi_h \in W$ for any linear combination h of the functions $\mathbf{1}_{A_n}$.

Define $\alpha :]0,1] \to]-\infty, 0]$ by $\alpha(s) = \log s$, $s \in]0,1]$, and $(Sf)(s) = f(\alpha(s))w(\alpha(s))/s$ for $f \in L_w^{1,-}(\mathbb{R})$. Then S is an isometric isomorphism between $L_w^{1,-}(\mathbb{R})$ and $L^1(0,1)$. In particular we have

$$J^m f = \int_0^1 (Sf)(s) \frac{(-\alpha(s))^{m-1}}{(m-1)!w(\alpha(s))} ds \quad \text{for all } m \in \mathbb{N}.$$

With

$$g(s) = \frac{(-\alpha(s))^{m-1}}{(m-1)!w(\alpha(s))}, \quad s \in]0,1], \quad \text{and} \quad g(0) = 0$$

we obtain

$$J^m f = \Phi_g(Sf). \tag{3.7}$$

In view of (1.5) the function g is continuous and hence uniformly continuous on $[0,1]$. This means that, for any $\epsilon > 0$, we find k and a linear combination g_ϵ of the characteristic functions $\mathbf{1}_{A_{2^k+j-1}}$, $j = 1, \dots, 2^k$, such that

$$\|g - g_\epsilon\|_\infty \le \epsilon,$$

and hence $\|\Phi_g - \Phi_{g_\epsilon}\| \le \epsilon$. Since $\Phi_{g_\epsilon} \in W$ for all ϵ we conclude $\Phi_g \in W$.

Finally, let $\tilde{e}_n^- = S^{-1}e_n$ for all n. Then $(\tilde{e}_n^-)_{n=1}^{\infty}$ is a Schauder basis of $L_w^{1,-}(\mathbb{R})$. The norm-closed linear span of the biorthogonal functionals is equal to $S^*W = \{w^* \circ S : w^* \in W\}$. From (3.7) we obtain $J^m \in S^*W$ for all $m \ge 1$ and therefore $(\tilde{e}_n^-)_{n=1}^{\infty}$ is shrinking for the functionals J^m (see the remark after Definition 2.3). $\square$

4. Proof of Theorem 1.1 in the case $N > 1$

Returning to the proof of Theorem 1.1, when $N > 1$, we assume that the weight $w : \mathbb{R}^N \to \mathbb{R}^+$ and $m \in \mathbb{N}$ are given. First we select a weight $\tilde{w} : \mathbb{R}^N \to \mathbb{R}^+$ such that $L^p_w(\mathbb{R}^N) \subset L^1_{\tilde{w}}(\mathbb{R}^N)$ and such that

$$\tilde{w}(x) = \prod_{j=1}^{N} v(x_j) \tag{4.1}$$

for some weights v on $\mathbb{R}$ satisfying the assumptions around (1.5) in the one-dimensional case. One can for example find $\tilde{w}$ as follows. Define for every $k \in \mathbb{N}$ the number B_k such that

$$B_k = \sup_{x \in \mathbb{R}^N} \frac{(1+|x|)^k}{w(x)} \tag{4.2}$$

and then set

$$\tilde{v}(x) = \prod_{j=1}^{N} \sum_{k=1}^{\infty} 2^{-k} B_k^{-1/N} (1+|x_j|)^{k/N}. \tag{4.3}$$

We then have

$$\begin{aligned} \tilde{v}(x) &\leq \prod_{j=1}^{N} \sum_{k=1}^{\infty} 2^{-k} B_k^{-1/N} (1+|x|)^{k/N} \\ &\leq \prod_{j=1}^{N} \sum_{k=1}^{\infty} 2^{-k} (1+|x|)^{k/N} \inf_{y \in \mathbb{R}^N} \frac{w(y)^{1/N}}{(1+|y|)^{k/N}} \leq \prod_{j=1}^{N} \sum_{k=1}^{\infty} 2^{-k} w(x)^{1/N} = w(x) \end{aligned} \tag{4.4}$$

If $p = 1$, we take $\tilde{w} = \tilde{v}$, and for $p > 1$ we set

$$\tilde{w}(x) = \tilde{v}(x)^{1/p} \prod_{j=1}^{N} (1+|x_j|)^{-2/p'} \tag{4.5}$$

where $p' = p/(p-1)$ is the dual exponent of p. Then, $\tilde{w}$ is still as in (1.5), and moreover, for every $f \in L^p_{\tilde{v}}(\mathbb{R}^N)$ we have by the Hölder inequality

$$\begin{aligned} \|f\|_{1,\tilde{w}} &= \int_{\mathbb{R}^N} |f| \tilde{w} dx \leq \int_{\mathbb{R}^N} |f(x)| \tilde{v}(x)^{1/p} \prod_{j=1}^{N} (1+|x_j|)^{-2/p'} dx \\ &\leq \Big(\int_{\mathbb{R}^N} |f(x)|^p \tilde{v}(x) dx \Big)^{1/p} \Big(\int_{\mathbb{R}^N} \prod_{j=1}^{N} (1+|x_j|)^{-2} dx \Big)^{1/p'} \leq C \|f\|_{p,\tilde{v}} \end{aligned} \tag{4.6}$$

so that $L^p_w(\mathbb{R}^N) \subset L^p_{\tilde{v}}(\mathbb{R}^N) \subset L^1_{\tilde{w}}(\mathbb{R}^N)$, and $\tilde{w}$ is of the form (4.1) with

$$v(x_j) = (1+|x_j|)^{-2/p'} \sum_{k=1}^{\infty} 2^{-k} B_k^{-1/N} (1+|x_j|)^{k/N}.$$

Using Theorem 1.1 with the weight v (in the place of w) we find the Schauder basis $(\tilde{e}_n)_{n=1}^{\infty}$ in $L^1_v(\mathbb{R})$ and the increasing sequence $(n_m)_{m=1}^{\infty}$ such that (1.8) holds. We may and do require that the basis $(\tilde{e}_n)_{n=1}^{\infty}$ is normalized in $L^1_v(\mathbb{R})$ so that $\|\tilde{e}_n\|_{1,v} = 1$ for every n. We denote the corresponding nth basis projection (2.11) by $\tilde{P}_n$ and the corresponding complementary projection $\tilde{Q}_n = \mathrm{id}_{L^1_v(\mathbb{R})} - \tilde{P}_n$.

Since the weighted Lebesgue measure $\tilde{w}dx$ on $\mathbb{R}^N$ is the product of the N measures vdx_j on $\mathbb{R}$ by (4.1), we can apply the theory tensor product norms and present the space $L^1_{\tilde{w}}(\mathbb{R}^N)$ as the N-fold projective tensor product

$$L^1_{\tilde{w}}(\mathbb{R}^N) = L^1_v(\mathbb{R})\widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi L^1_v(\mathbb{R}), \tag{4.7}$$

see [21], Ch. 46 and in particular Exercise 46.5. We need a few facts concerning (4.7): according to the definition of the projective tensor product, every $f \in L^1_{\tilde{w}}(\mathbb{R}^N)$ can be written as

$$f(x) = \sum_{k=1}^{\infty} \lambda_k f^{(1,k)}(x_1) f^{(2,k)}(x_2) \dots f^{(N,k)}(x_N) \tag{4.8}$$

where the numbers λ_k form an absolutely summable sequence,

$$\sum_{k=1}^{\infty} |\lambda_k| \le C \|f\|_{1,\tilde{w}} \tag{4.9}$$

and every $f^{(j,k)}$ belongs to the space $L^1_v(\mathbb{R})$ and has the bound

$$\|f^{(j,k)}\|_{1,v} \le 1. \tag{4.10}$$

It follows that the sum (4.8) converges absolutely in the space $L^1_{\tilde{w}}(\mathbb{R}^N)$.

The second fact is that the functions

$$e_{\bar{n}} = \tilde{e}_{n(1)} \otimes \dots \otimes \tilde{e}_{n(N)} \quad (\text{i.e. } e_{\bar{n}}(x) = \tilde{e}_{n(1)}(x_1) \dots \tilde{e}_{n(N)}(x_N),\ x \in \mathbb{R}^N) \tag{4.11}$$

where $\bar{n} = \big(n(1), n(2), \dots, n(N)\big) \in \mathbb{N}^N$ runs over all N-tuples, form a Schauder basis of the space $L^1_{\tilde{w}}(\mathbb{R}^N)$, see for example [8]. We will prove the theorem by using this basis. Here, we do not need to order the basis explicitly with an index in $\mathbb{N}$; nevertheless, every operator

$$P_n := \tilde{P}_n \otimes \dots \otimes \tilde{P}_n \quad , \quad n \in \mathbb{N},$$

is a basis projection with $\sup_n \|P_n\| < \infty$ in the operator norm of $L^1_{\tilde{w}}(\mathbb{R}^N)$, although not all basis projections of the basis $(e_n)_{n=1}^\infty$ are of this form. The complementary projection can be written as a finite sum

$$Q_n := I - P_n = \sum_{\sigma \in \mathbb{S}} \tilde{R}_{n,\sigma(1)} \otimes \dots \otimes \tilde{R}_{n,\sigma(N)} \quad , \quad n \in \mathbb{N} \tag{4.12}$$

where we write I for the identity operator on $L^1_{\tilde{w}}(\mathbb{R}^N)$ for brevity,

$$\sigma = (\sigma(1), \dots, \sigma(N)) \in \mathbb{S} := \prod_{j=1}^{N} \{1,2\} \setminus \{(1,1,\dots,1)\} \tag{4.13}$$

and

$$\tilde{R}_{n,k} = \begin{cases} \tilde{P}_n, & k = 1 \\ \tilde{Q}_n, & k = 2. \end{cases} \tag{4.14}$$

In other words, the sum (4.12) consists of exactly those terms, where at least one factor equals $\tilde{Q}_n$; the sum has $2^N - 1$ terms.

Lemma 4.1. *If $f \in L^1_{\tilde{w}}(\mathbb{R}^N)$ and $P_n f = 0$ for some $n \in \mathbb{N}$, then f has a representation* (4.8)*, where for every k at least one factor $f^{(j,k)}$ satisfies*

$$\tilde{P}_n f^{(j,k)} = 0 \quad , \quad \textit{equivalently}, \quad f^{(j,k)} = \tilde{Q}_n f^{(j,k)}, \tag{4.15}$$

and the bounds (4.9) *and* (4.10) *still hold true.*

Proof. If f is given and (4.15) does not already hold for its representation (4.8), we write using the definition of the tensor product operator (4.12) and the absolute convergence of the series (4.8)

$$\begin{aligned} f(x) = Q_n f(x) &= \sum_{k=1}^{\infty} \lambda_k \sum_{\sigma \in \mathbb{S}} \tilde{R}_{n,\sigma(1)} f^{(1,k)}(x_1) \dots \tilde{R}_{n,\sigma(N)} f^{(N,k)}(x_N) \\ &= \sum_{k=1}^{\infty} \sum_{\sigma \in \mathbb{S}} B^N \lambda_k \frac{1}{B} \tilde{R}_{n,\sigma(1)} f^{(1,k)}(x_1) \dots \frac{1}{B} \tilde{R}_{n,\sigma(N)} f^{(N,k)}(x_N), \end{aligned} \tag{4.16}$$

where $B > 0$ is a uniform bound for the operator norms of the projections $\tilde{P}_n$ and $\tilde{Q}_n$ in the space $L^1_v(\mathbb{R})$, and the double sequence $\big(B^N \lambda_k\big)_{k \in \mathbb{N}, \sigma \in \mathbb{S}}$ is still absolutely summable, since $\mathbb{S}$ has the fixed number 2^{N-1} of terms. Thus (4.16) is the desired representation of f. □

We show that the basis (4.11) satisfies the claim of Theorem 1.1. Let $m \in \mathbb{N}$ be given and let n_m be as chosen above; we write for brevity $n_m =: n$. As remarked above, P_n is a basis projection related to the basis (4.11). We assume that $f \in L^1_{\tilde{w}}(\mathbb{R}^N)$ is such that $P_n f = 0$, and take a representation (4.8) with the properties given by Lemma 4.1. We consider an arbitrary term of (4.8), with some abuse of notation in the variables:

$$\begin{aligned} & e^{t\Delta}\Big(\frac{1}{B} \tilde{R}_{n,\sigma(1)} f^{(1,k)}(x_1) \dots \frac{1}{B} \tilde{R}_{n,\sigma(N)} f^{(N,k)}(x_N) \Big) \\ &= B^{-N} e^{t\Delta}\Big(\Big(\prod_{j:\sigma(j)=1} \tilde{R}_{n,\sigma(j)} f^{(j,k)}(x_j) \prod_{j:\sigma(j)=2} \tilde{R}_{n,\sigma(j)} f^{(j,k)}(x_j) \Big)\Big) \\ &= B^{-N} \prod_{j:\sigma(j)=1} e^{t\partial_{x_j}^2} \tilde{R}_{n,\sigma(j)} f^{(j,k)}(x_j) \prod_{j:\sigma(j)=2} e^{t\partial_{x_j}^2} \tilde{R}_{n,\sigma(j)} f^{(j,k)}(x_j) \end{aligned}$$

Here, if $\sigma(j) = 1$, we have $\tilde{R}_{n,\sigma(j)} = \tilde{P}_n$, and thus by the uniform boundedness of the operator norms of $\tilde{P}_n$ in $L^1_v(\mathbb{R})$ and (4.10),

$$\big\| e^{t\partial_{x_j}^2} \tilde{R}_{n,\sigma(j)} f^{(j,k)} \big\|_\infty \le \frac{C}{t^{1/2}} \big\| \tilde{R}_{n,\sigma(j)} f^{(j,k)} \big\|_{1,v} \le \frac{C'}{t^{1/2}}. \tag{4.17}$$

However, if $\sigma(j) = 2$, we have $\tilde{R}_{n,\sigma(j)} = \tilde{Q}_n$, and thus the choices made above and (1.8) imply

$$\begin{aligned} \big\| e^{t\partial_{x_j}^2} \tilde{R}_{n,\sigma(j)} f^{(j,k)} \big\|_\infty &= \big\| e^{t\partial_{x_j}^2} \tilde{Q}_n f^{(j,k)} \big\|_\infty \\ &\le \frac{C}{t^m} \big\| \tilde{Q}_n f^{(j,k)} \big\|_{1,v} \le \frac{C'}{t^m} \end{aligned} \tag{4.18}$$

for $t \ge 1$.

By Lemma 4.1, every term in (4.8) has at least one factor with $\sigma(j) = 2$, hence,

$$\begin{aligned} \big\| e^{t\Delta} f \big\|_\infty &\le \sum_{k=1}^{\infty} |\lambda_k| B^N \Big\| e^{t\Delta}\Big(\frac{1}{B} \tilde{R}_{n,\sigma(1)} f^{(1,k)}(x_1) \dots \frac{1}{B} \tilde{R}_{n,\sigma(N)} f^{(N,k)}(x_N) \Big) \Big\|_\infty \\ &\le C \sum_{k=1}^{\infty} |\lambda_k| \|f\|_{1,\tilde{w}} \frac{1}{t^{m+(N-1)/2}} \le \frac{C'}{t^{m+(N-1)/2}} \|f\|_{1,\tilde{w}} \le \frac{C'}{t^m} \|f\|_{1,\tilde{w}} \end{aligned}$$

for $t \ge 1$. □

We conclude by a discussion. First, we remark that in the case $p = 2$ it is possible to use standard Hilbert space methods (Fréchet-Riesz theorem and Gram-Schmidt method) and give an existence proof for an orthonormal basis in $L_w^{2;\pm}(\mathbb{R})$ with the property (2.13) for the functionals $x_m^* := J^m$. This yields the existence of an *orthonormal* basis in Theorem 1.1.1°, for $L_w^2(\mathbb{R})$.

The heat equation is a classical albeit simplified model for the heat conduction or linear diffusion processes. Since our discovery is basically an existence proof, its possible physical relevance depends on concrete examples of Schauder basis and estimates of the magnitude of the numbers n_m. We pose the problem:

1°. *Given a weight w, $1 \le p < \infty$ and $m \in \mathbb{N}$, minimize the number n_m in Theorem 1.1.*

Of course, the result in higher dimensions should be improved.

2°. *Find a Schauder basis in the space $L_w^p(\mathbb{R}^N)$, $N > 1$, with the same properties as in Theorem 1.1,* 1°.

Finally, we ask if it is possible in the case $N > 1$ to use the one-dimensional result in such a way that the weight only needs to be fast growing in one coordinate direction and milder assumptions are sufficient in other directions. For example:

3°. *Does Theorem 1.1,* 2°*, hold for the weight*

$$w(x) = e^{|x_1|} \prod_{j=2}^{N} (1 + |x_j|)^2. \tag{4.19}$$

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