RADIATION CONDITIONS FOR THE LINEAR WATER-WAVE PROBLEM IN PERIODIC CHANNELS

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Abstract. We study the well-posedness of the linearized water-wave problem in a periodic channel with fixed or freely floating compact bodies. Floquet-Bloch-Gelfand-transform techniques lead to a generalized spectral problem with quadratic dependence on a complex parameter, and the asymptotics of the solutions at infinity can be described using Floquet waves. These are constructed from Jordan chains, which are related with the eigenvalues of the quadratic pencil and which are calculated in the paper in some typical cases.

Posing proper radiation conditions requires a careful study of spaces of incoming and outgoing waves, especially in the threshold situation. This is done with the help of a certain skew-Hermitian form q, which is closely related to the Umov-Poynting vector of energy transportation. Our radiation conditions make the problem operator into a Fredholm operator of index zero and provides natural (energy) classification of outgoing/incoming waves. They also lead to a novel, most natural properties and interpretation of the scattering matrix, which becomes unitary and symmetric even at threshold.

1. Introduction.

1.1. **Preamble.** We consider the linearized water-wave problem in a periodic water channel $\Pi \subset \mathbb{R}^3$, which contains fixed submerged and/or surface piercing obstacles Θ; see Section 4.1 for geometric details and Remark 2.1 for the case of freely floating objects. The problem consists of the Poisson equation in $\Omega := \Pi \setminus \overline{\Theta}$ for the unknown φ (velocity potential) and given f,

$$-\Delta_x \varphi(x) = f(x), \qquad x \in \Omega,$$

homogeneous Neumann (no penetration) conditions $\partial_\nu \varphi(x) = 0$ on the boundary except for the free water surface Γ, where a Steklov type spectral (kinematic) condition $\partial_\nu \varphi(x) = \lambda \varphi(x)$ is posed with λ as a spectral parameter. The problem as described here is not well-posed: think for example about the special case of a straight cylinder, where it is easy to observe that the above mentioned boundary conditions with $\lambda \in \mathbb{R}_+ = (0, +\infty)$ cannot guarantee the uniqueness or existence of the solution in Sobolev spaces, and the same remains true for general periodic domains. The

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main goal of the present paper thus becomes to supplement this linear water-wave problem with proper radiation conditions.

To that end, we write the problem as an equation $A_\beta(\lambda)\varphi = F$ containing a linear operator $A_\beta(\lambda) : W^1_\beta(\Omega) \to W^1_{-\beta}(\Omega)^*$, where $W^1_\beta(\Omega)$, $\beta \in \mathbb{R}$, denotes a weighted Sobolev space the elements of which have, roughly speaking, the growth ($\beta < 0$) or decay ($\beta > 0$) no more than $e^{-\beta|x_1|}$ in the unbounded x_1-directions of the channel. For small $\beta > 0$ the operators $A_{\pm\beta}$ are Fredholm (Theorem 4.1), however, the index of $A_\beta(\lambda)$ is strictly positive and that of $A_{-\beta}(\lambda)$ negative. The point will be a careful study of the solutions of the above written equation, more precisely, the description of incoming and outgoing waves in the preimage $A_{-\beta}(\lambda)^{-1}W^1_{-\beta}(\Omega)^*$ and their relation to the domain and kernel spaces of $A_{\pm\beta}$; this division of wave classes is the most intriguing issue especially at the threshold of the continuous spectrum, see Sections 4.2–4.3. It turns out that restricting the operator $A_{-\beta}(\lambda)$ to a subspace, which in particular does not contain incoming waves, makes the operator index equal zero, see Theorem 4.8. This is the appropriate radiation condition, which renders the original problem well-posed. This investigation is motivated by the well-known shortcomings of the Sommerfeld radiation principle in periodic domains or at thresholds.

As for the basic mathematical tools, our approach is based on the Floquet-Bloch-Gelfand-(FBG-)transform, see Section 3.2. This enables us to convert the problem in the unbounded quasicylinder (periodic set) into a problem in a bounded periodicity cell, depending on a complex parameter η, which is the dual variable of the FBG-transform. We end up with a two-fold spectral problem, since η can be treated as a new spectral parameter. However, the dependence of the problem on η is quadratic instead of linear: to each η-eigenvalue there may correspond a system of Jordan chains consisting of eigenfunctions and their associated functions $\phi_j^{q,p}$. The above mentioned incoming and outgoing waves $u_n^\pm$ can be constructed with help of the Floquet waves (3.16), which contain the functions $\phi_j^{q,p}$ as ingredients. The structure of the Jordan chains is in general complicated and it is hard to calculate explicitly the coefficients of the functions $\phi_j^{q,p}$ in the formulas of $u_n^\pm$, however, it has been proven in [20, Ch. 5] that it is always possible to find a proper normalization. In the final section of this paper we complete perform these calculations for Jordan chains in some particular cases.

Another key tool is the skew-Hermitian form q (defined in (4.7), or (2.18) in a simplified case), which acts in the above mentioned weighted spaces, the spaces of waves. The q-form will be used to distinguish between incoming and outgoing waves (Theorem 4.3), and thus also to find the correct domain space for the operator $A_\beta(\lambda)$. The natural connection of the q-form with the Umov-Poynting vector yields physical motivation for the radiation conditions, which we introduce by mathematical considerations. The conditions are of Mandelstam type: any incoming wave must bring energy form infinity, and on the contrary, outgoing waves radiate energy to infinity. We emphasize that in the threshold case, when Jordan chains of length larger than one appear, not every Floquet wave is able to drive energy along the waveguide. Accordingly, only a special choice of wave packets (linear combinations) yields an appropriate basis in the linear spaces of incoming and outgoing waves, so that the scattering matrix with respect to this basis becomes unitary and symmetric.

There are several known approaches to composing radiation conditions, and in some sense our way to supply the problem with a Fredholm operator is standard. Our methods are inspired by the work [20]; see also [21, Ch. 5], which contains both an account of the FBG-transform and q-form techniques. The method using the q-form is general and flexible, and it was also applied in [22] for the water-wave problem in a completely different geometric situation.

The paper is composed as follows. In Section 2 we review, for the sake of making an analogy to the periodic case, the much more simple case of a straight channel, where one can use the Fourier transform instead of the FBG-transform. We also make a remark on the case of freely floating objects in Section 2.1. The Sommerfeld principle is usually sufficient for rendering the problem well-posed, see Proposition 2.4. However, in the threshold case, the incoming and outgoing waves are rather wave packages determined by the q-form, as we show in Proposition 2.5. Thus, the q-form can provide a way to extend the physical radiation conditions to this case, though this is done in detail only in the general case of Section 4.

Section 3 contains a study of the linear water-wave problem in the unperturbed periodic channel. Here we use the FBG-transform and give the operator theoretic formulation of the problem in the bounded periodicity cell. Basic facts concerning the η-spectrum, Jordan chains and Floquet waves are introduced.

In Section 4, which contains the main results, we consider the full water-wave problem containing the fixed obstacle in the periodic channel and present a novel approach to radiation conditions. Starting with existing literature, we recall in Theorem 4.1 the connection of the Fredholm property of $A_\beta(\lambda)$ and the η-spectrum, and in Theorem 4.5, the known solutions of the linear water-wave equation. The analysis of the Fredholm properties of the problem operator $A_{-\beta}(\lambda) : W^1_{-\beta}(\Omega) \to W^1_\beta(\Omega)^*$ leads to the formulation of the radiation conditions via Theorem 4.8 and to the calculation of the (unitary) scattering matrix in Theorem 4.10. We also establish the connection of our radiation conditions with the Mandelstam radiation principle by determining the relation of the Umov-Poynting and the q-form, see Theorem 4.12. The investigation is completed by the final Section 5, where we calculate explicitly in some special cases the incoming and outgoing waves in terms of the Jordan chains and Floquet waves of Section 3.

2. Straight channels.

Before proceeding to the general case of periodic channels, we consider the water-wave problem in straight channels. The results of this section are in principle known, but we present them here in order to demonstrate the analogy with the case of periodic channels.

2.1. **Formulation of the problem.** We consider the boundary value problem of linearized water-wave theory (see, e.g., [11, 25]) in a cylindrical channel $\Pi := \mathbb{R}\times\omega = \{x = (x_1, x_2, x_3) = (x_1, x') : x_1 \in \mathbb{R}, x' \in \omega\}$ containing a fixed submerged or surface piercing obstacle Θ, the closure of which is a compact subset of $\overline{\Pi} \subset \mathbb{R}^3$.

It is assumed that $\omega \subset \mathbb{R}^2$ is bounded and Lipschitz. The boundary $\partial\Pi$ consists of the bottom and walls of the channel, $\Sigma^\natural = \partial\Pi \cap \{x : x_3 < 0\}$ and the water surface $\Gamma^\natural = \partial\Pi \cap \{x : x_3 = 0\}$. Furthermore, the water domain $\Omega = \Pi \setminus \overline{\Theta} \subset \mathbb{R}^3$ is also required to be connected and Lipschitz. We denote $\Sigma = \partial\Omega \cap \{x : x_3 < 0\}$ and $\Gamma = \partial\Omega \cap \{x : x_3 = 0\} \subset \Gamma^\natural$. Since the closure of Θ is compact, the domain Ω coincides with Π outside the set $\{x : |x| < R_0\}$ for some large enough $R_0 > 0$, which we fix now.

The water-wave problem is formulated in the domain Ω and on its boundaries as follows (see e.g. [11]):

$$-\Delta_x \varphi(x) = f(x), \qquad x \in \Omega, \tag{2.1}$$

$$\partial_\nu \varphi(x) = 0, \qquad x \in \Sigma, \tag{2.2}$$

$$\partial_z \varphi(x) = \lambda \varphi(x), \qquad x \in \Gamma. \tag{2.3}$$

Here, Δ_x is the Laplace operator in x, ∂_ν is the outward normal derivative defined almost everywhere on $\partial\Omega$, while $\partial_\nu = \partial_z = \partial/\partial z$ on Γ. Later we will use $\Delta_{x'}$ and $\nabla_{x'}$ for the Laplacian and gradient in the variable $x' = (x_2, x_3)$. Moreover, φ is the velocity potential, $\lambda = \kappa^2/g > 0$ is a spectral parameter with the oscillation frequency $\kappa > 0$ and the acceleration $g > 0$ due to gravity. The known function f expresses external effects and it will belong to a suitably chosen function space.

Remark 2.1. In the case of a freely floating body Θ, which is assumed to be connected for simplicity, the boundary condition (2.2) becomes inhomogeneous on $\partial\Omega \cap \partial\Theta$, due to possible rigid motions of Θ. These are described by a column vector $a \in \mathbb{R}^6$, which contains three translations and three rotations. The PDE-problem (2.1)– (2.3) must be coupled with an algebraic 6×6-system for a, which contains weighted integrals of $\partial_\nu \varphi$ over $\partial\Omega \cap \partial\Theta$, i.e. the momenta and torques of hydrodynamical forces acting on the body. The complete formulation of this problem can be found in [6, 7]. The variational formulation of the problem and its reduction to a self-adjoint Hilbert space operator are presented in [5], [23], and this approach allows to adapt all results of the present paper to the case of a freely floating body in both straight and periodic channels. The reason is that our approach only uses the parts of the channel which are of some distance from the body. We will not comment on this generalization in the sequel.

Consider the corresponding homogeneous problem in the unperturbed cylinder,

$$-\Delta_x \varphi(x) = 0, \qquad x \in \Pi, \tag{2.4}$$

$$\partial_\nu \varphi(x) = 0, \qquad x \in \Sigma^\natural, \tag{2.5}$$

$$\partial_z \varphi(x) = \lambda \varphi(x), \qquad x \in \Gamma^\natural. \tag{2.6}$$

The standard approach to this (and also to (2.1)–(2.3)) consists of separation of variables and the Fourier transform with respect to the variable x_1. This leads to the following model problem on the cross-section ω:

$$-\Delta_{x'} W(x') + \xi^2 W(x') = 0, \qquad x' \in \omega, \tag{2.7}$$

$$\partial_\nu W(x') = 0, \qquad x' \in \varsigma', \tag{2.8}$$

$$\partial_z W(x') = \lambda W(x'), \qquad x' \in \gamma', \tag{2.9}$$

where ς' and γ' are parts of the boundary $\partial\omega \subset \mathbb{R}^2$ corresponding to the channel bottom and walls, and the free water surface, respectively, and the parameter $\xi \in \mathbb{R}$ is the separation of variables constant . We take the new spectral parameter $\mu = -\xi^2$, and replace (2.7) by

$$-\Delta_{x'} W(x') = \mu W(x') \ , \quad x' \in \omega. \tag{2.10}$$

The weak formulation, see [14], formula II.2.6, of the problem (2.8)–(2.10) means finding $W \in H^1(\omega)$ such that

$$(\nabla_{x'} W, \nabla_{x'} V)_\omega - \lambda(W,V)_{\gamma'} = \mu(W,V)_\omega \quad \text{for all } V \in H^1(\omega). \tag{2.11}$$

This is derived from (2.7) by multiplying with an arbitrary $V \in H^1(\omega)$, integrating, using the Green formula and the boundary conditions (2.8) and (2.9). Let us denote by $B(\lambda)$ the sesquilinear form on $H^1(\omega) \times H^1(\omega)$ defined by the left hand side of (2.11). Due to the standard trace inequality

$$\|W; L^2(\gamma')\| \le \varepsilon \|\nabla_{x'} W; L^2(\omega)\| + C_\varepsilon(\omega, \gamma') \|W; L^2(\omega)\| \ , \tag{2.12}$$

valid for every $\varepsilon > 0$, the form $B(\lambda)$ is closed in $H^1(\omega)$ and lower semi-bounded, hence, by [1, Ch. 10], it determines a lower semi-bounded self-adjoint operator $A(\lambda)$ such that the problem (2.11) is equivalent to the spectral problem

$$A(\lambda) W = \mu(\lambda) W \ , \ W \in L^2(\omega).$$

Due to the compactness of the embedding $H^1(\omega) \hookrightarrow L^2(\gamma')$, there exists an increasing sequence of eigenvalues for A:

$$\mu_1(\lambda) < \mu_2(\lambda) \le \cdots \le \mu_j(\lambda) \le \cdots \to +\infty. \tag{2.13}$$

The functions $\lambda \mapsto \mu_j(\lambda)$ are monotone decreasing for $\lambda \in \mathbb{R}^+$, by [1, Thm. 10.2.4]. For all $j \in \mathbb{N}$ we denote by W_j the eigenfunction corresponding to μ_j, indexed and normalized such that

$$(W_j, W_l)_\omega = \delta_{j,l}, \tag{2.14}$$

where $\delta_{j,l}$ is the Kronecker symbol. By [1, Thm. 10.2.2], the eigenvalues μ_j can be computed from the max-min-principle

$$\mu_j(\lambda) = \sup_{H_j} \inf_{W \in H_j} \frac{\|\nabla_{x'} W; L^2(\omega)\|^2 - \lambda \|W; L^2(\gamma')\|^2}{\|W; L^2(\omega)\|^2},$$

where the supremum is taken over all $(j-1)$ -codimensional subspaces H_j of $H^1(\omega)$. Choosing $W = 1$ reveals that $\mu_1(\lambda) < 0$, and from (2.12) we find that there are finitely many non-positive eigenvalues μ_j, $j = 1, \ldots, J(\lambda) \in \mathbb{N}$. Since the functions $\lambda \mapsto \mu_j(\lambda)$ are decreasing, $J(\lambda)$ is an increasing function of λ.

2.2. **Waves.** Returning to the problem (2.4)–(2.6) for the unperturbed cylinder, for each $j \le J(\lambda)$ with $\mu_j(\lambda) \neq 0$, there are two oscillating wave solutions

$$\varphi_j^{\pm}(x) = e^{\pm i x_1 |\mu_j(\lambda)|^{1/2}} W_j(x') \tag{2.15}$$

where $\pm|\mu_j(\lambda)|^{1/2}$ is the wave number, and, according to the Sommerfeld principle, the wave φ_j^+ travels from $-\infty$ to $+\infty$ and the wave φ_j^- to the opposite direction. The physical explanation of this is as follows: the corresponding time-dependent problem, including the wave equation, has solutions

$$\phi_j^{\pm}(x,t) = e^{-i(\kappa t \mp i x_1 |\mu_j(\lambda)|^{1/2})} W_j(x'). \tag{2.16}$$

The direction of the propagation, as t increases, is determined by the sign of the wave number in (2.16). However, if $\mu_j(\lambda) = 0$, the wave (2.15) becomes standing and its direction remains ambiguous. This has the consequence that the Sommerfeld radiation condition fails, although it usually makes the problem (2.1)–(2.3) well-posed by excluding incoming waves from the solutions. However, we will show in Proposition 2.5 that incoming and outgoing waves can be identified as wave packages by using the skew-Hermitian Q-form of (2.18).

The solutions (2.15) are used for the study of the original problem (2.1)–(2.3) as follows. First, pick C^∞-smooth cut off functions $\chi_\pm$, depending on x_1 only, such that χ_+ has the properties $0 \le \chi_+ \le 1$, $\chi_+(x_1) = 0$ for $x_1 \le R_0$ and $\chi_+(x_1) = 1$ for $x_1 \ge R_0 + 1$, where R_0 is as in Section 2.1. Set $\chi_-(x_1) = \chi_+(-x_1)$. Assuming $\mu_j(\lambda) \neq 0$ for all j, let us denote

$$\begin{aligned} w_n^+(x) &= \frac{1}{\sqrt{2}|\mu_j(\lambda)|^{1/4}} \chi_\pm(x_1) e^{\pm i x_1 |\mu_j(\lambda)|^{1/2}} W_j(x') \;, \\ w_n^-(x) &= \frac{1}{\sqrt{2}|\mu_j(\lambda)|^{1/4}} \chi_\pm(x_1) e^{\mp i x_1 |\mu_j(\lambda)|^{1/2}} W_j(x'), \end{aligned} \tag{2.17}$$

where $j = 1, \ldots, J(\lambda)$ and the functions w_n^+ and w_n^-, $n = 1, \ldots, 2J(\lambda)$, are indexed in an unspecified order. From the point of view of the body Θ, the waves w_n^- are incoming, and w_n^+ outgoing, cf. the explanation above. As remarked in Proposition 2.4 below, this classification of waves is enough to make the Sommerfeld radiation condition work, if $\mu_j(\lambda) \neq 0$ for all j.

In order to classify the standing and resonance (i.e. linearly growing) waves occurring in the case $\mu_j(\lambda) = 0$, we introduce the form

$$Q(u,v) = \sum_{\pm} \pm \int_\omega \left(\overline{v(x)} \frac{\partial u}{\partial x_1}(x) - u(x) \overline{\frac{\partial v}{\partial x_1}(x)} \right) \Big|_{x_1 = \pm R} dx' \;,\; R \ge R_0, \tag{2.18}$$

which is defined for solutions u, v of the Helmholtz equation (2.1) belonging to $H^1_{\rm lOc}(\overline{\Omega})$. It is plain that Q is sesquilinear and anti-Hermitian, in short, skew-Hermitian: $Q(u,v) = -\overline{Q(v,u)}$. Notice that we will usually calculate Q for functions like (2.17), and in that case only one term $\pm$ in (2.18) may be nonzero, due to the cut-off functions. The integral in (2.18) can be replaced by an integral over a subdomain with positive volume; this is done in a more general setting in (4.8), and it is based on the following observation. We denote $\Omega_S = \{x \in \Pi : |x_1| > S\}$ for $S \geq R_0$.

Lemma 2.2. *Assume that both u and v satisfy (2.1)–(2.3) with $f = 0$ (or (2.4)–(2.6)) in Ω_{R_0}, $\Sigma \cap \overline{\Omega_{R_0}}$ and $\Gamma \cap \overline{\Omega_{R_0}}$ instead of Ω, Σ and Γ. Then the value of $Q(u,v)$ does not depend on the value of $R > R_0$.*

Proof. Let us denote for a moment by Q_R and Q_S the expressions (2.18) corresponding to the values $R > S > R_0$, respectively. The difference $Q_R(u,v) - Q_S(u,v)$ contains integrals over parts of the boundary of $\Omega_{R,S} := \Omega_R \setminus \overline{\Omega_S}$, and applying the Green formula in $\Omega_{R,S}$ hence yields

$$Q_R(u,v) - Q_S(u,v) = \int\limits_{\Sigma \cap \overline{\Omega_{R,S}}} \big(\overline{v}\partial_\nu u - u\overline{\partial_\nu v}\big) ds_x$$

$$+ \int\limits_{\Gamma \cap \overline{\Omega_{R,S}}} \big(\overline{v}\partial_\nu u - u\overline{\partial_\nu v}\big) ds_x + \int\limits_{\Omega_{R,S}} \big(\overline{v}\Delta u - u\overline{\Delta v}\big) dx. \tag{2.19}$$

The integrals in (2.19) vanish due to (2.1), (2.2), and (2.3). ⊠

A direct calculation shows the following property of Q for the waves (2.17):

Lemma 2.3. *For all $n, m = 1, \ldots, 2J(\lambda)$ we have*

$$Q(w_n^\pm, w_m^\pm) = \pm i\delta_{n,m} \quad \text{and} \quad Q(w_n^\pm, w_m^\mp) = 0. \tag{2.20}$$

Proof. Let j and l be the indices corresponding to n and m in (2.17). In the case of $Q(w_n^+, w_m^+)$, the integral over the cross-section $\omega \times \{R\}$ in (2.18) equals, by (2.17) and (2.14),

$$\frac{1}{2}\frac{1}{|\mu_j(\lambda)|^{1/4}|\mu_l|^{1/4}} \int\limits_\omega \Big(e^{-iR|\mu_l|^{1/2}} i|\mu_j(\lambda)|^{1/2} e^{iR|\mu_j(\lambda)|^{1/2}}$$

$$- e^{iR|\mu_j(\lambda)|^{1/2}}(-i)|\mu_l|^{1/2} e^{-iR|\mu_l|^{1/2}}\Big) W_j(x') W_l(x')\, dx' = i\delta_{j,l}.$$

Hence, $Q(w_n^+, w_m^+) = i\delta_{n,m}$. The other cases are similar. ⊠

We observe from (2.20) that the Q-form can identify outgoing and incoming waves: a wave v is outgoing, if $\operatorname{Im} Q(v,v) > 0$ and incoming, if $\operatorname{Im} Q(v,v) < 0$. Moreover, this classification extends to the situation when standing and resonance waves exist, i.e. $\mu_j(\lambda) = 0$ for some j, see Proposition 2.5. (The skew-Hermitian form will be put in full use in Section 4 with general periodic channels, where the much more complicated Floquet waves (3.16) with no obvious direction will replace the simple oscillating waves (2.15).)

2.3. **Radiation condition via Q-form for the threshold case in straight channels.** We first recall that exponentially decaying solutions of the homogeneous problem (2.1)–(2.3) may exist, and they are called *trapped modes* according to [8, 27] and others. Trapped modes are nothing but eigenfunctions of the problem (2.1)–(2.3) in the Sobolev space $H^1(\Omega)$, corresponding to the eigenvalue λ. Let $L_0 \subset H^1(\Omega)$ denote the subspace of trapped modes.

Let us first restrict to the case $\mu_j(\lambda) \neq 0$ for all j. It is known that given a function f which decays exponentially as $x_1 \to \pm\infty$, any bounded solution φ of the problem (2.1)–(2.3) can be written as

$$\varphi = \tilde{\varphi} + \sum_{n=1}^{2J(\lambda)} c_n^+ w_n^+ + \sum_{n=1}^{2J(\lambda)} c_n^- w_n^-$$

where c_n^+ and c_n^- are constants and $|\tilde{\varphi}(x)|$ decreases exponentially as $x_1 \to \pm\infty$. For the exact formulation of this result, see [10], also [21, § 5.1], or Theorem 4.5 below. The Sommerfeld radiation principle excludes incoming waves from the solution of the wave propagation problem, and under our assumption on the eigenvalues $\mu_j(\lambda)$, this is enough to make the problem well-posed: in particular, the following holds true.

Proposition 2.4. *Assume that the eigenvalues* (2.13) *of the problem* (2.4)–(2.6) *satisfy $\mu_j(\lambda) \neq 0$ for all j. Then, if φ is a solution of the homogeneous problem* (2.1)–(2.3) *subject to the Sommerfeld condition, i.e.*

$$\varphi = \widetilde{\varphi} + \sum_{n=1}^{2J(\lambda)} c_n w_n^+, \tag{2.21}$$

where $e^{\beta|x_1|}\widetilde{\varphi}, e^{\beta|x_1|}\nabla_x \widetilde{\varphi} \in L^2(\Omega)$ for some small $\beta > 0$, then $c_n = 0$ for all n so that $\varphi \in L_0$ is a trapped mode.

The non-homogeneous problem (2.1)–(2.3) with $e^{\beta|x_1|} f \in L^2(\Omega)$ will only be treated rigorously in the periodic case in Section 4.

Proof. If φ is a solution of the homogeneous problem as in (2.21), we get for any $R > R_0$ and $\Omega_R = \{x \in \Omega : |x_1| \geq R\}$, using the Green formula

$$\begin{aligned} 0 &= \int_{\Omega_R} \big((\Delta\varphi)\overline{\varphi} - \varphi \Delta \overline{\varphi} \big) dx = \int_{\partial\Omega_R} \big((\partial_\nu \varphi)\overline{\varphi} - \varphi \partial_\nu \overline{\varphi} \big) ds_x \\ &= \int_{\partial\Omega_R \setminus (\Sigma \cup \Gamma)} \big((\partial_\nu \varphi)\overline{\varphi} - \varphi \partial_\nu \overline{\varphi} \big) ds_x = Q(\varphi, \varphi) \\ &= Q(\widetilde{\varphi}, \widetilde{\varphi}) + Q\Big(\widetilde{\varphi}, \sum_n c_n w_n^+\Big) \\ &\quad + Q\Big(\sum_n c_n w_n^+, \widetilde{\varphi}\Big) + Q\Big(\sum_n c_n w_n^+, \sum_n c_n w_n^+\Big). \end{aligned} \tag{2.22}$$

Since $\widetilde{\varphi}$ is exponentially decaying, the first three terms on the right-hand side can be made arbitrarily small by increasing R. Since Q is independent of R, the identity

(2.20) yields

$$0 = Q\Big(\sum_n c_n w_n^+, \sum_n c_n w_n^+\Big) = \sum_{n,m} c_n \bar{c}_m Q(w_n^+, w_m^+) = i \sum_n |c_n|^2,$$

hence, $c_n = 0$ for all n. $\boxtimes$

We now change the above assumption on the eigenvalues and consider the case $\mu_j(\lambda) = 0$ in (2.13) for some j (there may be one or several such indices j), which means that standing and resonance waves occur. We first remark that the Sommerfeld principle fails in this case. Namely, instead of (2.15), an eigenvalue $\mu_j(\lambda) = 0$ corresponds to the solutions

$$W_j(x')\ , x_1 W_j(x').$$

(Since W_j now satisfies (2.7) with $\xi^2 = 0$, a function $g(x_1)W_j(x')$ satisfies (2.4), if and only if $g''(x_1) = 0$.) However, since $Q(W_j, W_j) = 0 = Q(x_1 W_j, x_1 W_j)$, none of these solutions can be used as such to define outgoing or incoming waves. However, a proper classification of incoming and outgoing waves can be made by using the Q-form as follows.

Proposition 2.5. *Defining the wave packages*

$$w_n^\pm(x) := \chi_+(x_1) \frac{1}{\sqrt{2}} (x_1 \mp i) W_j(x'), \tag{2.23}$$

we have $Q(w_n^+, w_n^+) = i$, $Q(w_n^-, w_n^-) = -i$ *and* $Q(w_n^+, w_n^-) = 0$,

Proof. The result follows from (2.14):

$$Q(w_n^\pm, w_n^\pm) = \frac{1}{2} \sum_\pm \int_\omega \Big(\overline{(x_1 \mp i)} - (x_1 \mp i)\Big) |W_j(x')|^2 \Big|_{x_1 = \pm R} dx' = \pm i. \ \boxtimes$$

Based on this and the remark after Lemma 2.3, we call w_n^+ as *outgoing* and w_n^- as an *incoming* wave.

The general case of periodic channels and the connection with radiation conditions will be considered in detail in Section 4. The above case corresponds, in the general setting of periodic channels, to the threshold case with Jordan chains of length 2; cf. also the example in Section 5.2.

Remark 2.6. When $w_n^\pm$ are defined as the waves packages (2.23), the solution (2.21) has linear growth as $x_1 \to \pm\infty$. On the other hand, the conventional physical radiation conditions at the threshold (i.e. in the presence of waves (2.23)) involve only staying waves $\chi_\pm(x) W_j(x')$, which lead to bounded solutions (2.21). However, the corresponding scattering cannot be determined properly, that is, with a unitary and symmetric scattering matrix. The reason for this is that neither staying nor resonance waves are able to drive energy along the channel, contrary to the packets (2.23). Moreover, in Section 4.5 we will show that the skew-Hermitian form (2.18) is proportional to the projection of the Umov-Poynting-vector onto x_1-axis [26, 24]. In this way our radiation conditions, based on the q-form, become Mandelstam-type conditions, which are related to the direction of energy transfer [16].

3. Periodic channels.

In this section we consider a periodic channel without an obstacle. For the straight channel it was possible to replace variable ξ by $\mu = -\xi^2$, reducing the model problem to a standard spectral problem for a positive self-adjoint operator with spectral parameter μ. However, in case of periodic channels it is necessary to use the FBG-transform and introduce Floquet waves, which involve a new spectral parameter η. The dependence of the problem operator on η is quadratic, a fact which crucially complicates the structure of the waves and makes a straightforward application of the Sommerfeld principle impossible in this setting, even in non-threshold cases.

3.1. **Formulation of the problem.** In the following, we use the same notation as in Section 2, although Π denotes now a different type of domain, namely an (unperturbed) periodic channel. The problem under consideration reads as

$$-\Delta_x \varphi(x) = f, \qquad x \in \Pi, \tag{3.1}$$

$$\partial_\nu \varphi(x) = 0, \qquad x \in \Sigma^\natural, \tag{3.2}$$

$$\partial_z \varphi(x) = \lambda \varphi(x), \qquad x \in \Gamma^\natural, \tag{3.3}$$

with a given function f specified later. As before, $\Sigma^\natural$ and $\Gamma^\natural$ are the bottom and surface of Π ($x_3 = 0$ for $x \in \Gamma^\natural$ and $x_3 < 0$ for $x \in \Pi$ and $\Sigma^\natural$), whereas the Lipschitz domain Π consists of the interior points of the set

$$\overline{\Pi} = \bigcup_{j \in \mathbb{Z}} \overline{\varpi}_j \tag{3.4}$$

where each ϖ_j is a translate of the form $\varpi_j := \{x : (x_1 - j, x') \in \varpi\}$, and the periodicity cell ϖ is a bounded Lipschitz-subdomain of $\{x : x_1 \in (0,1)\}$. The boundary components determining the water surface, respectively, bottom and walls of ϖ are defined by $\gamma = \Gamma^\natural \cap \overline{\varpi}$, respectively, by $\varsigma = \Sigma^\natural \cap \overline{\varpi}$. By rescaling, we make all the geometric parameters dimensionless, and especially the period is fixed to be one.

3.2. **FBG-transform.** We recall the definition of the FBG-transform (see [3] and, e.g. [12, 21, 13] for more details):

$$v(x) \mapsto \hat{v}(x;\eta) = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}} e^{-i\eta(x_1 + j)} v(x_1 + j, x'), \tag{3.5}$$

where $x \in \Pi$ on the left, $\eta \in [0, 2\pi)$, and $x \in \varpi$ on the right. For the convenience of the reader we recall the basic properties: if, for example $v \in C_0^\infty(\overline{\Pi})$, then the sum in (3.5) is finite, and it is easy to see the periodicity $\hat{v}(x_1 + 1, x') = \hat{v}(x)$ and the differentiation formula $\widehat{\nabla_x v} = (\partial_{x_1} + i\eta, \nabla_{x'})\hat{v}$. The inverse operator is given by

$$v(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{2\pi} e^{ix_1\eta} \widehat{v}(x_1 - [x_1], x') d\eta$$

where $[a]$ denotes the integer part of the real number a.

The FBG-transform is an isometric isomorphism from $L^2(\Pi)$ onto $L^2(0,2\pi;L^2(\varpi))$, as well as an isomorphism from $H^1(\Pi)$ onto $L^2(0,2\pi;H^1_{\rm per}(\varpi))$ (see e.g. [21, § 3.4] and [19, Cor. 3.4.3]). Here $L^2(0,2\pi;L^2(\varpi))$ consists of $L^2(\varpi)$–valued (complex) L^2–functions on $[0,2\pi]$, the space $L^2(0,2\pi;H^1_{\rm per}(\varpi))$ is defined analogously, and $H^1_{\rm per}(\varpi)$ is the space of $\mathbb{C}$–valued Sobolev–functions 1–periodic with respect to x_1. Using the FBG-transform, the problem (3.1)–(3.3) turns into the following parameter dependent spectral problem in the periodicity cell:

$$-\big((\partial_1+i\eta)^2+\Delta_{x'}\big)\widehat{\varphi}(x;\eta)=\widehat{f}(x;\eta),\qquad x\in\varpi, \tag{3.6}$$

$$(\partial_\nu+i\nu_1\eta)\widehat{\varphi}(x;\eta)=0,\qquad x\in\varsigma, \tag{3.7}$$

$$\partial_z\widehat{\varphi}(x;\eta)=\lambda\widehat{\varphi}(x;\eta),\qquad x\in\gamma. \tag{3.8}$$

$$\begin{gathered}\widehat{\varphi}(0,x';\eta)=\widehat{\varphi}(1,x';\eta)\ \text{ and}\\ \partial_1\widehat{\varphi}(0,x';\eta)=\partial_1\widehat{\varphi}(1,x';\eta)\qquad \text{for all } x',\eta\end{gathered} \tag{3.9}$$

where $\nu=(\nu_1,\nu_2,\nu_3)$ and (3.9) constitutes the periodicity conditions.

3.3. **Variational and operator formulation, Floquet waves.** The problem (3.6)–(3.9) will be interpreted as a two-fold spectral problem.

Let us first fix $\eta\in\mathbb{R}$ and consider the λ-spectrum. A weak solution of (3.6)–(3.9) means a function $\widehat{\varphi}(\cdot;\eta)\in H^1_{\rm per}(\varpi)$ satisfying

$$((\partial_1+i\eta)\widehat{\varphi},\partial_1+i\eta)\widehat{\psi})_\varpi+(\nabla_{x'}\widehat{\varphi},\nabla_{x'}\widehat{\psi})_\varpi-\lambda(\widehat{\varphi},\widehat{\psi})_\gamma=(\widehat{f},\widehat{\psi})_\varpi \tag{3.10}$$

for all $\widehat{\psi}\in H^1_{\rm per}(\varpi)$. This is obtained in the usual way from the boundary conditions (3.7)–(3.9); for details, see [14]. In the case $f=0$ the problem (3.10) reads as

$$((\partial_1+i\eta)\widehat{\varphi},(\partial_1+i\eta)\widehat{\psi})_\varpi+(\nabla_{x'}\widehat{\varphi},\nabla_{x'}\widehat{\psi})_\varpi=\lambda(\widehat{\varphi},\widehat{\psi})_\gamma. \tag{3.11}$$

Denoting

$$B_\eta(\widehat{\varphi},\widehat{\psi}):=((\partial_1+i\eta)\widehat{\varphi},\partial_1+i\eta)\widehat{\psi})_\varpi+(\nabla_{x'}\widehat{\varphi},\nabla_{x'}\widehat{\psi})_\varpi+(\widehat{\varphi},\widehat{\psi})_\gamma,$$

the equation

$$B_\eta(T(\eta)\widehat{\varphi},\widehat{\psi})=(\widehat{\varphi},\widehat{\psi})_\gamma\ \text{ for all } \widehat{\psi}\in H^1_{\rm per}(\varpi)$$

defines a continuous, positive self-adjoint operator $T(\eta):H^1_{\rm per}(\varpi)\to H^1_{\rm per}(\varpi)$, which is compact due to the compact embedding $H^1_{\rm per}(\varpi)\hookrightarrow L^2(\gamma)$. The problem (3.11) is equivalent to the spectral problem

$$T(\eta)\widehat{\varphi}=M\widehat{\varphi}, \tag{3.12}$$

and due to the connection $M=1/(1+\lambda)$ and well known properties of the eigenvalues of (3.12), the problem (3.11) has for every fixed η the discrete λ-spectrum

$$0\le\Lambda_1(\eta)<\Lambda_2(\eta)\le\ldots\le\Lambda_n(\eta)\le\cdots\to+\infty. \tag{3.13}$$

We remark that by [9, Ch. 9], the functions $\eta\mapsto\Lambda_n(\eta)$ are continuous. Moreover, they are 2π-periodic: any eigenpair (λ,U) with η gives rise to eigenpairs $(\lambda,e^{\pm i2\pi x_1}U)$ with $\eta\pm2\pi$. Furthermore, by [21, Th. 3.4.6] or [19, Th. 2.1], λ belongs to the essential spectrum of the problem (3.1)–(3.3), if and only if it coincides with a $\Lambda_n(\eta)$ for some

η and $n \in \mathbb{N}$. In this way the essential spectrum of (3.1)–(3.3) gets the band-gap structure,

$$\sigma_{\rm ess} = \bigcup_{n \in \mathbb{N}} \Upsilon_n \ , \ \Upsilon_n = \{\lambda \, : \, \lambda = \Lambda_n(\eta), \ \eta \in [0, 2\pi]\},$$

though it may happen that the intervals overlap, meaning that the essential spectrum becomes just the ray $[0, +\infty)$. We remark that in any case the essential spectrum of the problem (3.1)–(3.3) is contained in $\overline{\mathbb{R}_+}$.

Second, we may fix any $\Lambda \in \overline{\mathbb{R}_+}$ and reformulate the problem to find the η-spectrum. We define the operator

$$\mathfrak{A} = \mathfrak{A}(\eta; \Lambda) : H^1_{\rm per}(\varpi) \to H^1_{\rm per}(\varpi)^*$$

by

$$\langle \mathfrak{A} \, \phi, \psi \rangle = \big((\partial_1 + i\eta)\phi, (\partial_1 + i\overline{\eta})\psi \big)_\varpi + (\nabla_{x'}\phi, \nabla_{x'}\psi)_\varpi - \Lambda(\phi, \psi)_\gamma \tag{3.14}$$

and observe that

$$\mathfrak{A}(\eta; \Lambda) = \mathfrak{A}_0(\Lambda) + \eta \mathfrak{A}_1 + \eta^2 \mathfrak{A}_2 : H^1_{\rm per}(\varpi) \to H^1_{\rm per}(\varpi)^*$$

is a quadratic pencil in $\eta \in \mathbb{C}$ (therefore the complex conjugation in (3.14)).

By [4, Ch. 1], there are two possibilities for any fixed Λ:

I. There exists a point $\eta^0 \in \mathbb{C}$ such that $\mathfrak{A}(\eta^0; \Lambda)$ is an isomorphism, and moreover $\mathfrak{A}(\eta; \Lambda)$ is an isomorphism for any $\eta \in \mathbb{C}$ except for a countable set of η-eigenvalues with the only accumulation point at infinity.

II. The whole complex plane is covered by the η-spectrum: for any $\eta \in \mathbb{C}$ there exists $\phi(\eta) \in H^1_{\rm per}(\varpi)$, $\phi(\eta) \neq 0$, such that

$$\mathfrak{A}(\eta; \Lambda)\phi(\eta) = 0.$$

Notice that if II holds for some $\Lambda = \Lambda^0 \in \mathbb{R}_+$, then $\Lambda^0 = \Lambda_n(\eta)$ for some $n \in \mathbb{N}$ and all $\eta \in [0, 2\pi)$ so that $\Upsilon_n = \{\Lambda^0\}$. This is an issue of importance, because in this case $\lambda = \Lambda^0$ is an eigenvalue of the problem (3.1)–(3.3) having infinite multiplicity. It is an open problem, if the case II occurs for some $\lambda \in \overline{\mathbb{R}_+}$. If I holds for every Λ, then the spectrum of the problem (3.1)–(3.3) is fully continuous (since the essential spectrum consists of the continuous spectrum and eigenvalues of infinite multiplicity, see the remarks below (3.13)).

Let $\{\eta_j\}_{j \in \mathbb{N}}$ stand for the set of η-eigenvalues in the half-open strip $\{\eta \in \mathbb{C} : \operatorname{Re} \eta \in [0, 2\pi)\}$. To any η_j there corresponds a system of Jordan chains

$$\{\phi_j^{\ell,p} \, : \, \ell = 1, \ldots, d_j, \ p = 0, \ldots, \aleph_j^\ell - 1\},$$

where d_j is the geometric multiplicity of η_j and $\aleph_j^1, \ldots, \aleph_j^{d_j}$ are partial algebraic multiplicities with the total multiplicity $\aleph_j = \aleph_j^1 + \ldots + \aleph_j^{d_j}$. The functions $\phi_j^{1,0}, \ldots, \phi_j^{d_j,0}$ are eigenvectors, while $\phi_j^{\ell,p}$ with $p \geq 1$ are the associated vectors satisfying the equations

$$\mathfrak{A}(\eta_j; \Lambda)\phi_j^{\ell,p} = -\sum_{m=0}^{p} \frac{1}{m!} \frac{d^m \mathfrak{A}}{d\eta^m}(\eta_j; \Lambda)\phi_j^{\ell,p-m} \tag{3.15}$$

where $\ell = 1, \ldots, d_j$ and $p = 0, \ldots, \aleph_j^\ell - 1$.

Finally, Floquet waves are defined by

$$U_j^{\ell,p}(x) = e^{i\eta_j x_1} \sum_{m=0}^{p} \frac{1}{m!}(ix_1)^m \phi_j^{\ell,p-m}(x)\,,\ x \in \Pi, \tag{3.16}$$

where the functions $\phi_j^{\ell,p}$ are extended periodically for $x_1 \in \mathbb{R}$. Although the functions $U_j^{\ell,p}$ do not even belong to $L^2(\Omega)$, they will still be members of the spaces $W^1_{-\beta}(\Omega)$, $\beta > 0$, with exponentially decaying weights, see below for definition. By a direct computation one can verify that they satisfy the problem (3.1)–(3.3) with $f = 0$, or the integral identity

$$(\nabla_x U, \nabla_x V)_\Pi = \lambda(U, V)_{\Gamma^\natural} \ \text{ for all } V \in C_0^\infty(\Pi).$$

We will assume in the following that the number $\delta > 0$ is fixed so small that all eigenvalues $\eta \in [0, 2\pi) \times [-\delta, \delta] \subset \mathbb{C}$ are real. Distinct eigenvalues are denoted by $\eta_1, \ldots, \eta_N \in [0, 2\pi)$, $N = N(\lambda)$ being their number. In addition we denote the total multiplicity of them by $\aleph_{\rm tot} = \aleph_{\rm tot}(\lambda) := \aleph_1 + \ldots + \aleph_N$. This number is even, see (4.9) below.

Notice that there is no obvious way to decide on the direction of Floquet waves (3.16), since the connection to the time dependent problem is not so straightforward as in the case of the functions (2.15).

Finally, we remark that it will be shown in Section 5.1 (see the remark after Proposition 5.1) that Jordan chains of length 2 appear for sure in the (threshold) situation where $\eta_j = 0$ or $\eta_j = \pi$ is a simple eigenvalue.

4. Operator theoretic approach

We proceed to study the general case of a periodic channel containing a submerged fixed obstacle Θ; see Remark 2.1 for freely floating bodies. We will use at some points the fact that the channel has two outlets to infinity, as a consequence of the periodicity. Channels with only one outlet (having periodicity outside a bounded subdomain) could also be treated by introducing suitable cut-off functions and constructing a parametrix of the problem operator, however, we leave these evident modifications to the reader.

Following [17], [21, Ch. 3 & 5] we consider the problem in Sobolev spaces with exponentially increasing or decreasing weights and employ a skew-Hermitian form q (generalization of (2.18)) to distinguish between incoming and outgoing waves and to formulate the radiation conditions.

4.1. **The setting.** In this section we consider the water-wave problem in a Lipschitz domain Ω, which is a compact perturbation of the periodic channel Π of Section 3.1 by a fixed obstacle Θ, $\Omega := \Pi \setminus \overline{\Theta}$. Although the domain is different from that in Section 2, we use the same notation as far as possible. In particular the fixed number $R_0 > 0$ is large enough to satisfy $\Theta \subset \{x : |x| \le R_0\}$, and the bottom and walls of Ω and the free water surface are still denoted by Σ and Γ, respectively, so that the problem reads as

$$-\Delta_x\varphi(x) = f(x), \qquad x \in \Omega, \tag{4.1}$$

$$\partial_\nu\varphi(x) = 0, \qquad x \in \Sigma, \tag{4.2}$$

$$\partial_z\varphi(x) = \lambda\varphi(x), \qquad x \in \Gamma, \tag{4.3}$$

where f is a given function in a suitable function space, see below. The weak formulation of the problem is obtained from (4.1) for test functions $\psi \in C_0^\infty(\overline{\Omega})$ using the boundary conditions (4.2)–(4.3):

$$(\nabla_x\varphi, \nabla_x\psi)_\Omega - \lambda(\varphi,\psi)_\Gamma = (f,\psi)_\Omega, \tag{4.4}$$

see [14]. The problem will be considered in the weighted Sobolev spaces $W^1_\beta(\Omega)$, $\beta \in \mathbb{R}$, which are endowed with norms

$$\|\varphi; W^l_\beta(\Omega)\| := \Big(\sum_{k=0}^{1} \int\limits_\Omega e^{2\beta|x_1|}|\nabla_x^k\varphi|^2 dx\Big)^{1/2},$$

while the dual space $W^1_\beta(\Omega)^* = W^1_{-\beta}(\Omega)$ of $W^1_\beta(\Omega)$ is determined with respect to the dual pairing

$$\langle\varphi,\psi\rangle := \int\limits_\Omega \varphi\psi dx + \int\limits_\Omega (\nabla_x^k\varphi)(\nabla_x^k\psi)dx.$$

The standard trace inequality $\|\phi; L^2(\Gamma)\| \leq C\|\phi; H^1(\Omega)\|$ holds in Ω, since the periodic domain Ω has Lipschitz boundary. Applying this to the function $e^{\beta|x_1|}\phi$, and taking into account that the norm $\|e^{\beta|x_1|}\phi; H^1(\Omega)\|$ is comparable with $\|\phi; W^l_\beta(\Omega)\|$, yields

$$\|\phi; L^2_\beta(\Gamma)\| := \|e^{\beta|x_1|}\phi; L^2(\Gamma)\| \leq C\|\phi; W^l_\beta(\Omega)\|,$$

hence, the left-hand side of the identity (4.4) is properly defined for all $\varphi \in W^1_\beta(\Omega)$. Assuming that $f \in L^2_\beta(\Omega)$ in (4.1)–(4.4) and using a completion argument, the test functions in (4.4) can be taken as elements of $W^1_{-\beta}(\Omega)$. A solution to (4.1)–(4.3) is then defined as a function $\varphi \in W^1_\beta(\Omega)$ which satisfies

$$(\nabla_x\varphi, \nabla_x\psi)_\Omega - \lambda(\varphi,\psi)_\Gamma = F(\psi) \tag{4.5}$$

for all $\psi \in W^1_{-\beta}(\Omega)$; here $F \in W^1_{-\beta}(\Omega)^*$ is a continuous antilinear functional. The left hand side of (4.5) determines a functional on $W^1_{-\beta}(\Omega) \ni \psi$ and hence also a bounded linear operator

$$A_\beta(\lambda) : W^1_\beta(\Omega) \to W^1_{-\beta}(\Omega)^* \quad , \quad \langle A_\beta(\lambda)\varphi,\psi\rangle = (\nabla_x\varphi, \nabla_x\psi)_\Omega - \lambda(\varphi,\psi)_\Gamma. \tag{4.6}$$

The Fredholm properties of $A_\beta(\lambda)$ have been studied in case of straight, respectively, periodic cylinders, in [10], resp. [17], [19], Th. 2.1, cf. [21], Th. 5.1.4. The following is thus known:

Theorem 4.1. *The operator $A_\beta(\lambda) : W^1_\beta(\Omega) \to W^1_{-\beta}(\Omega)^*$ is Fredholm, if and only if the line segment $[0,2\pi) + i\beta$ in the complex plane does not contain an η-eigenvalue.*

4.2. **Asymptotics of solutions.** Let $\lambda \in \mathbb{R}_0^+$ be a point in the essential spectrum of (4.1)–(4.3) (which is the same as the essential spectrum of (3.1)–(3.3), since these two problems differ only by a compact perturbation of the domain; see a criterion for the Fredholm property in [17] and [20, Ch. 3 §4, Ch. 5 §1]), and moreover, assume that λ is not an eigenvalue of infinite multiplicity. Let the number $\delta > 0$ be such that all eigenvalues $\eta \in [0, 2\pi) \times [-\delta, \delta] \subset \mathbb{C}$ of the problem (3.6)–(3.9) are real, see Section 3.3. For the rest of the paper, we fix such a δ and consider values β with $0 < \beta < \delta$. In particular, by Theorem 4.1, both operators $A_\beta(\lambda)$ and $A_{-\beta}(\lambda)$ are Fredholm and also adjoint to each other.

We define a skew-Hermitian form for solutions u, v of the equation (4.1) in $H^2_{\mathrm{loc}}(\Omega)$ by

$$q(u, v) = \sum_{\pm} \pm \int_{\omega(\pm R)} \left(\overline{v(x)} \frac{\partial u}{\partial x_1}(x) - u(x) \overline{\frac{\partial v}{\partial x_1}(x)} \right) \Big|_{x_1 = \pm R} dx', \tag{4.7}$$

where $|R| > R_0$ is assumed. This is a generalization of (2.18), where the x_1-dependence of the cross-section $\omega(R) = \{(x_1, x') \in \Omega : x_1 = R\}$ is taken into account; more precisely, Q and q coincide in the case of a cylinder, when restricted to functions with support in the set $\{x : x_1 \le -R_0\}$ or $\{x : x_1 \ge R_0\}$.

The following can be verified using the Green formula as in Lemma 2.2, denoting again $\Omega_S = \{x \in \Pi : |x_1| > S\}$ for $S \ge R_0$:

Lemma 4.2. *If both u and v are solutions to* (4.1)–(4.3) *with $f = 0$ in Ω_{R_0}, $\Sigma \cap \overline{\Omega_{R_0}}$ and $\Gamma \cap \overline{\Omega_{R_0}}$ instead of Ω, Σ and Γ, then the value of $q(u, v)$ does not depend on the value of $R > R_0$.*

Using Lemma 4.2 and integrating the formula (4.7) with respect to x_1 over the set $[-R - 1, -R] \cup [R, R + 1]$ with $R \ge R_0$ leads to yet another expression for q, which is defined for solutions u, $v \in H^1_{\mathrm{loc}}(\Omega)$:

$$q(u, v) = \sum_{\pm} \pm \int_{\varpi_\pm(R)} \left(\overline{v(x)} \frac{\partial u}{\partial x_1}(x) - u(x) \overline{\frac{\partial v}{\partial x_1}(x)} \right) dx, \tag{4.8}$$

where $\varpi_\pm(R) = \{x = (x_1, x') \in \Omega : R \le \pm x_1 \le R + 1\}$. The symmetric position of the integration domain on both outlets to infinity will be used for example in the proof of Proposition 4.4.

We remark that the proof of Proposition 2.4 shows that the q-form vanishes in the space of solutions in $W^1_\beta(\Omega)$ and it would thus be well defined in the quotient space of solutions in $W^1_{-\beta}(\Omega)$, which is written as $\mathbf{W}_{-\beta}(\lambda)/W^1_\beta(\Omega)$ by using the notation to be introduced in (4.15). In this way, waves could be defined as equivalence classes which do not depend on the choice of the cut-off function $\chi_\pm$. However, we do not expose this aspect in the following.

Let us recall the notation of Section 3.3, especially the eigenvalues $\eta_1, \ldots, \eta_{N(\lambda)}$, the total multiplicity of the eigenvalues $\aleph_{\mathrm{tot}}(\lambda)$, and the sequence of eigenfunctions and their associated functions $\phi_j^{q,p} \in L^2(\varpi)$, see (3.16). Moreover, let χ_+ be the same cut-off function as in Section 2.2, i.e. $\chi_+(x_1) = 0$, if $x_1 \le R_0$ and $\chi_+(x_1) = 1$, if $x_1 \geq R_0 + 1$, and let $\chi_-(x_1) = \chi_+(-x_1)$. There are $\aleph_{\rm tot}(\lambda)$ linearly independent functions of the form

$$\chi_\pm(x_1) U_j^{q,p}(x)\,, \tag{4.9}$$

where the indices j, q and p are as in (3.16); in particular, $\aleph_{\rm tot}(\lambda)$ is an even number.

The next theorem is known and elementary (see e.g. [15, Ch. XIV §7,Th. 4, Cor. 1]), but we outline the proof for the convenience of the reader. The proof uses Sylvester's law of inertia: given a Hermitian $n \times n$-matrix A and any invertible $n \times n$-matrix S such that $S^{-1}AS = \widetilde{A}$ is diagonal, the number of positive, zero and negative entries (on the diagonal) of $\widetilde{A}$ does not depend on the choice of S.

Theorem 4.3. *In the $\aleph_{\rm tot}(\lambda)$-dimensional subspace $W(\lambda)$ of $W^1_{-\beta}(\Omega)$ spanned by the functions* (4.9)*, one can find a basis consisting of functions*

$$\begin{aligned} &u_n^+\ ,\ n = 1, \ldots, J^+(\lambda), \quad \textit{and} \\ &u_n^-\ ,\ n = 1, \ldots, J^-(\lambda) = \aleph_{\rm tot}(\lambda) - J^+(\lambda), \end{aligned} \tag{4.10}$$

such that the following holds for all n and m:

$$q(u_n^\pm, u_m^\pm) = \pm i \delta_{nm}\ ,\ \ q(u_n^\pm, u_m^\mp) = 0. \tag{4.11}$$

The numbers $J^\pm(\lambda)$ do not depend on the choice of the basis with properties (4.11)*.*

We call the functions u_n^+ *outgoing* and u_n^- *incoming*, and we denote $W^\pm(\lambda) := {\rm sp}\{u_n^\pm\ :\ n = 1, \ldots, J^\pm(\lambda)\} \subset W^1_{-\beta}(\Omega)$. This classification could be extended to outgoing and incoming waves defined as equivalence classes, as was explained after Lemma 4.2. In Section 5 we calculate the functions $u_n^\pm$ explicitly for Jordan chains of length 1 and 2.

Proof. Given any basis of $W(\lambda)$, i.e. a set of linearly independent elements w_n, $n = 1, \ldots, \aleph_{\rm tot}(\lambda) =: K$, the condition $q(w_n, w_m) = M_{nm}$ defines a matrix $M = (M_{nm})_{n,m=1}^K$ which, by the property $q(f,g) = -\overline{q(g,f)}$, is skew-Hermitian, or antihermitian; equivalently, the matrix $A := iM$ is Hermitian. We remark that if another basis of $W(\lambda)$ consisting of vectors $\widetilde{w}_n$ were chosen and a matrix $\widetilde{A}$ were defined accordingly, we would have $A = S\widetilde{A}S^{-1}$ for some invertible $K \times K$-matrix S. Then, according to the law of inertia, the numbers of positive and negative eigenvalues of $A = iM$ are independent of the choice of the above mentioned basis; they are denoted by $J^+(\lambda)$ and $J^-(\lambda)$, respectively. Null eigenvalues do not exist, since in [18] and [21, § 3,3, §5.1] it has been proven that for any $u \in W(\lambda)$ one can find $v \in W(\lambda)$ such that $q(u,v) = 1$. (In other words, q is a non-degenerate form on $W(\lambda)$.) Let us denote the eigenvalues of A by α_n, $n = 1, \ldots, K$, and agree on the indexing that $\alpha_n > 0$ for $n = 1, \ldots, J^+(\lambda)$.

The matrix A and thus also $\overline{A}$ are Hermitian, and they have the same eigenvalues. The basis we are looking for is obtained from the eigenvectors of $\overline{A}$ in a straightforward way. Let $v_n = \sum_{j=1}^K b_{nj} e_j$ be an orthonormal set of eigenvectors in $\mathbb{C}^K$ for the matrix $\overline{A}$ corresponding to the eigenvalues $\alpha_n \in \mathbb{R}$. We set

$$u_n^+ = \alpha_n^{-1/2} \sum_{j=1}^{K} b_{nj} w_j \quad \text{for } n = 1, \dots, J^+(\lambda),$$

$$u_n^- = (-\alpha_n)^{-1/2} \sum_{j=1}^{K} b_{nj} w_{j+J^+(\lambda)} \quad \text{for } n = 1, \dots, J^-(\lambda).$$

By standard matrix calculus, denoting transposition by $\top$,

$$\begin{aligned} q(u_n^+, u_m^+) &= \alpha_n^{-1/2}\alpha_m^{-1/2} \sum_{j,k=1}^{K} b_{nj}\overline{b}_{mk} q(w_j, w_k) = \alpha_n^{-1/2}\alpha_m^{-1/2} \sum_{j,k=1}^{K} b_{nj}\overline{b}_{mk} M_{jk} \\ &= i\alpha_n^{-1/2}\alpha_m^{-1/2} v_n A(\overline{v}_m)^\top = i\alpha_n^{-1/2}\alpha_m^{-1/2} v_n \overline{\overline{A}(v_m)^\top} \\ &= i\alpha_n^{-1/2}\alpha_m^{-1/2} v_n \overline{\alpha_m (v_m)^\top} = i\alpha_n^{-1/2}\alpha_m^{1/2} v_n (\overline{v}_m)^\top = i\delta_{nm}, \end{aligned}$$

The other cases in (4.11) are treated in the same way. ⊠

Proposition 4.4. *There holds the identity $J^+(\lambda) = J^-(\lambda) = \aleph_{\rm tot}(\lambda)/2 =: J(\lambda)$, so that the dimensions of both spaces $W^\pm(\lambda)$ are equal to $J(\lambda)$.*

Proof. All elements in the space $W(\lambda)$ of Theorem 4.3 are linear combinations of the vectors (4.9), which also form a basis of $W(\lambda)$. Thus, the linear mapping T which maps the functions $\chi_+(x_1)U_j^{p,q}(x)$ to $\chi_-(x_1)U_j^{p,q}(x)$ and functions $\chi_-(x_1)U_j^{p,q}(x)$ to $\chi_+(x_1)U_j^{p,q}(x)$, is a linear bijection of $W(\lambda)$ onto itself. Moreover, by the definition of the q-form (consider the integration domains there),

$$q\big(T(\chi_\pm U_j^{p,q}), T(\chi_\pm U_j^{p,q})\big) = -q(\chi_\pm U_j^{p,q}, \chi_\pm U_j^{p,q}), \tag{4.12}$$

for every j, p, q, and thus it follows that

$$\begin{aligned} q\big(Tu_n^\pm, Tu_m^\pm\big) &= -q(u_n^\pm, u_m^\pm) = \mp i\delta_{nm} \\ q\big(Tu_n^\pm, Tu_m^\mp\big) &= -q(u_n^\pm, u_m^\mp) = 0 \end{aligned}$$

for every n, m. Hence, the set consisting of all functions $Tu_n^\pm$ is a basis of $W(\lambda)$ which has $J^-(\lambda)$ outgoing and $J^+(\lambda)$ incoming functions in the above terminology. Since these numbers do not depend on the choice of basis, by Theorem 4.3, we have $J^+(\lambda) = J^-(\lambda)$. ⊠

We will need the following result.

Theorem 4.5. *Assume that $F \in W^1_{-\beta}(\Omega)^* \subset W^1_\beta(\Omega)^*$, and let φ be a solution to the problem (4.5) in the space $W^1_{-\beta}(\Omega)$. It can be written in the form*

$$\varphi(x) = \sum_{n=1}^{J(\lambda)} \Big(c_n^+ u_n^+(x) + c_n^- u_n^-(x) \Big) + \widetilde{\varphi}(x), \tag{4.13}$$

where the coefficients $c_j^\pm$ *and the remainder* $\widetilde\varphi \in W^1_\beta(\Omega)$ *satisfy the estimate*

$$(4.14)\ \ \|\widetilde\varphi; W^1_\beta(\Omega)\| + \sum_{n=1}^{J(\lambda)} |c_n^+| + \sum_{n=1}^{J(\lambda)} |c_n^-| \le c_\beta\big(\|F; W^1_{-\beta}(\Omega)^*\| + \|\varphi; W^1_{-\beta}(\Omega)\|$$

with a constant c_β *independent of* F *and* φ*.*

This theorem is proven in [17]. It also follows from Theorem 5.1.4 of [21] (recall the choice of the parameters δ and β in the beginning of this section), where any solution φ of (4.5) is written as

$$\varphi = \widetilde\varphi + \sum_{j,q,p} c_j^{q,p} \chi_\pm U_j^{q,p}$$

with some coefficients $c_j^{q,p}$, $\widetilde\varphi(x) \in W^1_\beta(\Omega)$ and indices as in (3.16). Clearly, (4.13) follows by using the basis property of the functions $u_n^\pm$. We remark that Theorem 4.5 does not imply surjectivity of $A_\beta(\lambda)$ or $A_{-\beta}(\lambda)$.

Theorem 4.5 shows that the pre-image

$$\mathbf{W}_{-\beta}(\lambda) := A_{-\beta}(\lambda)^{-1} W^1_{-\beta}(\Omega)^* \subset W^1_{-\beta}(\Omega) \tag{4.15}$$

is the direct sum

$$\mathbf{W}_{-\beta}(\lambda) = W^1_\beta(\Omega) \dot{+} W^+(\lambda) \dot{+} W^-(\lambda).$$

Taking into account the estimate (4.14) we endow the space $\mathbf{W}_{-\beta}(\lambda)$ with the norm

$$\|\varphi\| := \|\widetilde\varphi; W^1_\beta(\Omega)\| + \sum_{n=1}^{J(\lambda)} |c_n^+| + \sum_{n=1}^{J(\lambda)} |c_n^-| \tag{4.16}$$

where $c_n^\pm$ and $\widetilde\varphi$ are as in (4.13). This makes $\mathbf{W}_{-\beta}(\lambda)$ into a Banach space; actually the norm (4.16) is equivalent to a Hilbertian norm, but we will not need this fact later.

We still define the operator

$$\mathbf{A}_{-\beta}(\lambda) : \mathbf{W}_{-\beta}(\lambda) \to W^1_{-\beta}(\Omega)^* \tag{4.17}$$

as the restriction of the operator $A_{-\beta}(\lambda)$ to $\mathbf{W}_{-\beta}(\lambda)$. The Fredholm index of an operator T is denoted by $\mathrm{Ind}\, T = \dim\ker T - \dim\mathrm{coker}\, T$. If T is a Fredholm operator from a Banach space X into a dual Y^* of a Banach space Y, then by definition $\mathrm{coker}\, T$ is the quotient space $Y^*/T(X)$. We will use the following elementary fact.

Lemma 4.6. *For a Fredholm operator* $T : X \to Y^*$*, the dimension of coker*T *coincides with the dimension of the space*

$$\{y \in Y \,:\, \langle Tx, y\rangle_Y = 0 \text{ for all } x \in X\}, \tag{4.18}$$

where $\langle\cdot,\cdot\rangle_Y$ *denotes the dual pairing of* Y *and* Y^**.*

Indeed, vectors of Y^* which do not belong to $T(X)$ can be made into one-to-one-correspondence with vectors (4.18) of Y by using the Hahn-Banach theorem.

Theorem 4.7. *The operator* $\mathbf{A}_{-\beta}(\lambda)$ *is Fredholm, and*

$$\mathrm{Ind}\, \mathbf{A}_{-\beta}(\lambda) = \mathrm{Ind}\, A_{-\beta}(\lambda) = J(\lambda). \tag{4.19}$$

Proof. We recall that both operators $A_\beta(\lambda)$ and $A_{-\beta}(\lambda)$ are Fredholm, due to Theorem 4.1 and the choice of δ and β in the beginning of Section 4.2. Then, also $\mathbf{A}_{-\beta}(\lambda)$ is Fredholm, by its definition. As for the indices, we claim that

$$\operatorname{Ind} \mathbf{A}_{-\beta}(\lambda) = \operatorname{Ind} A_\beta(\lambda) + \aleph_{\mathrm{tot}}(\lambda) = \operatorname{Ind} A_\beta(\lambda) + 2J(\lambda). \tag{4.20}$$

To see this, both operators $A_\beta(\lambda)$ and $\mathbf{A}_{-\beta}(\lambda)$ have the same image space. The claim then follows from elementary linear algebra: "diminishing" the domain of a Fredholm operator by a subspace of dimension $d \in \mathbb{N}$ increases the index by d. (In detail: Let $k \in \mathbb{N}$, $0 \le k \le \aleph_{\mathrm{tot}}(\lambda)$, be the dimension of $(W^+(\lambda) + W^-(\lambda)) \cap \ker \mathbf{A}_{-\beta}(\lambda)$. Then, by the definitions of $\mathbf{A}_{-\beta}(\lambda)$ and $\mathbf{W}_{-\beta}(\lambda)$, we have $\dim\ker \mathbf{A}_{-\beta}(\lambda) = \dim\ker A_\beta(\lambda) + k$ and also $\dim\operatorname{coker} \mathbf{A}_{-\beta}(\lambda) = \dim\operatorname{coker} A_\beta(\lambda) - (\aleph_{\mathrm{tot}}(\lambda) - k)$.)

We next show that

$$\operatorname{Ind} \mathbf{A}_{-\beta}(\lambda) = \operatorname{Ind} A_{-\beta}(\lambda). \tag{4.21}$$

We first observe that the kernels of $\mathbf{A}_{-\beta}(\lambda)$ and $A_{-\beta}(\lambda)$ are the same, by Theorem 4.5. We need to prove the equality of the dimensions of the cokernels of these operators, denoted here by $\dim\operatorname{coker} \mathbf{A}_{-\beta}(\lambda) = \mathbf{k}$ and $\dim\operatorname{coker} A_{-\beta}(\lambda) = k$. By definition of $\operatorname{coker} \mathbf{A}_{-\beta}(\lambda)$, there exist $\mathbf{k}$ linearly independent elements $G \in W^1_{-\beta}(\Omega)^* = W^1_\beta(\Omega)$ such that the equation $\mathbf{A}_{-\beta}(\lambda)\varphi = G$ does not have a solution in $\mathbf{W}_{-\beta}(\lambda)$ (cf. (4.17)). But Theorem 4.5 shows that this equation then does not have a solution in $W^1_{-\beta}(\Omega)$ either, which implies that $k \ge \mathbf{k}$.

There remains to verify that $k \le \mathbf{k}$. Applying Lemma 4.6 to the operator $A_{-\beta}(\lambda)$, there exists k linearly independent functions $g \in W^1_\beta(\Omega)$ such that (see (4.6))

$$\langle A_{-\beta}(\lambda)\varphi, g\rangle = (\nabla\varphi, \nabla g)_\Omega - \lambda(\varphi, g)_\Gamma = 0 \ \text{ for all } \varphi \in W^1_{-\beta}(\Omega). \tag{4.22}$$

In particular (4.22) holds for all $\varphi \in \mathbf{W}_{-\beta}(\lambda)$, i.e.,

$$\langle \mathbf{A}_{-\beta}(\lambda)\varphi, g\rangle = 0. \tag{4.23}$$

Since the k linearly independent functions g in (4.23) belong to $W^1_\beta(\Omega) \subset W^1_{-\beta}(\Omega)$, a second application of Lemma 4.6 to the operator $\mathbf{A}_{-\beta}(\lambda) : \mathbf{W}_{-\beta}(\lambda) \to W^1_{-\beta}(\Omega)^*$ shows that $\operatorname{coker} \mathbf{A}_{-\beta}(\lambda)$ is at least of dimension k. Consequently, $k = \mathbf{k}$ and (4.21) holds.

As it was remarked in the beginning of Section 4.2, $A_{-\beta}(\lambda)$ equals the adjoint $A_\beta(\lambda)^*$. Hence, $\operatorname{Ind} A_\beta(\lambda) = -\operatorname{Ind} A_{-\beta}(\lambda)$; this, (4.21) and (4.20) imply $\operatorname{Ind} \mathbf{A}_{-\beta}(\lambda) = J(\lambda)$. ⊠

4.3. **Radiation conditions.** We denote by

$$\mathbf{A}^+_{-\beta}(\lambda) : \mathbf{W}^+_{-\beta}(\lambda) \to W^1_{-\beta}(\Omega)^*$$

the restriction of $A_{-\beta}(\lambda) : W^1_{-\beta}(\Omega) \to W^1_\beta(\Omega)^*$ (or that of $\mathbf{A}_{-\beta}(\lambda) : \mathbf{W}_{-\beta}(\lambda) \to W^1_{-\beta}(\Omega)^*$, see (4.15)–(4.17)), where

$$\mathbf{W}^+_{-\beta}(\lambda) = W^1_\beta(\Omega) \dot{+} W^+(\lambda) \subset \mathbf{W}_{-\beta}(\lambda),$$

consists of functions

$$\varphi = \sum_{n=1}^{J(\lambda)} c_n^+ u_n^+ + \widetilde{\varphi} \quad , \quad \widetilde{\varphi} \in W^1_\beta(\Omega).$$

Now, solving the equation

$$\mathbf{A}^+_{-\beta}(\lambda)\varphi = F \quad , \quad F \in W^1_{-\beta}(\Omega)^* \tag{4.24}$$

with a given F is equivalent to solving the original water-wave problem (4.1)–(4.3) in the space consisting of exponentially decaying and outgoing waves. Moreover, Theorem 4.8 shows that the problem (4.24) is well-posed. These definitions and results constitute our radiation conditions (for a physical interpretation using the Mandelstam (energy) principle, see Section 4.5). As incoming waves are excluded, this is analogous with the Sommerfeld radiation principle, but the physical and mathematical reasons are quite different.

Theorem 4.8. *The operator*

$$\mathbf{A}^+_{-\beta}(\lambda) : \mathbf{W}^+_{-\beta}(\lambda) \to W^1_{-\beta}(\Omega)^*$$

is Fredholm of index zero. The problem (4.24) *has a solution* $u \in \mathbf{W}^+_{-\beta}(\lambda)$*, if and only if the right hand side* $F \in W^1_{-\beta}(\Omega)^*$ *satisfies the compatibility conditions*

$$\langle F, v\rangle = 0 \quad \forall\, v \in \ker A_\beta(\lambda).$$

This solution is defined up to an addentum in ker $A_\beta(\lambda)$*, a trapped mode. If the orthogonality conditions*

$$\langle u, v\rangle = 0 \quad \forall\, v \in \ker A_\beta(\lambda)$$

are satisfied, then the solution u *is unique and has the bound*

$$\|u; \mathbf{W}^+_{-\beta}(\lambda)\| \le c\|F; W^1_{-\beta}(\Omega)^*\|.$$

Proof. That the index of $\mathbf{A}^+_{-\beta}(\lambda)$ is null, can be seen by a comparison with the operator $\mathbf{A}_{-\beta}(\lambda)$ in the same way as in the beginning of the proof of Theorem 4.7: the domain of $\mathbf{A}^+_{-\beta}(\lambda)$ differs from the domain of $\mathbf{A}_{-\beta}(\lambda)$ by a $J(\lambda)$-dimensional subspace, and the index of $\mathbf{A}_{-\beta}(\lambda)$ is $J(\lambda)$, by (4.19). The other statements are consequences of the Fredholm alternative, since we can replace $\ker A_\beta(\lambda)$ by $\ker \mathbf{A}^+_{-\beta}(\lambda)$ by using the next lemma. ⊠

Lemma 4.9. *We have* $\ker \mathbf{A}^+_{-\beta}(\lambda) = \ker A_\beta(\lambda)$.

Proof. By the definitions of the operators, $\ker A_\beta(\lambda) \subset \ker \mathbf{A}^+_{-\beta}(\lambda)$. A function $v \in \ker \mathbf{A}^+_{-\beta}(\lambda) \backslash \ker A_\beta(\lambda)$ is of the form $v = \sum_j c_j u_j^+ + \widetilde{\varphi}$ with $\widetilde{\varphi} \in W^1_\beta(\Omega)$. However, v is harmonic as an element in $\ker \mathbf{A}^+_{-\beta}(\lambda)$, and thus, $q(v,v) = 0$, by the definition of q and the Green formula. The proof of Proposition 2.4 shows that all coefficients c_j must be 0, i.e. $v \in \ker A_\beta(\lambda)$. ⊠

4.4. **Scattering matrix.** We next consider scattering of incoming waves in problem (4.1)–(4.3).

Theorem 4.10. *The dimension of the space* $L(\lambda) := \ker A_{-\beta}(\lambda) \ominus \ker A_{\beta}(\lambda)$ *equals* $J(\lambda)$*, and it has a basis* $\zeta_1, \ldots, \zeta_{J(\lambda)}$ *such that*

$$\zeta_j = u_j^- + \sum_{k=1}^{J(\lambda)} s_{jk} u_j^+ + \widetilde{\varphi}_j \ , \tag{4.25}$$

where $\widetilde{\varphi}_j \in W^1_\beta(\Omega)$ *and the coefficients* s_{jk} *form a unitary* $J(\lambda) \times J(\lambda)$*-matrix* s*, the scattering matrix.*

Proof. We have $\dim \ker \mathbf{A}_{-\beta}(\lambda) = \dim \ker \mathbf{A}^+_{-\beta}(\lambda) + J(\lambda)$, since the image spaces of these two operators are the same and the domains differ by a subspace of dimension $J(\lambda)$. Moreover, the kernels of $A_{-\beta}(\lambda)$ and $\mathbf{A}_{-\beta}(\lambda)$ are the same, by Theorem 4.1. Hence, the claim about the dimension of $L(\lambda)$ follows from Lemma 4.9.

Given any orthonormal basis $(\mathbf{e}_j)_{j=1}^{J(\lambda)}$ of $L(\lambda)$, there exist numbers b^-_{jk}, b^+_{jk} such that it can be written as

$$\mathbf{e}_j = \sum_{k=1}^{J(\lambda)} b^-_{jk} u_k^- + \sum_{k=1}^{J(\lambda)} b^+_{jk} u_k^+ + \widetilde{\varphi}'_j. \tag{4.26}$$

for some constants $b^{\pm}_{jk}$ and some $\widetilde{\varphi}'_j \in W^1_\beta(\Omega)$. We remark that the $J(\lambda)$ elements $\sum_{k=1}^{J(\lambda)} b^-_{jk} u_k^-$ must form a linearly independent set. Indeed, if these were linearly dependent, it would be possible to construct a nonzero element $y \in L(\lambda) \subset \ker A_{-\beta}(\lambda)$ as a linear combination $y = \sum_{k=1}^{J(\lambda)} a_k u_j^+ + \widetilde{y}$ with $\widetilde{y} \in W^1_\beta(\Omega)$. However, this would lead to a contradiction by the proof of Proposition 2.4 or Lemma 4.9. Applying the Gram-Schmidt method to the coefficient matrix (b^-_{jk}) makes the basis $(\mathbf{e}_j)_{j=1}^{J(\lambda)}$, (4.26), into the basis (4.25) of $L(\lambda)$.

It suffices to prove the unitarity of s, i.e., the equality $s^* = s^{-1}$. Similarly to (2.22) we obtain

$$\begin{aligned}
q(\zeta_j, \zeta_k) &= q\Big(u_j^- + \sum_{n=1}^{J(\lambda)} s_{jn} u_n^+ + \widetilde{\varphi}_j, u_k^- + \sum_{m=1}^{J(\lambda)} s_{km} u_m^+ + \widetilde{\varphi}_k\Big) \\
&= q\Big(u_j^- + \sum_{n=1}^{J(\lambda)} s_{jn} u_n^+, u_k^- + \sum_{m=1}^{J(\lambda)} s_{km} u_m^+\Big) \\
&= q(u_j^-, u_k^-) + \sum_{n,m=1}^{J(\lambda)} s_{jn} \overline{s_{km}} q(u_n^+, u_m^+) \\
&= -i\delta_{j,k} + i \sum_{n,m=1}^{J(\lambda)} s_{jn} \overline{s_{km}} \delta_{n,m} = i\Big(\sum_{n=1}^{J(\lambda)} s_{jn} \overline{s_{kn}} - \delta_{j,k}\Big).
\end{aligned} \tag{4.27}$$

Again, the functions ζ_j are harmonic, as solutions of the homogeneous problem (4.1). Thus $q(\zeta_j, \zeta_k) = 0$ and the unitarity of s follows from (4.27). ⊠

The scattering matrix s of Theorem 4.10 of course depends on the choice of the bases $\{u_1^\pm, \dots, u_{J(\lambda)}^\pm\}$ in $W^\pm(\lambda)$, but we want to show that s always defines an isometry $W^-(\lambda) \cong W^+(\lambda)$, once the spaces $W^\pm(\lambda) = \mathrm{sp}\{u_n^\pm : n = 1, \dots, J(\lambda)\}$ are fixed (formula (4.31) below). To this end it is convenient to use vector/matrix notation and denote the rows of solutions and waves by

$$\zeta = (\zeta_1, \dots, \zeta_{J(\lambda)}) \ , \quad u^\pm = (u_1^\pm, \dots, u_{J(\lambda)}^\pm). \tag{4.28}$$

Then, decompositions (4.25) can be written briefly as

$$\zeta = u^- + u^+ s + \widetilde{\zeta}. \tag{4.29}$$

Let $t^\pm$ be unitary $J(\lambda) \times J(\lambda)$-matrices. We set

$$\mathbf{u}^\pm = (\mathbf{u}_1^\pm, \dots, \mathbf{u}_{J(\lambda)}^\pm) = u^\pm t^\pm \tag{4.30}$$

and obtain

$$\boldsymbol{\zeta} = \zeta t^- = \mathbf{u}^- + \mathbf{u}^+ (t^+)^* s t^- + \widetilde{\boldsymbol{\zeta}}.$$

Thus, the new scattering matrix $\mathbf{s}$ in the new basis (4.30) equals

$$\mathbf{s} = (t^+)^* s t^-, \tag{4.31}$$

which is in accordance with the transformations $u^+ \mapsto u^+ t^+$ in $W^+(\lambda)$ and $u^- \mapsto u^- t^-$ in $W^-(\lambda)$.

Finally we remark that, although the choice of the combined basis $\{u^+, u^-\}$ in the entire space $W(\lambda)$ of waves is not unique, the next lemma importantly shows that the scattering matrix is symmetric,

$$s = s^\top, \tag{4.32}$$

provided the relation

$$u^- = \overline{u^+} \tag{4.33}$$

holds; here $\top$ stands for the transposition, $(s^\top)_{kj} = s_{jk}$. (Note that s is in general not Hermitian.)

Lemma 4.11. *If the rows of waves $u^\pm$,* (4.28)*, satisfy* (4.11) *and* (4.33)*, then the unitary scattering matrix becomes symmetric, see* (4.32).

Proof. By (4.29) and (4.33) we write

$$\overline{\zeta}(\overline{s})^{-1} = \big(\overline{u^-} + \overline{u^+}\overline{s}\big)(\overline{s})^{-1} + \overline{\widetilde{\zeta}}(\overline{s})^{-1} = u^+(\overline{s})^{-1} + u^- + \overline{\widetilde{\zeta}}(\overline{s})^{-1}.$$

Thus, there are no outgoing waves in the decomposition

$$\zeta - \overline{\zeta}(\overline{s})^{-1} = u^+\big(s - (\overline{s})^{-1}\big) + \widetilde{\zeta} - \overline{\widetilde{\zeta}}(\overline{s})^{-1},$$

the difference $\zeta - \overline{\zeta}(\overline{s})^{-1}$ decays exponentially and hence falls into $\ker A_\beta(\lambda)$. Therefore, $s = (\overline{s})^{-1} = (\overline{s})^* = s^\top$. $\boxtimes$

4.5. **Umov-Poynting vector.** In this section we will discover a relation between the q-form (4.7) and the Umov-Poynting vector. The latter was introduced in [26] for problems in acoustics and elasticity and in [24] for electro-magnetism.

Let G be a bounded subdomain of Ω with the surfaces

$$\Sigma_G = \Sigma \cap \partial G \ , \quad \Gamma_G = \Gamma \cap \partial G \quad \text{and} \quad \Upsilon = \partial G \setminus \big(\Sigma_G \cup \Gamma_G\big).$$

The total energy, i.e. the sum of kinetic and potential energy, contained in G equals (see [11])

$$E(\phi; t) = \int\limits_G |\nabla_x \mathrm{Re}\phi(x,t)|^2 dx + g \int\limits_{\Gamma_G} \mathrm{Re}\phi(x,t)\ \partial_t^2 \mathrm{Re}\phi(x,t) ds$$

where

$$\phi(x,t) = e^{-i\omega t}\varphi(x).$$

The Umov-Poynting vector $J(\phi; x, t)$ describes the flux density of energy out of the domain G:

$$\int\limits_\Upsilon \nu(x)^\top J(\phi; x,t) ds(x) = -\partial_t E(\phi; t).$$

The right-hand side becomes (since $\partial_t(\mathrm{Re}(e^{-i\omega t}\varphi)) = \mathrm{Re}(-i\omega e^{-i\omega t}\varphi) = \omega \mathrm{Im}(e^{-i\omega t}\varphi)$ etc.)

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(- \int\limits_G |\nabla_x \mathrm{Re}\phi(x,t)|^2 dx + g\omega^2 \int\limits_{\Gamma_G} |\mathrm{Re}\phi(x,t)|^2 ds(x)\Big) \\
&= 2\omega \Big(- \int\limits_G \big(\nabla_x \mathrm{Re}\phi(x,t)\big)^\top \nabla_x \mathrm{Im}\phi(x,t)\, dx + \lambda \int\limits_{\Gamma_G} \mathrm{Re}\phi(x,t) \mathrm{Im}\phi(x,t)\, ds(x)\Big) \\
&= -2\omega \int\limits_\Upsilon \mathrm{Im}\phi(x,t) \partial_\nu \mathrm{Re}\phi(x,t)\, ds(x),
\end{aligned}$$

where an integration by parts was performed and the homogeneous equations (4.1)–(4.3) were used. Consequently, we find the Umov-Poynting vector to be

$$J(\phi; x,t) = -2\omega \mathrm{Im}\phi(x,t) \nabla_x \mathrm{Re}\phi(x,t). \tag{4.34}$$

According to the Mandelstam radiation principle [16] the direction of a propagating wave ϕ is determined by the sign of the integral

$$\int\limits_\Upsilon \nu(x)^\top \widehat{J}(\phi; x)\, ds(x), \tag{4.35}$$

where $\widehat{J}(\phi; x)$ is the mean value of $J(\phi; x, t)$ over the time interval $(0, 2\pi/\omega) \ni t$. Thus, Theorems 4.3, 4.8 and the following result show that our radiation condition is of Mandelstam type.

Theorem 4.12. *The Umov-Poynting vector $J(\phi; x, t)$ and the q-form are related by*

$$q(\varphi,\varphi) = i\frac{2}{\omega}\int\limits_{\Upsilon} \nu(x)^\top \widehat{J}(\phi; x)\, ds(x),$$

Proof. By (4.34), we obtain that (4.35) coincides with

$$\begin{aligned}
&\frac{\omega}{2\pi}\int\limits_0^{2\pi/\omega}\int\limits_{\Upsilon} \nu(x)^\top J(\phi; x, t)\, ds(x)dt \\
&= -\frac{\omega^2}{\pi}\int\limits_0^{2\pi/\omega}\int\limits_{\Upsilon} \mathrm{Im}\phi(x,t)\partial_\nu \mathrm{Re}\phi(x,t)\, ds(x)dt \\
&= -\frac{\omega^2}{4\pi i}\int\limits_0^{2\pi/\omega}\int\limits_{\Upsilon} \big(e^{-i\omega t}\varphi - e^{i\omega t}\overline{\varphi}\big)\big(e^{-i\omega t}\partial_\nu\varphi + e^{i\omega t}\partial_\nu\overline{\varphi}\big)\, dsdt \\
&= -\frac{\omega}{2i}\int\limits_{\Upsilon}\big(\varphi\partial_\nu\overline{\varphi} - \overline{\varphi}\partial_\nu\varphi\big)\, ds = -\frac{i\omega}{2}\int\limits_{\Upsilon}\big(\overline{\varphi}\partial_\nu\varphi - \varphi\partial_\nu\overline{\varphi}\big)\, ds. \qquad (4.36)
\end{aligned}$$

Taking $G = \{x = (x_1, x') \in \Omega : |x_1| < R\}$, and accordingly $\Upsilon = \{x \in \Omega : |x_1| = R\}$, we find that (4.36) coincides with $-\frac{i\omega}{2}q(\varphi,\varphi)$. ⊠

We see that the classification of waves introduced in Theorem 4.3 by using the skew-Hermitian form q coincides with the classification by the Mandelstam energy principle, based on the Umov-Poynting vector for water-waves.

5. Appendix: Structure of Jordan chains.

In this section we calculate explicitly the incoming and outgoing waves $u_n^\pm$, (4.10), as suitable linear combinations of the truncated Floquet waves, cf. (3.16), (4.9). We only consider the special case where the length of the Jordan chains is either 1 or 2. This restriction is for technical simplicity only; longer chains could be treated by similar, although more cumbersome, calculations. The notation is as around (3.15)–(3.16) of Section 3.3, although we do not display the dependence of any expressions on the spectral parameter Λ or λ (which is assumed to be fixed and of the case I in Section 3.3),

We take advantage of Lemma 4.2 and formula (4.8) with $R > R_0$ large enough so that in the calculation of the q-form for functions (4.9), the cut-off function can be put equal to 1. In view of (3.16) we thus get the relation

$$q(U_j^{\ell,p}, U_{j'}^{\ell',p'}) = 0$$

for $j \neq j'$ and $\ell \neq \ell'$, but $q(U_j^{\ell,p}, U_j^{\ell,p'})$ can still be nonzero for $p \neq p'$ due to the monomials of x_1 in (3.16). This observation leads to look for the functions $u_n^\pm$ as linear combinations of functions corresponding to the same Jordan chain, i.e, for any n and sign $\pm$, there should exist fixed j and ℓ and coefficients $b_j^{\ell,p} \in \mathbb{C}$ such that

$$u_n^\pm = \chi \sum_{p=0}^{\aleph_j^\ell - 1} b_j^{\ell,p} U_j^{\ell,p}, \tag{5.1}$$

where $\chi = \chi_+$ or χ_- (sign not related to the sign-index of $u_n^\pm$). We next calculate the numbers $b_j^{\ell,p}$.

5.1. **Case with eigenvectors only.** Assume that $j \in \{1, \dots, N\}$ and $\eta_j \in [0, 2\pi)$ is a simple eigenvalue (hence $d_j = 1 = \ell$) and also $\aleph_j^\ell = 1$. Thus, η_j has the eigenvector $\phi_j^{\ell,0}$, $\ell \in \{1, \dots, d_j\}$, but corresponding associated vectors $\phi_j^{\ell,1}$ do not exist. The equation (3.15) with $p = 1$ can be written as the formally self-adjoint boundary value problem

$$-\big((\partial_1 + i\eta_j)^2 + \Delta_{x'}\big)\phi_j^{\ell,1}(x) = 2i(\partial_1 + i\eta_j)\phi_j^{\ell,0}(x), \qquad x \in \varpi, \tag{5.2}$$

$$\partial_z \phi_j^{\ell,1}(x) = \Lambda \phi_j^{\ell,1}(x), \qquad x \in \gamma, \tag{5.3}$$

$$(\partial_\nu + i\eta_j\nu_1)\phi_j^{\ell,1}(x) = -i\nu_1\phi_j^{\ell,0}(x), \qquad x \in \varsigma, \tag{5.4}$$

$$\phi_j^{\ell,1}(0,x') = \phi_j^{\ell,1}(1,x') \ \text{ and } \ \partial_1\phi_j^{\ell,1}(0,x') = \partial_1\phi_j^{\ell,1}(1,x') \qquad \text{for all } x', \tag{5.5}$$

where $\phi_j^{\ell,1}$ is understood as unknown and $\phi_j^{\ell,0}$ as given. By assumption, this problem does not have a solution. Moreover, the equation (3.15), with $p = 0$ and solution $\phi_j^{\ell,0}(x)$, is the homogeneous adjoint problem corresponding to (3.15) with $p = 1$, or (5.2)–(5.5). Hence, in view of the Fredholm alternative, the mentioned non-existence of solution means that

$$0 \neq a^0 = a^0(j,\ell) := 2i\big((\partial_1 + i\eta_j)\phi_j^{\ell,0}, \phi_j^{\ell,0}\big)_\varpi - i\big(\nu_1\phi_j^{\ell,0}, \phi_j^{\ell,0}\big)_\varsigma. \tag{5.6}$$

Applying the divergence formula to the vector field $(|\phi_j^{\ell,0}|^2, 0, 0)$ in ϖ yields the identity

$$(\partial_1\phi_j^{\ell,0}, \phi_j^{\ell,0})_\varpi = \int_\varpi \partial_1\big(|\phi_j^{\ell,0}|^2\big)dx - (\phi_j^{\ell,0}, \partial_1\phi_j^{\ell,0})_\varpi = \big(\nu_1\phi_j^{\ell,0}, \phi_j^{\ell,0}\big)_\varsigma - (\phi_j^{\ell,0}, \partial_1\phi_j^{\ell,0})_\varpi,$$

where also periodicity conditions of the type (5.5) were used for $\phi_j^{\ell,0}$. Hence, (5.6) turns into

$$\begin{aligned} &-2\eta_j\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 + i\big((\partial_1\phi_j^{\ell,0}, \phi_j^{\ell,0})_\varpi - (\phi_j^{\ell,0}, \partial_1\phi_j^{\ell,0})_\varpi\big) \\ &= -2\eta_j\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 - 2\mathrm{Im}\big((\partial_1\phi_j^{\ell,0}, \phi_j^{\ell,0})_\varpi\big), \end{aligned} \tag{5.7}$$

thus, a^0 is a nonzero real number. On the other hand, for the Floquet wave $U_j^{\ell,0} = e^{i\eta_j x_1}\phi_j^{\ell,0}(x)$ we calculate using (4.8) and replacing there ϖ_R by ϖ due to periodicity,

$$\begin{aligned} q(U_j^{\ell,0}, U_j^{\ell,0}) &= \int_\varpi \Big(\overline{\phi_j^{\ell,0}(x)}(\partial_1 + i\eta_j)\phi_j^{\ell,0} - \phi_j^{\ell,0}\overline{(\partial_1 + i\eta_j)\phi_j^{\ell,0}}\Big)dx \\ &= i2\eta_j\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 + (\partial_1\phi_j^{\ell,0}, \phi_j^{\ell,0})_\varpi - (\phi_j^{\ell,0}, \partial_1\phi_j^{\ell,0})_\varpi = -ia^0. \end{aligned} \tag{5.8}$$

Hence, the incoming and outgoing waves can be characterized as follows.

Proposition 5.1. *Let the number $a^0 = a^0(j,\ell) \in \mathbb{R}$ be determined by* (5.6) *and the function χ be as in* (5.1)*. The waves*

$$\frac{1}{\sqrt{|a^0(j,\ell)|}}\,\chi(x_1)U_j^{\ell,0}(x) =: u_n^-$$

with $a^0(j,\ell) > 0$ are outgoing, and the waves

$$\frac{1}{\sqrt{|a^0(j,\ell)|}}\,\chi(x_1)U_j^{\ell,0}(x) =: u_n^+$$

with $a^0(j,\ell) < 0$ are incoming.

We remark that if the simple eigenvalue η_j equals 0, then the corresponding eigenvector is real and the length of the associated Jordan chain is necessarily at least two: if it were one, the identity (5.8) would imply $a^0 = -\eta_j\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 = 0$, a contradiction with (5.6). Thus, the forthcoming consideration of Section 5.2 becomes quite important.

Moreover, if the simple eigenvalue η_j is π, then $\psi(x) = e^{i\pi x_1}\phi(x)$ is a real function (it is harmonic and satisfies the anti-periodicity conditions $\psi(0,x') = -\psi(1,x')$, $\partial_{x_1}\psi(0,x') = -\partial_{x_1}\psi(1,x')$). We get

$$\begin{aligned} -ia^0 \;&=\; i2\pi\|e^{-i\pi x_1}\psi; L^2(\varpi)\|^2 + \big(\partial_1 e^{-i\pi x_2}\psi, e^{-i\pi x_1}\psi\big)_\varpi \\ &\quad - \big(e^{-i\pi x_2}\psi, \partial_1 e^{-i\pi x_1}\psi\big)_\varpi = (\partial_1\psi,\psi)_\varpi - (\psi,\partial_1\psi)_\varpi = 0, \end{aligned} \tag{5.9}$$

which leads to the same conclusion as the above case $\eta_j = 0$.

5.2. **Case with Jordan chains of length two.** Next we consider the case of $j \in \{1,\dots,N\}$ such that $d_j = 1 = \ell$ and $\aleph_j^\ell = 2$, i.e., we have the Jordan chain $\{\phi_j^{\ell,0}, \phi_j^{\ell,1}\}$. Accordingly, the equation

$$\mathfrak{A}(\eta_j)\phi_j^{\ell,2} = -\frac{d\mathfrak{A}}{d\eta}(\phi_j^{\ell,1}) - \frac{1}{2}\frac{d^2\mathfrak{A}}{d^2\eta}(\phi_j^{\ell,0})$$

does not have a solution. The corresponding differential problem reads as

$$\begin{aligned} -\big((\partial_1 + i\eta_j)^2 + \Delta_{x'}\big)\phi_j^{\ell,2}(x) &= 2i(\partial_1 + i\eta_j)\phi_j^{\ell,1}(x) - \phi_j^{\ell,0}(x), && x \in \varpi, \\ \partial_2\phi_j^{\ell,2}(x) &= \Lambda\phi_j^{\ell,2}, && x \in \gamma, \\ (\partial_\nu + i\eta_j\nu_1)\phi_j^{\ell,2}(x) &= -i\nu_1\phi_j^{\ell,1}(x), && x \in \varsigma, \end{aligned}$$

supplemented with periodicity conditions similar to (5.5). By the nonexistence of a solution and the Fredholm alternative we again obtain that the number

$$\begin{aligned} a^1 = a^1(j,\ell) \;:=\; &-\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 \\ &+ 2i\big((\partial_1 + i\eta_j)\phi_j^{\ell,1}, \phi_j^{\ell,0}\big)_\varpi - i(\nu_1\phi_j^{\ell,1}, \phi_j^{\ell,0})_\varsigma \end{aligned} \tag{5.10}$$

is nonzero. On the other hand (5.2)–(5.5) has a solution $\phi_j^{\ell,1}$, which means, cf. (5.7), that

$$2\big((\partial_1 + i\eta_j)\phi_j^{\ell,0}, \phi_j^{\ell,0}\big)_\varpi - \big(\nu_1\phi_j^{\ell,0}, \phi_j^{\ell,0}\big)_\varsigma = 0. \tag{5.11}$$

(Below, we will need the fact that the associated vector $\phi_j^{\ell,1}$ is defined only up to a summand,

$$\phi_j^{\ell,1} = \widetilde{\phi}_j^{\ell,1} + c\phi_j^{\ell,0}, \tag{5.12}$$

where $\widetilde{\phi}_j^{\ell,1}$ is a particular solution of (5.2)–(5.5) and $c \in \mathbb{R}$ is an arbitrary constant. The value of a^1 does not depend on c because of (5.11).) Again, the number a^1 is real, since (5.10) and (5.2) imply

$$\begin{aligned}
a^1 &= -\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 + 2i\big((\partial_1 + i\eta_j)\phi_j^{\ell,1}, \phi_j^{\ell,0}\big)_\varpi - i(\nu_1\phi_j^{\ell,1}, \phi_j^{\ell,0}\big)_\varsigma \\
&= -\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 - 2i\big(\phi_j^{\ell,1}, (\partial_1 + i\eta_j)\phi_j^{\ell,0}\big)_\varpi + i(\phi_j^{\ell,1}, \nu_1\phi_j^{\ell,0}\big)_\varsigma \\
&= -\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 + \big(\phi_j^{\ell,1}, -((\partial_1 + i\eta_j)^2 + \Delta_{x'})\phi_j^{\ell,1}\big)_\varpi + (\phi_j^{\ell,1}, (\partial_\nu + i\eta_j\nu_1)\phi_j^{\ell,1}\big)_\varsigma \\
&= -\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 - \int\limits_\varpi \partial_1\big(\phi_j^{\ell,1}\overline{(\partial_1 + i\eta_j)\phi_j^{\ell,1}}\big)dx + \int\limits_\varpi (\partial_1\phi_j^{\ell,1})\overline{(\partial_1 + i\eta_j)\phi_j^{\ell,1}}dx \\
&\quad + \int\limits_\varpi \phi_j^{\ell,1} i\eta_j\overline{(\partial_1 + i\eta_j)\phi_j^{\ell,1}}dx - \big(\phi_j^{\ell,1}, \Delta_{x'}\phi_j^{\ell,1}\big)_\varpi + \big(\phi_j^{\ell,1}, (\partial_\nu + i\eta_j\nu_1)\phi_j^{\ell,1}\big)_\varsigma \\
&= -\|\phi_j^{\ell,0}; L^2(\varpi)\|^2 + \|(\partial_1 + i\eta)\phi_j^{\ell,1}; L^2(\varpi)\|^2 + \|\nabla_{x'}\phi_j^{\ell,1}; L^2(\varpi)\|^2.
\end{aligned}$$

Here, the last line was reached by (5.5) and an application of the divergence formula to the vector field $(\phi_j^{\ell,1}\overline{(\partial_1 + i\eta_j)\phi_j^{\ell,1}}, 0, 0)$,

$$-\int\limits_\varpi \partial_1\big(\phi_j^{\ell,1}\overline{(\partial_1 + i\eta_j)\phi_j^{\ell,1}}\big)dx + (\phi_j^{\ell,1}, (\nu_1\partial_1 + i\eta_j\nu_1)\phi_j^{\ell,1}\big)_\varsigma = 0,$$

and the Green formula

$$\big(\phi_j^{\ell,1}, \Delta_{x'}\phi_j^{\ell,1}\big)_\varpi + \|\nabla_{x'}\phi_j^{\ell,1}; L^2(\varpi)\|^2 = \big(\phi_j^{\ell,1}, (\nu_2\partial_2 + \nu_3\partial_3)\phi_j^{\ell,1}\big)_\varsigma.$$

In addition to $U_j^{\ell,0} = e^{i\eta_j x_1}\phi_j^{\ell,0}$ (cf. above) we now consider the Floquet wave $U_j^{\ell,1} = e^{i\eta_j x_1}\big(ix_1\phi_j^{\ell,0} + \phi_j^{\ell,1}\big)$. We remark that this solves the homogeneous problem (3.1), since

$$\begin{aligned}
&\Delta_x\big(e^{i\eta_j x_1}(ix_1\phi_j^{\ell,0} + \phi_j^{\ell,1})\big) = e^{i\eta_j x_1}\big((\partial_1 + i\eta_j)^2 + \Delta_{x'}\big)(ix_1\phi_j^{\ell,0} + \phi_j^{\ell,1}) \\
&= e^{i\eta_j x_1}\Big(ix_1\big((\partial_1 + i\eta_j)^2 + \Delta_{x'}\big)\phi_j^{\ell,0}\big) \\
&+ \big((\partial_1 + i\eta_j)^2 + \Delta_{x'}\big)\phi_j^{\ell,1} + 2i(\partial_1 + i\eta_j)\phi_j^{\ell,0}\Big),
\end{aligned}$$

which is null due to (5.2) and the fact that $\phi_j^{\ell,0}$ satisfies the homogeneous equation (3.6). Moreover,

$$\begin{aligned}
&\partial_\nu e^{i\eta_j x_1}(ix_1\phi_j^{\ell,0} + \phi_j^{\ell,1}) = e^{i\eta_j x_1}(\partial_\nu + i\eta_j\nu_1)(ix_1\phi_j^{\ell,0} + \phi_j^{\ell,1}) \\
&= e^{i\eta_j x_1}\big(ix_1(\partial_\nu + i\eta_j\nu_1)\phi_j^{\ell,0} \\
&+ i\nu_1\phi_j^{\ell,0} + (\partial_\nu + i\eta_j\nu_1)\phi_j^{\ell,1},
\end{aligned}$$

which also vanishes in view of (3.7) and (5.4).

Proposition 5.2. *Let $a^1 = a^1(j,\ell)$ be defined by (5.10) and χ as in (5.1). If the indices j and ℓ are such that $a^1(j,\ell) > 0$, then the wave packet*

$$\frac{\chi(x_1)}{\sqrt{2|a^1(j,\ell)|}}(U_j^{\ell,0} - U_j^{\ell,1}) =: u_n^+ \quad \textit{(respectively, } \frac{\chi(x_1)}{\sqrt{2|a|^1(j,\ell)}}(U_j^{\ell,0} + U_j^{\ell,1}) =: u_n^- \text{)}$$

is incoming (resp. outgoing). If $a^1(j,\ell) < 0$ holds, then the wave packet

$$\frac{\chi(x_1)}{\sqrt{2|a|^1(j,\ell)}}(U_j^{\ell,0} + U_j^{\ell,1}) =: u_n^+ \quad \textit{(respectively, } \frac{\chi(x_1)}{\sqrt{2|a|^1(j,\ell)}}(U_j^{\ell,0} - U_j^{\ell,1}) =: u_n^- \text{)}$$

is incoming (resp. outgoing).

Proof. We calculate the q-form of some Floquet waves. As remarked above, in the present case the problem (5.2)–(5.5) has a solution, which means that the expression $a^0(j,\ell)$ vanishes, see (5.7). From (5.8) we readily obtain

$$q(U_j^{\ell,0}, U_j^{\ell,0}) = 0.$$

Concerning $U_j^{\ell,1}$, we write

$$\begin{aligned}
q(U_j^{\ell,1}, U_j^{\ell,1}) &= \int\limits_{\omega(R)} \Big(\overline{(ix_1\phi_j^{\ell,0} + \phi_j^1)}(\partial_1 + i\eta_j)(ix_1\phi_j^{\ell,0} + \phi_j^{\ell,1}) \\
&\quad - (ix_1\phi_j^{\ell,0} + \phi_j^{\ell,1})\overline{(\partial_1 + i\eta_j)(ix_1\phi_j^{\ell,0} + \phi_j^{\ell,1})}\Big) dx' \\
&= R^2 \int\limits_{\omega(R)} \Big(\overline{i\phi_j^{\ell,0}}(\partial_1 + i\eta_j)(i\phi_j^{\ell,0} - i\phi_j^{\ell,0}\overline{(\partial_1 + i\eta_j)i\phi_j^{\ell,0}}\Big) dx' \\
&\quad + R \int\limits_{\omega(R)} \Big(\overline{i\phi_j^{\ell,0}}\big((\partial_1 + i\eta_j)\phi_j^{\ell,1} + i\phi_j^{\ell,0}\big) + \overline{\phi_j^{\ell,1}}(\partial_1 + i\eta_j)i\phi_j^{\ell,0} \\
&\quad - i\phi_j^{\ell,0}\overline{\big((\partial_1 + i\eta_j)\phi_j^{\ell,1} + i\phi_j^{\ell,0}\big)} - \phi_j^{\ell,1}\overline{(\partial_1 + i\eta_j)i\phi_j^{\ell,0}}\Big) dx' \\
&\quad + \int\limits_{\omega(R)} \Big(\overline{\phi_j^{\ell,1}}\big((\partial_1 + i\eta_j)\phi_j^{\ell,1} + i\phi_j^{\ell,0}\big) - \phi_j^{\ell,1}\big(\overline{(\partial_1 + i\eta_j)\phi_j^{\ell,1} + i\phi_j^{\ell,0}}\big)\Big) dx'.
\end{aligned}$$

The terms with factors R^2 and R must vanish since $q(U_j^{\ell,1}, U_j^{\ell,1})$ is independent of R. An integration over $(N, N+1) \ni R$ yields

$$\begin{aligned}
q(U_j^{\ell,1}, U_j^{\ell,1}) &= \int\limits_{\varpi} \Big(\overline{\phi_j^{\ell,1}}\big((\partial_1 + i\eta_j)\phi_j^{\ell,1} + i\phi_j^{\ell,0}\big) \\
&\quad - \phi_j^{\ell,1}\big(\overline{(\partial_1 + i\eta_j)\phi_j^{\ell,1} + i\phi_j^{\ell,0}}\big)\Big) dx
\end{aligned} \tag{5.13}$$

Let us write $\phi_j^{\ell,1} = \widetilde{\phi}_j^{\ell,1} + c\phi_j^{\ell,0}$ as in (5.12). We now want to fix c so as to make (5.13) null. Indeed,

$$q(U_j^{\ell,1}, U_j^{\ell,1}) = \big((\partial_1 + i\eta)\phi_j^{\ell,1}, \phi_j^{\ell,1}\big)_\varpi - \big(\phi_j^{\ell,1}, (\partial_1 + i\eta)\phi_j^{\ell,1}\big)_\varpi$$

$$
\begin{aligned}
&+ i(\phi_j^{\ell,0},\phi_j^{\ell,1})_\varpi + i(\phi_j^{\ell,1},\phi_j^{\ell,0})_\varpi \\
&= |c|^2\Big(\big((\partial_1+i\eta)\phi_j^{\ell,0},\phi_j^{\ell,0}\big)_\varpi - \big(\phi_j^{\ell,0},(\partial_1+i\eta)\phi_j^{\ell,0}\big)_\varpi\Big) \\
&+ c\Big(\big((\partial_1+i\eta)\phi_j^{\ell,0},\widetilde{\phi}_j^{\ell,1}\big)_\varpi - \big(\phi_j^{\ell,0},(\partial_1+i\eta)\widetilde{\phi}_j^{\ell,1}\big)_\varpi + i(\phi_j^{\ell,0},\phi_j^{\ell,0})_\varpi\Big) \\
&+ \overline{c}\Big(\big((\partial_1+i\eta)\widetilde{\phi}_j^{\ell,1},\phi_j^{\ell,0}\big)_\varpi - \big(\widetilde{\phi}_j^{\ell,1},(\partial_1+i\eta)\phi_j^{\ell,0}\big)_\varpi + i(\phi_j^{\ell,0},\phi_j^{\ell,0})_\varpi\Big) \\
&+ \big((\partial_1+i\eta)\widetilde{\phi}_j^{\ell,1},\widetilde{\phi}_j^{\ell,1}\big)_\varpi - \big(\widetilde{\phi}_j^{\ell,1},(\partial_1+i\eta)\widetilde{\phi}_j^{\ell,1}\big)_\varpi. \qquad (5.14)
\end{aligned}
$$

Here, in the coefficient of $|c|^2$, we again apply the divergence formula (as after (5.6)) to the second term. From (5.11) we deduce that this coefficient is null. By a similar calculation and (5.10), the coefficient of $\overline{c}$ equals $-ia^1$, which in particular is purely imaginary, since a^1 is real. This implies that the coefficient of c also equals $-ia^1$, since

$$
\begin{aligned}
&\big((\partial_1+i\eta)\widetilde{\phi}_j^{\ell,1},\phi_j^{\ell,0}\big)_\varpi - \big(\widetilde{\phi}_j^{\ell,1},(\partial_1+i\eta)\phi_j^{\ell,0}\big)_\varpi \\
&= -\overline{\big((\partial_1+i\eta)\widetilde{\phi}_j^{\ell,1},\phi_j^{\ell,0}\big)_\varpi} + \overline{\big(\widetilde{\phi}_j^{\ell,1},(\partial_1+i\eta)\phi_j^{\ell,0}\big)_\varpi} \\
&= -\big(\phi_j^{\ell,0},(\partial_1+i\eta)\widetilde{\phi}_j^{\ell,1}\big)_\varpi + (\partial_1+i\eta)\phi_j^{\ell,0},\widetilde{\phi}_j^{\ell,1}\big)_\varpi
\end{aligned}
$$

where the fact that the first line is purely imaginary was used. Also the last line of (5.14) is purely imaginary, denote it by iT with $T\in\mathbb{R}$. Thus, $q(U_j^{\ell,1},U_j^{\ell,1}) = c(-ia^1)+\overline{c}(-ia^1)+iT$, and choosing $c = T/(2a^1)$ yields

$$q(U_j^{\ell,1},U_j^{\ell,1}) = 0.$$

Finally,

$$
\begin{aligned}
q(U_j^{\ell,0},U_j^{\ell,1}) &= \int\limits_\varpi \Big(\overline{(ix_1\phi_j^{\ell,0}+\phi_j^{\ell,1})}(\partial_1+i\eta_j)\phi_j^{\ell,0} - \phi_j^{\ell,0}\overline{(\partial_1+i\eta_j)(ix_1\phi_j^{\ell,0}+\phi_j^{\ell,1})}\Big)dx \\
&= \int\limits_\varpi \Big(\overline{\phi_j^{\ell,1}}(\partial_1+i\eta_j)\phi_j^{\ell,0} - \phi_j^{\ell,0}\overline{(\partial_1+i\eta_j)\phi_j^{\ell,1})} + \phi_j^{\ell,0}\overline{i\phi_j^{\ell,0}}\Big)dx \\
&\quad + \int\limits_\varpi (-ix_1)\Big(\overline{\phi_j^{\ell,0}}(\partial_1+i\eta_j)\phi_j^{\ell,0} - \phi_j^{\ell,0}\overline{(\partial_1+i\eta_j)\phi_j^{\ell,0})}\Big)dx.
\end{aligned}
$$

The integral on the last line vanishes, since on all cross-sections

$$0 = q(U_j^{\ell,0},U_j^{\ell,0}) = \int\limits_{\omega(R)} \big(\overline{U_j^{\ell,0}}\partial_1 U_j^{\ell,0} - U_j^{\ell,0}\partial_1\overline{U_j^{\ell,0}}\big)dx'.$$

We get

$$
\begin{aligned}
q(U_j^{\ell,0},U_j^{\ell,1}) &= \big((\partial_1+i\eta_j)\phi_j^{\ell,0},\phi_j^{\ell,1}\big)_\varpi \\
&\quad - \big(\phi_j^{\ell,0},(\partial_1+i\eta_j)\phi_j^{\ell,1}\big)_\varpi + i\|\phi_j^{\ell,0};L^2(\varpi)\|^2 = -ia,
\end{aligned}
$$

and also $q(U_j^{\ell,1},U_j^{\ell,0}) = -\overline{q(U_j^{\ell,0},U_j^{\ell,1})} = -ia$. $\square$

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