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Two-sided estimates for eigenfrequencies in the John problem on freely floating body.

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The two-dimensional problem on oblique incident waves and a freely floating cylinder is reduced to the study of the spectrum of a suitable self-adjoint operator in Hilbert space. Using tools from spectral measure theory we estimate the difference between eigenfrequencies of the original problem and a problem on an inert body, which does not react to the buoyancy forces. We give the localization of eigenfrequencies of the freely floating body, and in addition derive a sufficient condition for the existence of the point spectrum in the corresponding boundary value problem.

Keywords: surface waves, trapped modes, freely floating body, comparison principle.

1. Goals and methods of investigation. F. John [1] formulated in 1949 the mathematical problem on the interaction of surface waves with a freely floating body in a layer of ideal liquid. In addition to the velocity potential φ this problem involves an unknown column a with components $a_1, \ldots, a_6$ describing (small) rigid motions of the body, namely three translations and three rotations. The model consists of a boundary value problem for the function φ and a system of linear algebraic equations for the column a; it includes the spectral parameter ω^2 (the square of the oscillation frequency), and it has "cross-terms" with the factor ω. In this way the spectral problem gives rise to a quadratic pencil

$$\omega \quad \mapsto \quad \mathfrak{A}(\omega) = \mathfrak{A}_0 + \omega \mathfrak{A}_1 + \omega^2 \mathfrak{A}_2, \tag{1.1}$$

which makes its investigation rather difficult, both theoretically and numerically. Maybe, this was the very reason why the general problem was forgotten for many years, while publications mainly concerned fixed (non-floating, $a = 0$) bodies. Many interesting and important results have been obtained for these (see the reviews [2, 3], the monographs [4, 5] and other publications).

During the last decade the John problem has arisen renewed interest. However, still in the papers [6, 7, 8, 9, 10] and in many others, formal calculations are

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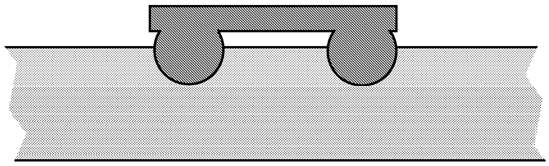

Figure 1.1: Surface-piercing freely floating body.

performed and/or a priori assumptions on the motion of the body are accepted: mainly the heaving "up'nd-down" motion is allowed. On the other hand, in the paper [11] the author observed that some of the examples [12], [4, § 4.2.2.4] of surface waves trapped by two identical surface-piercing bodies (cylinders, due to the two-dimensional formulation), also fit to the freely floating body (fig. 1), because the hydrodynamical forces over the wetted surfaces have null principal vector and torque.

A new approach to the problem on freely floating bodies was worked out in the paper [13] (see also Section 3 below). In particular the authors developed a sufficient condition for the existence of trapped modes, or localized solutions, and gave examples of concrete structures supporting at least four linear independent trapped modes in a symmetric channel. In the paper [15] the approaches of [14] and [13] were combined and examples with any given number of trapped modes were constructed; these involve surface-piercing and submerged freely floating bodies in symmetric channels. The method of [13] is based on the notion of trace operator [14], and it consists of the reduction of the pencil (1.1) to the equation

$$\mathcal{A}\mathcal{Y} = \alpha\mathcal{Y} \quad \text{in} \quad \mathcal{H}, \tag{1.2}$$

for a self-adjoint bounded operator $\mathcal{A}$ in a specially constructed Hilbert space $\mathcal{H}$. It is applied in Section 3 of our paper as well, namely, we compare the spectra of the problem on a freely floating body and the problems on fixed obstacles, as described in Sections 2 and 4.

The structure of the spectra of operator pencils and self-adjoint operators may differ crucially, and this causes difficulties for the study of the spectrum in case of a freely floating body. Even if the operators $\mathfrak{A}_q$ are self-adjoint, the pencil (1.1) may have, first, complex eigenvalues with non-zero imaginary parts, and, second, associated vectors, which together with eigenvectors form non-trivial Jordan chains. One can find such a complication of spectral structures even if the matrices $\mathfrak{A}_q$ in (1.1) are real and symmetric.

The eigenvalue $\omega = 0$ in the problem under consideration is certainly not algebraically simple, since it has Jordan chains of length two (see Remark 2). However, the spectrum of the two-dimensional John problem is real and algebraically simple outside the point $\omega = 0$. This is a concomitant result of the reduction procedure [13], applied in Section 3. In a general situation the anomalies of the spectrum, which contradict the physical nature of the problem, have not been refuted yet. We emphasize that the associated vectors corresponding to $\omega = 0$ do not lead to any irretrievable consequences (see Remark 2 again).

Another nuisance caused by the appearance of the quadratic pencil is the absence of a tool for comparing the spectra of pencils (in contrast, cf. [16, § 10.2], for a partial ordering of semi-bounded self-adjoint operators in Hibert space). This fact prompted the authors to write the present paper, extending the paper [17] where an *incomplete* comparison principle was obtained for the spectra of freely floating and fixed bodies.

In the following, we deal with a freely floating cylinder in an infinite layer of liquid. Since a comparison principle can exist only for discrete spectrum, we shall investigate oblique waves, for which the two-dimensional problem gets an interval free of continuous spectrum and therefore open for the discrete spectrum.

In the case of a two-dimensional problem on submerged body, the traditional comparison principle [18] establishes that increasing the body leads to diminishing eigenvalues and, hence, they are kept inside the discrete spectrum. This principle was adapted in [19] to surface-piercing bodies. The papers [14] and [20, 21] contain a new method based on trace operators and elementary theory of self-adjoint operators in Hilbert space. This gives simple proofs to all facts mentioned above and also provides numerous sufficient conditions and comparison principles for miscellaneous problems on water-waves, however, only in case of fixed bodies and obstacles. In Section 4 we formulate Propositions 4 and 5 about the auxiliary problem (4.3)–(4.6) derived by this method. However, to obtain information on the discrete spectrum of the problem (2.3)–(2.7) we use a very different idea. Namely, the reduction [13] of the pencil (1.1) to a self-adjoint operator $\mathcal{A}$ gives an opportunity to apply the spectral measure theory (see, e.g., [16, Capters 5 and 6]). This is made in Section 5 where using a simple consideration (cf., relations (5.7)–(5.9)) and thus proving the two-sided estimates for eigenvalues in the John problem. Additional calculations and the final Theorem 6 are presented in Section 6 where Corrolary 7 is formulated which gives a sufficient condition for trapping a surface wave.

Concerning freely floating bodies, the two-dimensional problem for oblique waves hitting an infinite cylinder surely has the simplest formulation, and this case is thus chosen to demonstrate our new approach for localizing the position of an eigenfrequency. However, a physical interpretation of the problem is restricted due to the necessity to assume that the cylinder has zero bending resistance. Our method works also for the John problem in other formulations but requires much more complicated calculations.

It is surprising that the spectrum of the problem (4.3)–(4.5), (4.7) on the fixed body does not become a "good approximation" for the spectrum of the problem (2.3)–(2.7) on the moving body. In Section 4, an integro-differential boundary condition (4.6) on the wetted surface appears in a natural way. This was introduced heuristically in the paper [17], and it corresponds to a body possessing inertia but unaffected by buoyancy forces. It is just the spectrum of the new problem (4.3)–(4.6) on the obstacle figures in Theorem 6, the main result in the paper.

2. The mathematical formulation of the problem. Let $\Pi = \mathbb{R} \times (-d, 0) \ni (y, z)$ be a strip of width $d > 0$ and let $\Theta \subset \mathbb{R}^2$ be a domain with Lipschitz boundary $\partial\Theta$ and compact closure $\overline{\Theta} = \Theta \cup \partial\Theta$. We assume that the set Θ_- defined by the formula

$$\Theta_\pm = \{(y, z) \in \Theta : \pm z > 0\}, \tag{2.1}$$

is non-empty and $\overline{\Theta}$ intersects the y-axis along the union θ of a finite family of closed segments (in fig.1 there are two). We consider the cylinder $\mathbb{R} \times \Theta$ with cross-section Θ floating freely in the layer $\mathbb{R}^2 \times (-d, 0)$ of incompressible and inviscid, i.e., ideal liquid, water for example. Moreover, we neglect the surface tension and the bend resistance of the cylinder.

The velocity potential, more precisely, the last multiplier in its representation

$$\boldsymbol{\varphi}(x, y, z, t) = \mathrm{Re}\left(e^{-i\omega t + ikx}\, \varphi(y, z)\right), \tag{2.2}$$

satisfies the Helmholtz equation

$$-\Delta\varphi(y, z) + k^2\varphi(y, z) = 0, \ (y, z) \in \Omega := \Pi \setminus \overline{\Theta}, \tag{2.3}$$

and the boundary conditions: the linearized kinematic condition

$$-\partial_z\varphi(y, 0) = g^{-1}\omega^2\varphi(y, 0), \ (y, z) \in \Gamma, \tag{2.4}$$

and the homogeneous Neumann condition, excluding the flow through the bottom Υ,

$$\partial_\nu\varphi(y, -d) = 0, \ (y, z) \in \Upsilon = \mathbb{R} \times \{-d\}. \tag{2.5}$$

Here, Δ is the Laplace operator, $\omega > 0$ the frequency of oscillations, $k > 0$ the wave number in the x-direction, $\Gamma = \{(y, z) : z = 0, y \notin \overline{\theta}\}$ the free surface of the liquid and $g > 0$ the acceleration due to gravity. The inequality $k > 0$ forbids the propagation of waves in the direction perpendicular to the cylinder axis.

Formula (2.2) means that we deal with time harmonic waves hitting the cylinder at the angle $\gamma \neq \pi/2$. For small amplitude oscillations of the body, the boundary condition on its wetted surface $\Sigma = \{(y, z) \in \partial\Theta : z < 0\}$ takes the form

$$\partial_\nu\varphi(y, z) = -i\omega\nu(y, z)^\top D(y - y^\bullet, z - z^\bullet)a, \ (y, z) \in \Sigma, \tag{2.6}$$

while the column $a = (a_1, a_2, a_3)^\top$ of rigid motions of the body Θ satisfies the system of three linear algebraic equations

$$gKa - i\omega S\varphi = \omega^2 Ma. \tag{2.7}$$

Let us explain the notation. First, $\nu = (\nu_1, \nu_2)^\top$ is the unit outward normal vector and $\partial_\nu = \nu^\top\nabla$ is the directional derivative along ν defined almost everywhere on

$\partial\Theta$, while $\top$ stands for transposition, $\nabla = (\partial_y, \partial_z)^\top$ is the gradient, and $\partial_y = \partial/\partial y$, $\partial_z = \partial/\partial z$. Second, D is a linear function matrix of size 2×3 determining rigid motions,

$$D(y,z) = \begin{pmatrix} 1 & 0 & -z \\ 0 & 1 & y \end{pmatrix}, \tag{2.8}$$

and $(y^\bullet, z^\bullet)$ is the mass centre,

$$(y^\bullet, z^\bullet) = m^{-1} \int_\Theta (y,z)\rho(y,z)\, dydz, \quad m = \int_\Theta \rho(y,z)\, dydz, \tag{2.9}$$

$\rho > 0$ and m are the density and the total mass. The density of the liquid assumed to be equal to one. Third, the column a is extracted from the following representation of (small amplitude) oscillation motion of the cylinder $\mathbb{R} \times \Theta$:

$$D(y - y^\bullet, z - z^\bullet)\mathbf{a}(x,t) = \operatorname{Re}\left(e^{-i\omega t + ikx} D(y - y^\bullet, z - z^\bullet) a(t)\right). \tag{2.10}$$

In agreement with formula (2.2), the appearance of the factor $e^{-i\omega t+ikx}$ on the right-hand side of (2.10) means that points of the generator lines on the cylindrical surface are shifted non-uniformly (the body coils like a snake). For the physical interpretation of the problem (2.3)–(2.7) it is necessary that elastic properties of the cylinder do not impede bending deformations of the body. We emphasize that the problem is posed in the unchanging domain $\mathbb{R} \times \Omega$ corresponding to the equilibrium position of the body, since inessential changes of its shape can be ignored, thanks to the assumed smallness of amplitude oscillations.

The system (2.7) is obtained by inserting the expression $\mathbf{a}(x,t) = \operatorname{Re}\left(e^{-i\omega t+ikx} a(t)\right)$ from (2.10) into the system of ordinary differential equations

$$M\partial_t^2 \mathbf{a} = -S\partial_t \boldsymbol{\varphi} - gK\mathbf{a} \tag{2.11}$$

which comes from the conservation law for the linear and angular momenta. Let us describe the matrices in (2.7). The inertia matrix

$$M = \int_\Theta D(y - y^\bullet, z - z^\bullet)^\top D(y - y^\bullet, z - z^\bullet)\rho(y,z)\, dydz \tag{2.12}$$

is a Gram matrix, symmetric and positive definite. The integral operator S with values in the space of columns of height three is given by the formula

$$S\varphi = \int_\Sigma D(y - y^\bullet, z - z^\bullet)^\top \nu(y,z)\varphi(y,z)\, ds, \tag{2.13}$$

where ds is the arc length element. According to the paper [1] (see, e.g., the book

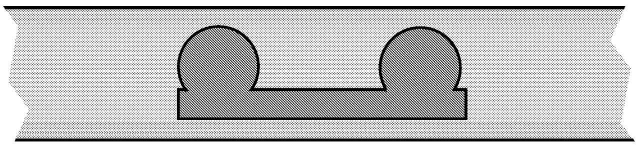

Figure 2.2: Submerged freely floating body.

[22]) the (3×3)-matrix K in the systems (2.7) and (2.11) is of the form

$$K = K^\theta + K^\Theta, \quad K^\theta = \int\limits_\theta d(y - y^\bullet)^\top d(y - y^\bullet)\, dy,$$
$$K^\Theta = \operatorname{diag}\{0, 0, I_z^\Theta\}, \quad I_z^\Theta = \int\limits_{\Theta_-} (z - z^\bullet)\, dydz, \tag{2.14}$$

while $d(y)$ is the lower row in the rigid motion matrix (2.8). The matrix K is related to the buoyancy of the body and its structure

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bullet & \bullet \\ 0 & \bullet & \bullet \end{pmatrix} \tag{2.15}$$

with nulls in the first column and row shows that a horizontal motion a_1 of the body does not influence the buoyancy forces. We emphasize that while going over to the two-dimensional formulation we take into account the same effect for translations along the x-axis. Rotations around the z- and y-axes are forbidden by the problem formulation for waves hitting the cylinder obliquely. As a result, the column $a = (a_1, \ldots, a_6)^\top$ of rigid motions in the three-dimensional problem has been reduced to the column $a = (a_1, a_2, a_3)^\top$ in the two-dimensional problem.

A freely floating object must be in the state of a stable equilibrium. According to the Archimedean law we have

$$m = v, \quad \text{where} \quad v = \int\limits_{\Theta_-} dydz \tag{2.16}$$

(the total mass m of the body Θ equals the volume v of the displaced liquid; we recall that the liquid density is one). The integration in the last integral in (2.14) and in the integral (2.16) is performed along the submerged part Θ_- of the body (see definition (2.1)). The classical Euler conditions [23] of the stability of a floating object (see also [1], [22] and others) require first of all that the buoyancy centre

$$(y^\ominus, z^\ominus) = v^{-1} \int\limits_{\Theta_-} (y, z)\, dydz, \tag{2.17}$$

stays on the same vertical line as the mass centre $(y^\bullet, z^\bullet)$ and strictly above it in case of the submerged body $\theta = \varnothing$. The latter means that the matrix K is positive but its rank κ equals one. If the body is surface-piercing and $\theta \neq \varnothing$, then the Euler stability condition becomes much more complicated; the matrix K must still be positive but of rank two (see, e.g., the paper [1] and the book [22]). We write these restrictions as

$$K \geq 0,\ \kappa = \operatorname{rank} K = 1 \text{ (assuming } z^\bullet < z^\ominega) \text{ in case } \theta = \varnothing,$$
$$K \geq 0,\ \kappa = \operatorname{rank} K = 2 \text{ for } \theta \neq \varnothing. \tag{2.18}$$

In what follows we just need the information (2.18) on the matrix K. A more detailed description of the second line of (2.18) can be found in [1, 22] and others.

3. Reduction of the problem on freely floating body to self-adjoint operator. We multiply the equation (2.3) with a smooth, compactly supported test function ψ and integrate by parts in the domain Ω taking into account the boundary conditions (2.4)–(2.6). We add to the result the system (2.7), multiplied scalarly by a test column $b \in \mathbb{C}^3$, using the evident relation

$$\int_\Sigma \overline{\psi(y,z)}\nu(y,z)^\top D(y - y^\bullet, z - z^\bullet)a\, ds = (a, S\psi)_{\mathbb{C}}, \tag{3.1}$$

where $(\ ,\ )_{\mathbb{C}}$ is the scalar product in the complex space $\mathbb{C}^3$ and S is the operator (2.13), we derive the integral identity, which is the variational formulation [13] of the problem (2.3)–(2.7):

$$(\nabla\varphi, \nabla\psi)_\Omega + k^2(\varphi,\psi)_\Omega + g(Ka, b)_{\mathbb{C}} + i\omega\left((a, S\psi)_{\mathbb{C}} - (S\varphi, b)_{\mathbb{C}}\right) =$$
$$= \omega^2\left((\varphi,\psi)_\Gamma + (Ma, b)_{\mathbb{C}}\right), \quad (\psi, b) \in H^1(\Omega) \times \mathbb{C}^3. \tag{3.2}$$

Here, $(\ ,\ )_\Omega$ and $(\ ,\ )_\Gamma$ are the natural scalar products in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively.

We endow the Sobolev space $H^1(\Omega)$ with the scalar product

$$\langle\varphi,\psi\rangle = (\nabla\varphi,\nabla\psi)_\Omega + k^2(\varphi,\psi)_\Omega \tag{3.3}$$

and introduce the trace operator T by the formula

$$\langle T\varphi,\psi\rangle = (\varphi,\psi)_\Gamma, \quad \varphi,\psi \in H^1(\Omega); \tag{3.4}$$

obviously T is positive, continuous and symmetric, therefore, self-adjoint. As known (see, e.g., [21]), the continuous spectrum of the trace operator covers the half-open interval $(0, \beta_\dagger]$ where

$$\beta_\dagger = \lambda_\dagger^{-1}, \quad \lambda_\dagger = k\,\frac{1 - e^{-2kd}}{1 + e^{-2kd}}. \tag{3.5}$$

The point $\beta = 0$ is an eigenvalue of infinite multiplicity, and the eigenspace consists of functions $\varphi \in H^1(\Omega)$ which vanish on the free liquid surface Γ. Thus, the segment $[0, \beta_\dagger]$ is the essential spectrum of the operator T.

By virtue of (3.3) and (3.4), the integral identity (3.2) converts into the abstract equation (system)

$$\begin{pmatrix} \mathbb{I} & 0 \\ 0 & gK \end{pmatrix} \begin{pmatrix} \varphi \\ a \end{pmatrix} + \omega \begin{pmatrix} 0 & iS^* \\ -iS & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ a \end{pmatrix} = \omega^2 \begin{pmatrix} g^{-1}T & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \varphi \\ a \end{pmatrix} \qquad (3.6)$$

in the Hilbert space

$$\mathcal{H} = H^1(\Omega) \times \mathbb{C}^3. \qquad (3.7)$$

Here, $\mathbb{I}$ is the identity operator in $H^1(\Omega)$, S^* stands for the adjoint of the operator

$$S: \, H^1(\Omega) \ \to \ \mathbb{C}^3, \qquad (3.8)$$

defined by the equation (2.13). Moreover, both S and S^* are compact because Σ is a union of finite arcs and therefore the embedding $H^1(\Omega) \subset L_2(\Sigma)$ is compact.

We omit in the system (3.6) the terms including the operators S and S^* so that the system turns into the combination of two equations

$$\begin{aligned} \varphi &= \lambda T \varphi, \\ Ka &= \lambda M a \end{aligned} \qquad (3.9)$$

with the new spectral parameter

$$\lambda = g^{-1}\omega^2. \qquad (3.10)$$

Since the first equation (3.9) takes the form $T\varphi = \beta\varphi$ after the change $\lambda \mapsto \beta = 1/\lambda$, the continuous λ-spectrum coincides with $[\lambda_\dagger, +\infty)$, where the cut-off $\lambda_\dagger > 0$ is given in (3.5). The second equation (3.9) is an algebraic system and, hence, its λ-spectrum is fully discrete. Thus, according to the relationship (3.10) of the spectral parameters λ and ω the problem (3.9) has the following continuous spectrum:

$$(-\infty, -\omega_\dagger] \cup [\omega_\dagger, +\infty) \qquad (3.11)$$

where

$$\omega_\dagger = \sqrt{g\lambda_\dagger} = \sqrt{gk\,\frac{1 - e^{-2kd}}{1 + e^{-2kd}}}. \qquad (3.12)$$

The system (3.6) differs from the combination (3.9) only by a compact "perturbation" and therefore the continuous ω-spectrum (3.11) remains the same for the operator form (3.6) of the problem (3.2).

Remark 1. The first equation (3.9) corresponds to the problem of interaction of surface waves with fixed obstacle Θ, and it is composed from the equation (2.3) and the boundary conditions (2.4), (2.5) together with

$$\partial_\nu \varphi(y,z) = 0,\ (y,z) \in \Sigma. \tag{3.13}$$

The cut-off $\lambda_\dagger$, (3.5), of the continuous spectrum in (2.3)–(2.5), (3.13) was computed for the first time in the paper [24] (see also [4] and others). Furthermore, singular Weyl sequences detecting the essential spectrum (see, e.g., [16, § 9.1]) are constructed in the paper [21] using the eigenfunction

$$\phi_\dagger(z) = e^{kz} + e^{-k(z+2d)} \tag{3.14}$$

of the following model problem

$$-\partial_z^2 \phi(z) + k^2 \phi(z) = 0,\ z \in (-d,0), \quad \partial_z \phi(0) = \lambda \phi(0),\ -\partial_z \phi(-d) = 0 \tag{3.15}$$

on the interval$(-d,0)$, i.e. the cross-section of the strip Π. ⊠

The system (3.6) may be interpreted as the quadratic pencil (1.1). The point $\omega = 0$ belongs to its spectrum and, due to the structure (2.14) of the matrix K, its eigenspace has dimension $3 - \kappa = 3 - \operatorname{rank} K$, and it is spanned by the columns

$$\begin{gathered} e^{(1)},\ e^{(2)} \text{ in case } \theta = \varnothing \text{ and } \operatorname{rank} K = 1, \\ e^{(1)} \text{ in case } \theta \neq \varnothing \text{ and } \operatorname{rank} K = 2 \end{gathered} \tag{3.16}$$

(cf. (2.18); here, $e^{(j)} = (\delta_{1,j}, \delta_{2,j}, \delta_{3,j})^\top \in \mathbb{R}^3$).

Remark 2. As mentioned in Section 1, the eigenvectors indicated in formula (3.16) have associated vectors, so that the eigenvalue $\omega = 0$ of the pencil (3.6) is not algebraically simple. For the submerged body, the eigenvectors $X^{j,0} = (0, e^{(j)}) \in \mathcal{H}$, $j = 1,2$, thus get the associated vectors $X^{j,1} = (\varphi^{(j)}, 0) \in \mathcal{H}$ which satisfy the equation

$$\mathfrak{A}(0) X^{j,1} = -\frac{d\mathfrak{A}}{d\omega}(0) X^{j,0}$$

and therefore involve solutions of the Neumann problem

$$\begin{gathered} -\Delta \varphi(y,z) + k^2 \varphi(y,z) = 0,\ (y,z) \in \Omega, \quad -\partial_\nu \varphi(y,z) = 0,\ (y,z) \in \Gamma \cup \Upsilon, \\ \partial_\nu \varphi(y,z) = i\nu(y,z)^\top D(y - y^\bullet, z - z^\bullet) e^{(j)},\ (y,z) \in \Sigma. \end{gathered}$$

In the case of the surface-piercing body the only retained Jordan chain is composed of the eigenvector $(0, e^{(1)})$ and the associated vector $(\varphi^{(1)}, 0)$. One can verify that in both cases there is no Jordan chain with length longer than two. This information is not used in the sequel. ⊠

Let $T^{1/2}$ be the positive square root of the positive self-adjoint operator T (see, e.g., [16, § 10.3]) and N is the inverse matrix for the matrix M, real symmetric and positive definite. We introduce the function and the column

$$\eta = \omega g^{-1/2} T^{1/2} \varphi, \quad f = \omega M a$$

and rewrite the quadratic pencil (3.6) as the linear pencil

$$\begin{pmatrix} \mathbb{I} & 0 & 0 & 0 \\ 0 & \mathbb{I} & 0 & 0 \\ 0 & 0 & gK & 0 \\ 0 & 0 & 0 & N \end{pmatrix} \begin{pmatrix} \varphi \\ \eta \\ a \\ f \end{pmatrix} = \omega \begin{pmatrix} 0 & g^{-1/2}T^{1/2} & -iS^* & 0 \\ g^{-1/2}T^{1/2} & 0 & 0 & 0 \\ iS & 0 & 0 & \mathbb{I} \\ 0 & 0 & \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \eta \\ a \\ f \end{pmatrix}. \tag{3.17}$$

Matric operators on the left- and right-hand sides of (3.17) appear to be self-adjoint in the Hilbert space

$$\mathcal{H}^2 = H^1(\Omega)^2 \times (\mathbb{C}^3)^2 \tag{3.18}$$

(cf. formula (3.7)). The information given above on the eigenvalue $\omega = 0$ of the problem (3.6) holds true for the problem (3.17) as well; one needs to make simple changes in the definition of the eigen- and associated vectors. In addition, the appearance of the eigenvalue $\omega = 0$ itself is caused by the degeneracy of the matrix K. We restrict the problem (3.17) to the subspace $\mathcal{H}^\natural = H^1(\Omega)^2 \times (\mathbb{C}^\kappa)^2$ of the space (3.18). To this end, we set

$$a = \begin{pmatrix} a_\circ \\ a_\natural \end{pmatrix}, \quad \begin{cases} a_\circ = (a_1, a_2)^\top, \ a_\natural = a_3 \text{ in case of } \kappa = \operatorname{rank} K = 1, \\ a_\circ = a_1, \ a_\natural = (a_2, a_3)^\top \text{ in case of } \qquad \kappa = 2 \end{cases} \tag{3.19}$$

(cf. formulas (3.16) and (2.18)) and split the matrix N as follows:

$$N = \begin{pmatrix} N_{\circ\circ} & N_{\circ\natural} \\ N_{\natural\circ} & N_{\natural\natural} \end{pmatrix}. \tag{3.20}$$

Here, $N_{\circ\circ}$ and $N_{\natural\natural}$ are blocks of size $(3-\kappa)\times(3-\kappa)$ and $\kappa \times \kappa$, respectively, while one of them is a scalar.

If $\omega \neq 0$ and $(\varphi, \eta, a, f)^\top$ is the corresponding eigenvector of the system (3.17), then its third line provides the equality

$$iS_\circ \varphi + f_\circ = 0 \tag{3.21}$$

where $S_\circ\varphi$, and further also $S_\natural\varphi$ are the fragments of the column (2.13), defined as in (3.19). By the splitting (3.20), the fourth line in the system reads componentwise as follows:

$$\begin{aligned} N_{\circ\circ} f_\circ + N_{\circ\natural} f_\natural &= \omega a_\circ, \\ N_{\natural\circ} f_\circ + N_{\natural\natural} f_\natural &= \omega a_\natural. \end{aligned} \tag{3.22}$$

Now using the first equality (3.22) yields

$$-i\omega S_\circ^* a_\circ = -i\omega S_\circ^*(N_{\circ\circ} f_\circ + N_{\circ\natural} f_\natural) = -i\omega S_\circ^* N_{\circ\circ} S_\circ \varphi - i\omega S_\circ^* N_{\circ\natural} f_\natural.$$

Hence, the first line of (3.17) converts into

$$\varphi + S_\circ^* N_{\circ\circ} S_\circ \varphi + i\omega S_\circ^* N_{\circ\natural} f_\natural = \omega \left(g^{-1/2} T^{1/2} \eta - i S_\natural^* f_\natural \right).$$

Taking (3.21) and (2.14) into account, we shorten the third line as follows:

$$g K_{\natural\natural} a_\natural = i\omega S_\natural \varphi + \omega f_\natural.$$

Finally the fourth line, thanks to relations (3.21) and (3.22), can be replaced by the single equation

$$-i N_{\natural\circ} S_\circ \varphi + N_{\natural\natural} f_\natural = \omega a_\natural.$$

Summing up above calculations, we have turned the system (3.17) into

$$\begin{pmatrix} \mathbb{I} + S_\circ^* N_{\circ\circ} S_\circ & 0 & 0 & i S_\circ^* N_{\circ\natural} \\ 0 & \mathbb{I} & 0 & 0 \\ 0 & 0 & g K_{\natural\natural} & 0 \\ -i N_{\natural\circ} S_\circ & 0 & 0 & N_{\natural\natural} \end{pmatrix} \begin{pmatrix} \varphi \\ \eta \\ a_\natural \\ f_\natural \end{pmatrix} = \omega \begin{pmatrix} 0 & g^{-1/2} T^{1/2} & -i S_\natural^* & 0 \\ g^{-1/2} T^{1/2} & 0 & 0 & 0 \\ i S_\natural & 0 & 0 & \mathbb{I} \\ 0 & 0 & \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \eta \\ a_\natural \\ f_\natural \end{pmatrix}. \tag{3.23}$$

The short form of the problem (3.23)

$$\mathcal{B}\mathcal{X} = \omega \mathcal{D}\mathcal{X} \tag{3.24}$$

involves the vector

$$\mathcal{X} = (\varphi, \eta, a_\natural, f_\natural)^\top \in \mathcal{H}^\natural$$

and two continuous operators $\mathcal{B}$ and $\mathcal{D}$ in the Hilbert space

$$\mathcal{H}^\sharp = H^1(\Omega)^2 \times (\mathbb{C}^\kappa)^2$$

(cf. definitions (3.18) and (3.19)). It is straightforward to verify the self-adjointness of both the operators. Furthermore, $\mathcal{B}$ is positive definite because

$$\begin{aligned} (\mathcal{B}\mathcal{X}, \mathcal{X})_{\mathcal{H}^\natural} &= \langle \varphi, \varphi \rangle + \langle S_\circ^* N_{\circ\circ} S_\circ \varphi, \varphi \rangle + i \langle S_\circ^* N_{\circ\natural} f_\natural, \varphi \rangle + \langle \eta, \eta \rangle + \\ &+ g(K_{\natural\natural} a_\natural, a_\natural)_{\mathbb{C}} - i(N_{\natural\circ} S_\circ \varphi, f_\natural)_{\mathbb{C}} + (N_{\natural\natural} f_\natural, f_\natural)_{\mathbb{C}} = \\ &= \langle \varphi, \varphi \rangle + \langle \eta, \eta \rangle + g(K_{\natural\natural} a_\natural, a_\natural)_{\mathbb{C}} + \left(N \begin{pmatrix} -i S_\circ \varphi \\ f_\natural \end{pmatrix}, \begin{pmatrix} -i S_\circ \varphi \\ f_\natural \end{pmatrix} \right)_{\mathbb{C}} \geq \\ &\geq C_{\mathcal{B}} \left(\|\varphi; H^1(\Omega)\|^2 + \|\eta; H^1(\Omega)\|^2 + \|a_\natural; \mathbb{C}\|^2 + \|f_\natural; \mathbb{C}\|^2 \right), \quad C_{\mathcal{B}} > 0. \end{aligned} \tag{3.25}$$

We have used here the positive definiteness of the matrices $K_{\natural\natural}$ and $N = M^{-1}$. Thus, introducing the notation

$$\mathcal{A} = \mathcal{B}^{-1/2}\mathcal{D}\mathcal{B}^{-1/2}, \quad \mathcal{Y} = \mathcal{B}^{1/2}\mathcal{X}, \quad \alpha = \frac{1}{\omega}, \tag{3.26}$$

transforms the equation (3.24) into the equation (1.2) with a bounded self-adjoint operator $\mathcal{A}$.

The above reasoning is exactly the content of the reduction procedure [13].

4. The problem on inert body. Following the paper [17], we set formally $K = 0$ in the algebraic system (2.7) and insert the obtained formula

$$a = -i\omega^{-1}M^{-1}S\varphi \tag{4.1}$$

into the boundary condition (2.6), reducing it to the form

$$\partial_\nu\varphi(y,z) = -\omega\nu(y,z)^\top D(y-y^\bullet, z-z^\bullet)M^{-1}S\varphi, \ (y,z)\in\Sigma. \tag{4.2}$$

The main Theorem 6 is formulated just for the boundary condition (4.2), which corresponds to the inert body Θ unaffected by the buoyancy forces (see Remark 3). However, we also consider the following auxiliary problem with an arbitrary symmetric and positive matrix P:

$$-\Delta\Phi(y,z) + k^2\Phi(y,z) = 0, \ (y,z)\in\Omega, \tag{4.3}$$

$$-\partial_z\Phi(y,0) = \Lambda\Phi(y,0), \ (y,z)\in\Gamma, \tag{4.4}$$

$$\partial_\nu\Phi(y,-d) = 0, \ (y,z)\in\Upsilon, \tag{4.5}$$

$$\partial_\nu\Phi(y,z) = -\nu(y,z)^\top D(y-y^\bullet, z-z^\bullet)PS\Phi, \ (y,z)\in\Sigma. \tag{4.6}$$

The boundary condition (4.6) emerged as a result of the change $M^{-1}\mapsto P$ on the right-hand side of (4.2). The equation (4.3) and the conditions (4.4), (4.5) came from the formulas (2.3) and (2.4), (2.5), and finally, S was defined by (2.13) and (3.8).

Remark 3. 1) If P is the null matrix, the condition (4.6) turns into

$$\partial_\nu\Phi(y,z) = 0, \ (y,z)\in\Sigma, \tag{4.7}$$

which corresponds to a fixed obstacle (cf. Remark 1) and means that the normal component $v_n = n^\top v$ of the velocity vector $v = \nabla\varphi$ vanishes on the wetted part Σ of the surface of the body Θ (compare with the no-flow condition (4.5) through the bottom Υ). The equality (4.7) can be obtained formally by passing to the limit $M\to\infty$, which however contradicts the Archimedean law (2.16). Thus the problem (4.3)–(4.5), (4.7) indeed describes a body fixed externally.

2) The right-hand side of (4.2) includes the principal vector and torque $S\Phi$ of hydrodynamic forces, while the vector $M^{-1}S\Phi$ corresponds to the linear and angular accelerations generated by these forces. Finally the value $\omega\nu(y,z)^\top D(y-y^\bullet, z-z^\bullet)M^{-1}S\varphi$ is proportional to the normal velocity of points on the surface Σ due to motion under action of the liquid flow. The given interpretation of the boundary condition (4.2) impels to call (2.3)–(2.5), (4.2) the problem on an inert body unaffected by the buoyancy forces. In the case of the submerged body the latter is possible in the case of the coincidence of the mass $(y^\bullet, z^\bullet)$ and buoyancy $(y^\ominus, z^\ominus)$ centra, since in the relation (2.14) both components K^θ and K^Θ of the matrix K become null. In the case of the surface-piercing body the equality $K=0$ is not possible because $K^\theta \geq 0$ and $\operatorname{rank} K^\theta = 2$ but $\operatorname{rank} K^\Theta = 1$. ⊠

We have introduced the general matrix $P \geq 0$ in order to include both above variants of boundary conditions into our consideration.

According to the equality (3.1) the variational formulation of the problem (4.3)–(4.6) looks as follows:

$$(\nabla\Phi, \nabla\Psi)_\Omega + k^2(\Phi,\Psi)_\Omega + (PS\Phi, S\Psi)_{\mathbb{C}} = \Lambda(\Phi,\Psi)_\Gamma, \quad \Psi \in H^1(\Omega). \tag{4.8}$$

We take the sesquilinear form $\langle\Phi,\Psi\rangle_P$ on the left-hand side of (4.8) as a scalar product in the Sobolev space $H^1(\Omega)$ and, in analogy with (3.4), introduce the trace operator

$$\langle T_P\Phi, \Psi\rangle_P = (\Phi,\Psi)_\Gamma, \quad \Phi,\Psi \in H^1(\Omega). \tag{4.9}$$

As a result the variational problem (4.8) transforms into the abstract equation

$$T_P\Phi = B\Phi \quad \text{in} \quad H^1(\Omega) \tag{4.10}$$

with the new spectral parameter

$$B = \Lambda^{-1}.$$

Since the scalar products $\langle\Phi,\Psi\rangle$ and $\langle\Phi,\Psi\rangle_P$ differ only by the term $(PS\Phi, S\Psi)_{\mathbb{C}}$ including the finite-dimensional projection (3.8), the modified trace operator (4.9) has the same essential spectrum $[0,\beta_\dagger]$ as the operator T in the formula (3.4).

As in [20, 21], the reduction of the variational problem (4.8) (or (4.3)–(4.6) in the differential form) to the standard spectral problem (4.10) for a self-adjoint bounded operator T_P in the Hilbert space $H^1(\Omega)$ allows by means of a simple calculation to derive sufficient conditions for the existence of the discrete spectrum (read: trapped modes) and to verify appropriate comparison principles. We formulate a couple of simple assertions. Proofs of them can be found in the paper [17], and with slightly modified considerations also in the paper [21].

Proposition 4. *Let the inequality*

$$\int_\Theta |\nabla\phi_\dagger(z)|^2\, dydz \geq \lambda_\dagger \int_\theta |\nabla\phi_\dagger(0)|^2\, dy + (PS\phi_\dagger, S\phi_\dagger)_{\mathbb{C}},$$

be valid, where $\phi_\dagger$ *and* $\lambda_\dagger$ *are the eigenfunction and eigenvalue* (3.14) *and* (3.5) *of the model problem* (3.15)*. Then the operator* T_P *has an eigenvalue* $B_1 > \beta_\dagger$ *in its discrete spectrum while the variational problem* (4.8) *(or the boundary value problem* (4.3)–(4.6)*) has the eigenvalue* $\Lambda_1 = B_1^{-1} < \lambda_\dagger$*. This gives rise to a trapped mode* $\Phi_1 \in H^1(\Omega)$*, which possesses finite total (kinetic plus potential) energy and therefore decays exponentially as* $y \to \pm\infty$*.*

Apart from the eigenpairs $\{\Lambda_j, \Phi_j\}$ of the problem (4.8), the next assertion involves eigenpairs of the same problem but with another symmetric and positive 3×3-matrix $\mathbf{P}$. This is assumed to satisfy

$$P \geq \mathbf{P}, \tag{4.11}$$

which implies that all eigenvalues of the matrix $P - \mathbf{P}$ are non-negative. (Objects related to the modified problem will be denoted by bold letters in the following.)

Proposition 5. *Let the problem* (4.8) *(or the problem* (4.3)–(4.6)*) have the eigenvalues*

$$0 < \Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_J < \lambda_\dagger, \tag{4.12}$$

composing its discrete spectrum, and let $\mathbf{P}$ *be as above. If* (4.11) *holds, then the problem* (4.8) *with the matrix* $\mathbf{P}$ *replacing* P *has at least* J *eigenvalues* $\boldsymbol{\Lambda}_j$ *subject to the inequalities*

$$\boldsymbol{\Lambda}_j \leq \Lambda_j, \quad j = 1, \ldots, J.$$

If $\mathbf{P}$ is the null matrix, then Proposition 5 concerns the fixed obstacle. Hence, non-emptyness of the discrete spectrum of the problem (4.3)–(4.6) guarantees non-emptyness of the discrete spectrum of the problem (4.3)–(4.5), (4.7) while the total multiplicity of the latter is not less than the number J in (4.12).

5. Spectral measure and localization of eigenfrequencies in case of freely floating body. In Section 3 we have constructed a bounded self-adjoint operator $\mathcal{A}$ in the Hilbert space $\mathcal{H}^\natural$. Since the operators $\mathcal{B}$ and $\mathcal{D}$ on the left- and right-hand sides of (3.23) differ from the operators

$$\mathcal{B}^0 = \begin{pmatrix} \mathbb{I} & 0 & 0 & 0 \\ 0 & \mathbb{I} & 0 & 0 \\ 0 & 0 & gK_{\natural\natural} & 0 \\ 0 & 0 & 0 & N_{\natural\natural} \end{pmatrix}, \qquad \mathcal{D}^0 = \begin{pmatrix} 0 & g^{-1/2}T^{1/2} & 0 & 0 \\ g^{-1/2}T^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I} \\ 0 & 0 & \mathbb{I} & 0 \end{pmatrix},$$

only by compact perturbations, the essential spectrum of the operator $\mathcal{A} = \mathcal{B}^{-1/2}\mathcal{D}\mathcal{B}^{-1/2}$ coincides with the segment

$$\left[-\omega_\dagger^{-1}, \omega_\dagger^{-1}\right] = \left[-(g\lambda_\dagger)^{-1/2}, (g\lambda_\dagger)^{-1/2}\right]. \tag{5.1}$$

This follows from the relation of ω and α, see (3.11) and (3.26), and from the information on the trace operator T presented in Section 3. The bound $\omega_\dagger$ is computed from (3.12). The set

$$\left(-\infty, -\omega_\dagger^{-1}\right) \cup \left(\omega_\dagger^{-1}, +\infty\right)$$

may contain discrete spectrum. This happens if and only if the norm $\|\mathcal{A}\|$ of the operator meets the inequality

$$\|\mathcal{A}\| = \sup_{\mathcal{Y}\in\mathcal{H}^\natural} \left| \frac{(\mathcal{A}\mathcal{Y}, \mathcal{Y})_{\mathcal{H}^\natural}}{(\mathcal{Y}, \mathcal{Y})_{\mathcal{H}^\natural}} \right| > \frac{1}{\omega_\dagger}. \tag{5.2}$$

Namely, the relation $\|\mathcal{A}\| < \omega_\dagger^{-1}$ is impossible, and if equality holds in (5.2), then the whole spectrum of $\mathcal{A}$ belongs to the segment (5.1) and therefore its discrete component is empty. In the case (5.2) the point $\|\mathcal{A}\|$, which certainly belongs to the spectrum, must fall into the discrete spectrum.

The above reasoning was used in the papers [20, 21] and then also in [13, 17] to obtain sufficient conditions for trapping waves: a concrete test function was inserted into the formula (5.2) and a condition was derived to ensure the strict inequality. In what follows we use a completely different idea to compare the spectra of the problems (2.1)–(2.7) and (4.3)–(4.6).

According to the spectral theorem (cf. [16, Theorem 6.1.1]) the bounded self-adjoint operator $\mathcal{A}$ generates the spectral measure[2] $E_\mathcal{A}$ which in its turn associates to any element $\mathcal{U} \in \mathcal{H}^\natural$ the scalar measure $\mu_{\mathcal{U},\mathcal{U}} = (E_\mathcal{A}\mathcal{U}, \mathcal{U})_{\mathcal{H}^\natural}$ on the line $\mathbb{R}$. We shall only need a couple of simple formulas, which can be found for example in the proof of Theorem 6.1.3 of [16]:

$$\|\mathcal{U}; \mathcal{H}^\natural\|^2 = (\mathcal{U}, \mathcal{U})_{\mathcal{H}^\natural} = \int_\mathbb{R} d\mu_{\mathcal{U},\mathcal{U}}(t) \quad \text{for} \quad \mathcal{U} \in \mathcal{H}^\natural \tag{5.3}$$

and

$$\|\mathcal{A}\mathcal{U} - \alpha\mathcal{U}; \mathcal{H}^\natural\|^2 = \int_\mathbb{R} (t-\alpha)^2 d\mu_{\mathcal{U},\mathcal{U}}(t) \quad \text{for} \quad \mathcal{U} \in \mathcal{H}^\natural,\ \alpha \in \mathbb{C}. \tag{5.4}$$

Let us choose some

$$\mathcal{Y} \in \mathcal{H}^\natural, \quad \alpha \in \mathbb{C} \tag{5.5}$$

and let us assume that the segment

$$\upsilon(\delta) = [\alpha - \delta, \alpha + \delta] \tag{5.6}$$

is free of the spectrum of the operator $\mathcal{A}$ that is $E_\mathcal{A}(\upsilon(\delta)) = 0$. Then, by the formulas

[2]For reader's convenience we keep the notation used in the book [16].

(5.4) and (5.3), we obtain

$$\|\mathcal{AY}-\alpha\mathcal{Y};\mathcal{H}^\natural\|^2=\int\limits_{\mathbb{R}}(t-\alpha)^2d\mu_{\mathcal{Y},\mathcal{Y}}(t)=\int\limits_{\mathbb{R}\setminus\upsilon(\delta)}(t-\alpha)^2d\mu_{\mathcal{Y},\mathcal{Y}}(t)\geq$$
$$\geq\delta^2\int\limits_{\mathbb{R}\setminus\upsilon(\delta)}d\mu_{\mathcal{Y},\mathcal{Y}}(t)=\delta^2\int\limits_{\mathbb{R}}d\mu_{\mathcal{Y},\mathcal{Y}}(t)=\delta^2\|\mathcal{Y};\mathcal{H}^\natural\|^2. \tag{5.7}$$

However, if

$$\delta>\delta(\alpha,\mathcal{Y}):=\|\mathcal{Y};\mathcal{H}^\natural\|^{-1}\|\mathcal{AY}-\alpha\mathcal{Y};\mathcal{H}^\natural\| \tag{5.8}$$

the relation (5.7) is absurd and therefore our assumption on the absence of the spectrum in the segment (5.6) is wrong. Assuming

$$\delta(\alpha,\mathcal{Y})<\alpha-\omega_\dagger^{-1}, \tag{5.9}$$

the segment (5.6) with half-length $\delta\in\big(\delta(\alpha,\mathcal{Y}),\alpha-\omega_\dagger^{-1}\big)$ must contain a point of the discrete spectrum. We have thus detected an eigenvalue α_1 of the operator $\mathcal{A}$ which satisfies

$$|\alpha-\alpha_1|\leq\delta(\alpha,\mathcal{Y}). \tag{5.10}$$

Since the number α and the vector $\mathcal{Y}$ in the formula (5.5) are regarded as known and fixed, the relation (5.10) is to be considered as the localization of the position of an eigenvalue of the operator $\mathcal{A}$. It is also worth to make the value $\delta(\alpha,\mathcal{Y})$ as small as possible. This amounts to diminishing the norm $\|\mathcal{AY}-\alpha\mathcal{Y};\mathcal{H}^\natural\|$ while requiring $\|\mathcal{Y};\mathcal{H}^\natural\|=1$, i.e., we are forced to treat α and $\mathcal{Y}$ as approximate eigenvalue and eigenvector of the operator $\mathcal{A}$

6. Calculations and final theorem. Let the auxiliary problem problem (4.8) with some matrix $P\geq 0$ have the eigenvalue and the corresponding eigenfunction

$$\Lambda\in(0,\lambda_\dagger),\quad \Phi\in H^1(\Omega). \tag{6.1}$$

In accordance with formulas (3.10), (3.26) and (4.1), (6.1) we set

$$\omega=\sqrt{g\Lambda},$$
$$\varphi=\Phi,\quad \eta=\omega g^{-1/2}T^{1/2}\varphi,\quad a=-i\omega^{-1}M^{-1}S\varphi, \tag{6.2}$$
$$f=\omega Ma=-iS\varphi, \tag{6.3}$$

and

$$\alpha=\frac{1}{\omega},\quad \mathcal{Y}=\mathcal{B}^{1/2}\mathcal{X},\quad \mathcal{X}=(\varphi,\eta,a_\natural,f_\natural)^\top.$$

We have

$$\|\mathcal{Y};\mathcal{H}^\natural\|^2=(\mathcal{Y},\mathcal{Y})_{\mathcal{H}^\natural}=(\mathcal{BX},\mathcal{X})_{\mathcal{H}^\natural},$$

due to relations (3.25) and (3.3), (3.4), hence

$$\|\mathcal{Y};\mathcal{H}^\natural\|^2 = \langle\varphi,\varphi\rangle + g^{-1}\omega^2\langle T\varphi,\varphi\rangle + (M^{-1}S\varphi,S\varphi)_{\mathbb{C}} + g(Ka,a)_{\mathbb{C}} =$$

$$= (\nabla\Phi,\nabla\Phi)_\Omega + k^2(\Phi,\Phi)_\Omega + \Lambda(\Phi,\Phi)_\Gamma + (M^{-1}S\Phi,S\Phi)_{\mathbb{C}} + \Lambda^{-1}(KM^{-1}S\Phi,M^{-1}S\Phi)_{\mathbb{C}}. \tag{6.4}$$

By the representation (2.15) of the matrix K, the equality $(Kb,b)_{\mathbb{C}} = (K_\natural b_\natural, b_\natural)_{\mathbb{C}}$ is valid, and this was used at the end of the calculation (6.4). Moreover,

$$\|\mathcal{AY}-\alpha\mathcal{Y};\mathcal{H}^\natural\| = \sup_{\substack{\mathcal{W}\in\mathcal{H}^\natural:\\ \|\mathcal{W};\mathcal{H}^\natural\|=1}} |(\mathcal{AY}-\alpha\mathcal{Y},\mathcal{W})_{\mathcal{H}^\natural}| =$$

$$= \frac{1}{\omega}\sup_{\substack{\mathcal{Z}\in\mathcal{H}^\natural:\\ \|\mathcal{B}^{1/2}\mathcal{Z};\mathcal{H}^\natural\|=1}} |(\omega\mathcal{DX}-\mathcal{BY},\mathcal{Z})_{\mathcal{H}^\natural}|\,.$$

Let us compute the components of the vector

$$(F,Y,A,G)^\top = \mathcal{BY}-\omega\mathcal{DX}.$$

First of all, owing to the structure of matrices $\mathcal{B}$ and $\mathcal{D}$ in the system (3.17) and also to formulas (6.2) and (6.3) for η, a and f, we notice that

$$Y = \eta - \omega g^{-1/2}T^{1/2}\varphi = \omega g^{-1/2}T^{1/2}\varphi - \omega g^{-1/2}T^{1/2}\varphi = 0,$$

$$G = -iN_{\natural\circ}S_\circ\varphi + N_{\natural\natural}f_\natural - \omega a_\natural = -iN_{\natural\circ}S_\circ\varphi - iN_{\natural\natural}S_\natural\varphi + i(M^{-1}S\varphi)_\natural = 0,$$
$$A = gK_{\natural\natural}a_\natural - i\omega S_\natural\varphi + \omega f_\natural = gK_{\natural\natural}a_\natural = -i\omega^{-1}gK_{\natural\natural}(M^{-1}S\varphi)_\natural.$$

Finally,

$$\begin{gathered} F = \varphi + S_\circ^* N_{\circ\circ}S_\circ\varphi + iS_\circ^* N_{\circ\natural}f_\natural - \omega g^{-1/2}T^{1/2}\eta + i\omega S_\natural^* a_\natural = \\ = \varphi - g^{-1}\omega^2 T\varphi + S_\circ^*(NS\varphi)_\circ + S_\natural^*(NS\varphi)_\natural = \Phi + S^*M^{-1}S\Phi - \Lambda T\Phi. \end{gathered} \tag{6.5}$$

Using the definitions (3.4) and (4.9) of the trace operators T and T_P one finds that (6.5) is equal to

$$S^*(M^{-1}-P)S\Phi.$$

Thus,

$$\|\mathcal{AY}-\alpha\mathcal{Y};\mathcal{H}^\natural\| = \omega^{-1}\sup\left|((M^{-1}-P)S\Phi,S\psi)_{\mathbb{C}} - ig\omega^{-1}(K_{\natural\natural}(M^{-1}S\Phi)_\natural,b_\natural)_{\mathbb{C}}\right|, \tag{6.6}$$

where the supremum is computed over all vectors $\mathcal{Z} = (\psi,\xi,b_\natural,g_\natural)^\top \in \mathcal{H}^\natural$ such that

$$1 = (\mathcal{BZ},\mathcal{Z})_{\mathcal{H}^\natural} = \langle\psi,\psi\rangle + \langle\xi,\xi\rangle + g(K_{\natural\natural}b_\natural,b_\natural)_{\mathbb{C}} + \left(N\begin{pmatrix}-iS_\circ\psi\\ g_\natural\end{pmatrix},\begin{pmatrix}-iS_\circ\psi\\ g_\natural\end{pmatrix}\right)_{\mathbb{C}}. \tag{6.7}$$

Let us outline some further issues. First, due to the structure of the matrix (2.15), the last term in the equality (6.6) can be replaced by $(KM^{-1}S\Phi, b)_{\mathbb{C}}$. Second, the expression (6.6) does not involve the function ξ and, therefore, the positivity of each term on the right of (6.7) permits us to put $\xi = 0$. Finally, since the components ψ and b are independent and the multiplication of b with any unimodular complex number does not effect the value (6.7), we conclude that

$$\|\mathcal{A}\mathcal{Y} - \alpha\mathcal{Y}; \mathcal{H}^\natural\| = \omega^{-1} \sup \left(\left| ((M^{-1} - P)S\Phi, S\psi)_{\mathbb{C}} \right| + g\omega^{-1} \left| (KM^{-1}S\Phi, b)_{\mathbb{C}} \right| \right). \tag{6.8}$$

The supremum now is computed over all vectors $(\psi, b, g_\natural)^\top \in H^1(\Omega) \times \mathbb{C}^3 \times \mathbb{C}^\tau$ normalized as follows:

$$\langle \psi, \psi \rangle + g(Kb, b)_{\mathbb{C}} + \left(N \begin{pmatrix} -iS_\circ\psi \\ g_\natural \end{pmatrix}, \begin{pmatrix} -iS_\circ\psi \\ g_\natural \end{pmatrix} \right)_{\mathbb{C}} = 1. \tag{6.9}$$

The value (6.4) can be calculated easily with the help of the integral identity (4.8) and the test function $\Psi = \Phi$:

$$\|\mathcal{Y}; \mathcal{H}^\natural\|^2 = 2\Lambda(\Phi, \Phi)_\Gamma + ((M^{-1} - P)S\Phi, S\Phi)_{\mathbb{C}} + \Lambda^{-1}(KM^{-1}S\Phi, M^{-1}S\Phi)_{\mathbb{C}}. \tag{6.10}$$

On the other hand, computing the supremum (6.8) becomes a complicated task. The only simple case is $P = M^{-1}$. Indeed, the first term in the supremum disappears while

$$\|g^{1/2}K_{\natural\natural}^{1/2}(M^{-1}S\Phi)_\natural; \mathbb{C}\| = \sup_{\substack{b_\natural \in \mathbb{C}^\kappa: \\ \left\|g^{1/2}K_{\natural\natural}^{1/2} b_\natural; \mathbb{C}\right\| \le 1}} \left| \left(g^{1/2}K_{\natural\natural}^{1/2}(M^{-1}S\Phi)_\natural, g^{1/2}K_{\natural\natural}^{1/2} b_\natural \right)_{\mathbb{C}} \right|$$

and, moreover, $\|g^{1/2}K_{\natural\natural}^{1/2} b_\natural; \mathbb{C}\|^2 = g(K_{\natural\natural} b_\natural, b_\natural)_{\mathbb{C}} \le 1$ due to the restriction (6.9). Hence,

$$\|\mathcal{A}\mathcal{Y} - \alpha\mathcal{Y}; \mathcal{H}^\natural\| = \omega^{-2}\|g^{1/2}K_{\natural\natural}^{1/2}(M^{-1}S\Phi)_\natural; \mathbb{C}\| =$$

$$= \omega^{-2}g^{1/2}(K_{\natural\natural}(M^{-1}S\Phi)_\natural, (M^{-1}S\Phi)_\natural)_{\mathbb{C}}^{1/2} = \omega^{-1}\Lambda^{-1/2}(KM^{-1}S\Phi, M^{-1}S\Phi)_{\mathbb{C}}^{1/2}, \tag{6.11}$$

so that we have

$$\boldsymbol{\delta}(\Lambda, \Phi) := \delta(\alpha, \mathcal{Y}) = \frac{1}{\sqrt{g\Lambda}} \sqrt{\frac{\Lambda^{-1}(KM^{-1}S\Phi, M^{-1}S\Phi)_{\mathbb{C}}}{2\Lambda(\Phi, \Phi)_\Gamma + \Lambda^{-1}(KM^{-1}S\Phi, M^{-1}S\Phi)_{\mathbb{C}}}}, \tag{6.12}$$

by (6.10), (6.11) and the condition $P = M^{-1}$.

We have thus proven the following assertion.

Theorem 6. *Assume the problem* (4.8) *(or* (4.3)–(4.6)*) on the inert body, with* $P = M^{-1}$*, has an eigenvalue* $\Lambda \in (0, \lambda_\dagger)$ *and the corresponding eigenfunction* $\Phi \in H^1(\Omega)$ *meets the inequality*

$$\boldsymbol{\delta}(\Lambda, \Phi) < \frac{1}{\sqrt{g\Lambda}} - \frac{1}{\sqrt{g\lambda_\dagger}} \tag{6.13}$$

where $\boldsymbol{\delta}(\Lambda, \Phi)$ *is the value* (6.12)*. Then the problem* (3.2)*(or* (2.3)–(2.7)*) on the freely floating body has an eigenfrequency* ω_1 *which satisfies the estimate*

$$\left| \frac{1}{\omega_1} - \frac{1}{\sqrt{g\Lambda}} \right| \leq \boldsymbol{\delta}(\Lambda, \Phi).$$

The eigenfunction Φ of the problem (4.8) can of course be normalized by the condition $\|\Phi; L_2(\Gamma)\|^2 = 1$ (cf. the relation (3.10)). If the value

$$\Lambda^{-1}(KM^{-1}S\Phi, M^{-1}S\Phi)_{\mathbb{C}} \tag{6.14}$$

is sufficiently small, then the expression (6.12) becomes small as well, and hence the condition (6.13) is satisfied. In other words, Theorem 6 indeed establishes the existence of an eigenfrequency $\omega \in (0, \omega_\dagger)$ of the John problem in the case of a small buoyancy matrix K. The matrix K appears by no means in the auxiliary problem (4.3)–(4.6); therefore, changing it while preserving the inertia matrix (2.12) does not influence the eigenpair $\{\Lambda, \Phi\}$. If the body is submerged, then redistributing the density $\rho(y, z)$ within Θ may preserve the inertia matrix M but shift the mass centre (2.9) close to the buoyancy centre (2.17). This can make I_z^Θ as well as the matrix $K = K^\Theta$ arbitrarily small. Of course, it is necessary to preserve the Archimedean law (2.16) and the stability condition $z^\bullet < z^\ominus$. Changing the density of the body, we also may use Proposition 5 which guarantees that the discrete spectrum of the problem (4.3)–(4.6) is preserved and which also gives a one-sided estimate of its eigenvalue, if the matrix P varies in the integro-differential boundary condition (4.6).

If the body is surface-piercing, we additionally have to assume the smallness of the cross-section θ of the body Θ with the plane $\{z = 0\}$. This assures the smallness of the second term K^θ in the representation (2.14) of the buoyancy matrix K.

Let us formulate a sufficient condition for trapping a wave by a freely floating body as an inequality for the value (6.14).

Corollary 7. *Assume that the problem* (4.8) *on the inert body with* $P = M^{-1}$ *has an eigenvalue* $\Lambda \in (0, \lambda_\dagger)$ *and the inequality*

$$\Lambda^{-1}(KM^{-1}S\Phi, M^{-1}S\Phi)_{\mathbb{C}} < 2\sqrt{\Lambda\lambda_\dagger} \left(2 - \sqrt{\frac{\Lambda}{\lambda_\dagger}}\right)^{-1} \left(1 - \sqrt{\frac{\Lambda}{\lambda_\dagger}}\right)^2$$

is satisfied, where Φ *is the corresponding eigenfunction normalized by the condition* $\|\Phi; L_2(\Omega)\| = 1$*. Then the problem* (3.2) *on the freely floating body has an eigenfrequency* $\omega_1 \in (0, \omega_\dagger)$*.*

We mention that a simple sufficient condition for the existence of a trapped mode in the problem (4.3)–(4.6) is given by Proposition 4. Finally, if ω is an eigenfrequency in the John problem, then, by an evident argument, $-\omega$ is also an eigenfrequency.

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Two-sided estimates for eigenfrequencies in the John problem for freely floating body.

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