LINEAR WATER-WAVE PROBLEM IN A POND WITH A SHALLOW BEACH

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Abstract. We consider the linear water-wave problem in a bounded water-basin with shallow beaches. In spite of the boundedness of the domain, the spectrum of the problem may have a continuous component, if the beach of the basin has a cuspidal form. Following the approach and methods of [11] we improve the results of the citation by proving the existence of a continuous spectrum under much weaker geometric assumptions. In particular we solve a borderline case left open in the citation.

Keywords: linear water-wave problem, Steklov condition, continuous spectrum, shallow beach

1991 Mathematics Subject Classification: primary 35J20, secondary 35P99, 76B15

1. Introduction and main result.

The linear model for water-waves is realistic in many situations, especially in case the wave amplitude is relatively small. Within this theory, water-waves are described by a mixed boundary value problem for the Laplace equation with Steklov spectral boundary condition on the horizontal water surface (see (1.2)–(1.4) below, and [14], [3], [4] for the physical background). It is well known that the wave propagation phenomenon occurs when the Steklov spectral parameter λ belongs to the continuous spectrum σ_c of the problem. It is not surprising that this is the case for unbounded domains, but in [11] it was shown that the continuous spectrum may be nonempty even in a bounded three-dimensional pond. This may happen in case the pond has shallow beaches, where the water volume has geometric forms like rotational cusps or peak shaped edges.

The aim of the present work is to generalize the results of [11], especially its Theorem 2.1, concerning the discreteness of the spectrum of the linear water-wave problem. As in the citation, the bounded domain is assumed to have a shallow beach which can be geometrically described as a cuspidal edge. The structure of the spectrum depends on the geometry as follows. In the reference it was proven that if a geometric parameter $\mathbf{m}$, which describes the sharpness of the cuspidal edge, is larger than 2, then continuous spectrum appears and contains the point 0. In case $\mathbf{m} < 2$ the spectrum was shown to be discrete, and the borderline case $\mathbf{m} = 2$ remained unsolved. In this paper we solve the open case by proving that set $[1/4, \infty)$ is contained in the essential spectrum, if $\mathbf{m} = 2$; the method of proof is a refinement of the arguments in [11]. Moreover, the assumptions of [11] are weakened by allowing them to be of more local nature. In addition to [11], our methods are based on the previous works [10], [13] on Steklov and Neumann problems in cuspidal domains. The construction of the Weyl sequence originally emerges from the asymptotic ansätze for thin rods and plates, see for example [6], [7], [8].

Let us describe the domain (bounded water-basin) $\Omega \subset \mathbb{R}^3$ in detail; see Figure 1. A point in $\mathbb{R}^3$ is usually denoted by $x = (y, z)$, where $y = (y_1, y_2) \in \mathbb{R}^2$. The boundary $\partial\Omega$ of Ω contains the free water surface (denoted by $\Lambda \subset \{x = (y, z) : z = 0\} =: \Pi$) and the bottom $\{x = (y, z) : z = -\mathbf{h}(y_1, y_2)\}$, where the smooth function $\mathbf{h} > 0$ is the depth of the pond. In addition $\partial\Omega$ may contain walls, which are smooth surfaces parallel to z-axis connecting the free surface Λ and the bottom; bottom and walls together form the set Σ, see Figure 1. The edges of Λ and Σ coincide on a piecewise C^2-smooth, closed simple contour $\gamma \subset \Pi$, hence $\partial\Omega$ equals $\Lambda \cup \Sigma \cup \gamma$ and the domain Ω consists of the points (y, z) with

$$-\mathbf{h}(y) < z < 0. \tag{1.1}$$

It is assumed that the origin $\mathcal{O} = (0, 0, 0)$ is contained in the contour γ and that γ is C^2-smooth at $\mathcal{O}$.

We consider the linearized water-wave problem, [14], [3], [4], which reads as

$$-\Delta_x\Phi(x) = 0 \ , \quad x \in \Omega, \tag{1.2}$$

$$\partial_n\Phi(x) = 0 \ , \quad x \in \Sigma, \tag{1.3}$$

$$\partial_n\Phi(x) = \lambda\Phi(x) \ , \quad x \in \Lambda. \tag{1.4}$$

where Δ_x is the Laplacian, ∂_n is the derivative along the outward normal n, and Φ denotes the velocity potential and λ the spectral parameter proportional to the square of the frequency of harmonic oscillations.

To give exact meaning for the spectral concepts, we shall formulate the problem (1.2)–(1.4) as a standard spectral problem for a bounded operator K_Ω in a Hilbert space. Being self-adjoint and positive, its spectrum is contained in the positive real axis. The problem whether or not it is discrete, is answered by our main result:

Theorem 1.1. *Assume that the depth function* $\mathbf{h}(\mathbf{y})$ *of the pond* Ω *equals* y_1^2 *asymptotically at* $\mathcal{O} = (0, 0, 0)$ *(more precisely, (i)–(iii) hold, see just below). Then the ray* $[1/4, +\infty)$ *belongs to the essential spectrum of the problem* (1.2)–(1.4).

In the above theorem we are assuming that in a neighbourhood $\mathcal{U} \subset \Pi$ of $\mathcal{O}$ the following hold:
(i) γ is tangential to the y_2 axis,
(ii) Λ is contained in the right half-plane $\{(y, 0) : y_1 > 0\}$ of Π and contains the parabola $\{y : y_1 > y_2^2\}$,
(iii) there exist constants $0 < c < C$ such that $cy_1^2 \le \mathbf{h}(y) \le Cy_1^2$ and $|\nabla_y \mathbf{h}(y)| \le Cy_1$ for all $y \in \mathcal{U} \cap \Lambda$.

The last condition corresponds to the choice $\mathbf{m} = 2$ in the geometric assumption made in [11] that the depth function behaves like $\nu^{\mathbf{m}}$, where ν is the distance of a point on the water surface Λ to the boundary γ, or beach. However, the present assumption is weaker, since the contour γ does not need to coincide with the y_2-axis, hence, the depth function needs to vanish only at one point, the origin. (Notice the different notation: the present domain Ω corresponds to Ξ of the reference.)

The assumption on the parabola in *(ii)* is superfluous, since the other assumptions would imply that a smaller parabola would anyway be contained in Λ. That would be sufficient for the next construction, though at the cost of adding some extra parameters. Finally, it would be enough to assume *(iii)* only on that parabola.

So, the next task is to introduce the operator theoretic interpretation of the problem under investigation; as for the literature, see [5] and [2]. Due to the boundary irregularities, it is useful to consider the weak formulation of (1.2)–(1.4) in the Hilbert space $\mathcal{H}(\Omega,\Lambda) = H^1(\Omega)\cap L^2(\Lambda)$, which is defined as the completion of the space $C_c^\infty(\overline{\Omega}\setminus\gamma)$ with respect to the norm $\langle\Phi,\Phi\rangle_\Omega^{1/2}$, where

$$\langle\Phi,\Psi\rangle_\Omega = (\nabla_x\Phi,\nabla_x\Psi)_\Omega + (\Phi,\Psi)_\Lambda, \tag{1.5}$$

and $(f,g)_\Omega = \int_\Omega fg\,dx$ as well as $(f,g)_\Lambda = \int_\Lambda fg\,dy$. The weak formulation is then obtained by multiplying the equation (1.2) by a function $\Psi\in\mathcal{H}(\Omega,\Lambda)$, integrating over Ω and using the first Green formula together with the boundary conditions (1.3)–(1.4):

$$(\nabla_x\Phi,\nabla_x\Psi)_\Omega = \lambda(\Phi,\Psi)_\Lambda\ ,\quad \Psi\in\mathcal{H}(\Omega,\Lambda). \tag{1.6}$$

We define the operator $K_\Omega:\mathcal{H}(\Omega,\Lambda)\to\mathcal{H}(\Omega,\Lambda)$ using the formula

$$\langle K_\Omega\Phi,\Psi\rangle_\Omega = (\Phi,\Psi)_\Lambda\ ,\quad \Phi,\Psi\in\mathcal{H}(\Omega,\Lambda). \tag{1.7}$$

This operator has unit norm, and it is symmetric, therefore self–adjoint, and positive. The problem (1.6) is transformed into the following standard spectral problem for K_Ω:

$$K_\Omega\Phi = \mu\Phi\quad,\quad \Phi\in\mathcal{H}(\Omega,\Lambda). \tag{1.8}$$

The spectral parameter μ is related to λ by the identity $\mu = (1+\lambda)^{-1}$. Because of the above mentioned properties, the μ-spectrum of K_Ω is contained in the interval $[0,1]$, and actually 1 is an eigenvalue with constant eigenfunction.

Concerning the notation of this paper, C, C', c and so on denote positive constants which may vary from place to place. The norm of an element f in a Banach function space X is denoted by $\|f;X\|$. As for the literature on the present topic, we refer to citations mentioned in [12].

2. Proof of the main result.

The rest of the paper is occupied by the proof of Theorem 1.1, which is a refinement of the argument given in [11], Section 4. Given a number $\lambda\in[1/4,+\infty)$, the aim is to construct a Weyl sequence $(\Psi_j)_{j=1}^\infty\subset\mathcal{H}(\Omega,\Lambda)$ for the operator K_Ω corresponding to the number $(1-\lambda)^{-1}$. By definition, such a sequence $(\Psi_j)_{j=1}^\infty$ should have the following properties:
(i) for some positive constants c and C, $c\le\|\Psi_j;\mathcal{H}(\Omega,\Lambda)\|\le C$ for all j,
(ii) the sequence converges to 0 weakly in $\mathcal{H}(\Omega,\Lambda)$,
(iii) $\|K_\Omega\Psi_j-(1+\lambda_j)^{-1}\Psi_j;\mathcal{H}(\Omega,\Lambda)\|\to 0$ as $j\to\infty$.

Having this done, $\mu := (1+\lambda_j)^{-1}$ is shown to belong to the essential spectrum of K_Ω, which is equivalent to the statement of Theorem 1.1, by the explanation at the end of the previous section.

To construct the required Weyl sequence we shall use a collection of rectangular subdomains of Λ and Ω. The elements of the Weyl sequence will be supported on these subdomains. This leads to a simplification of the calculations, since the exact form of the boundary curve γ does not enter the ordinary differential equation defining the Weyl sequence; it will be enough to apply a separation of Cartesian coordinates to a model elliptic PDE (2.5).

Let us denote

$$r_j = e^{-2^j}\ ,\ j=1,2,3,\dots, \tag{2.1}$$

hence, the relation $r_{j+1} = r_j^2$ holds. Moreover,

$$A_j = \{y \in \mathbb{R}^2 \ : \ r_{j+1} \le y_1 \le r_j \, , \ -r_j \le y_2 \le r_j\}, \tag{2.2}$$

$$B_j = \{y \in \mathbb{R}^2 \ : \ 2r_{j+1} \le y_1 \le r_j/2 \, , \ -r_j \le y_2 \le r_j\}, \tag{2.3}$$

$$C_j = \{y \in \mathbb{R}^2 \ : \ r_{j+1} \le y_1 \le r_j \, , \ -r_j/1 \le y_2 \le r_j/2\}, \tag{2.4}$$

For large enough j, say $j \ge J_0$, the sets A_j, B_j and C_j are contained in Λ (due to $r_j^2 = r_{j+1}$ and the assumption (ii) in Section 1). The corresponding boldface letters $\mathbf{A}_j$, $\mathbf{B}_j$, and $\mathbf{C}_j$ denote the respective subsets of Ω, for example $\mathbf{A}_j = \{x = (y, z) : y \in A_j \, , \ 0 \ge z \ge -\mathbf{h}(y)\}$ and similarly for the other sets.

Another ingredient for the construction of the Weyl sequence is gotten from solutions of the following model problem:

$$-\nabla_y \cdot y_1^2 \nabla_y U(y) = \lambda U(y) \ , \quad y \in \Lambda. \tag{2.5}$$

Similar model equations have been used in [8], [10], [11], [13], and they have been derived from the asymptotic theory of elliptic problems in thin domains. To find the solutions we use separation of variables and write $U(y) := F(y_1)u(y_2)$ with

$$u'' = Lu \quad , \ L < 0, \tag{2.6}$$

and

$$y_1^2 F'' + 2y_1 F' + (Ly_1^2 + \lambda)F = 0. \tag{2.7}$$

This is an ODE with polynomial coefficients, and $y_1 = 0$ is a regular singular point. Using the Frobenius method, see [1], the indicial equation becomes

$$r^2 + r + \lambda = 0. \tag{2.8}$$

Thus we find that if $\lambda \ge 1/4$, then the one or two solutions of (2.8) are $r = -1/2 + i\tilde{r}$ with $\tilde{r} \in \mathbb{R}$, and (2.7) has the Frobenius solutions

$$F(y_1) = y_1^r \sum_{n=0}^{\infty} c_n y_1^n = y_1^{-1/2} e^{i\tilde{r}\log y_1} \sum_{n=0}^{\infty} c_n y_1^n, \tag{2.9}$$

where the coefficients c_n are complex valued. Putting (2.9) into (2.7) we obtain the recursion formula for the coefficients c_n, $n \ge 2$,

$$c_{n+2} = \frac{Lc_n}{2r(n+2) + (n+2)(n+3)} = \frac{Lc_n}{(n+2)(2r+n+1)} \quad , \quad n = 0, 1, 2, \ldots .$$

The radius of the convergence of this power series is obviously ∞. Moreover, the coefficient of c_0 vanishes, since it equals $r^2 + r + \lambda$. We can thus choose $c_0 = 1$. The coefficient of c_1 becomes $r^2 + 3r + 2 + \lambda$, which may be nonzero, hence, we set $c_1 = 0$. With these choices, (2.9) satisfies (2.7).

A solution of (2.6) is $u = e^{i\sqrt{-L}y_2}$. Since the aim is only to construct a Weyl sequence, its elements do not need to satisfy boundary conditions. In particular, there does not appear a condition for the parameter L , which we thus set equal -1.

With these definitions the solution $U(y) = F(y_1)u(y_2)$ of the model problem (2.5) has the properties

$$|U(y)| \le Cy_1^{-1/2} \ , \ |\partial_{y_1} U(y)| \le Cy_1^{-3/2} \ , \ |\partial_{y_2} U(y)| \le Cy_1^{-1/2} \tag{2.10}$$

for some constant $C > 0$, for all $y \in \Lambda \cap \mathcal{U}$. By possibly diminishing the set $\mathcal{U}$ we can even assume

$$|U(y)| \ge C' y_1^{-1/2} \tag{2.11}$$

for some constant $C' > 0$, for $y \in \Lambda \cap \mathcal{U}$, since the power series $\sum_{n=0}^{\infty} c_n y_1^n$ does not have zeros in a neighbourhood of $y_1 = 0$ due to $c_0 = 1$; cf. (2.9).

The Weyl sequence consists of functions

$$\Psi_j(x) = a_j X_j(y_1) Y_j(y_2) U(y), \tag{2.12}$$

where $j \geq J_0$ and $a_j > 0$ is a normalization factor to be defined later. Each Ψ_j is supported in A_j, since $X_j \in C^\infty(\mathbb{R})$ is a cut–off function such that $0 \leq X_j(t) \leq 1$ for all t, $X_j(t) = 1$ for $2r_{j+1} \leq t \leq r_j/2$ and $X_j(t) = 0$ for $t \leq r_{j+1}$ and $t \geq r_j$. Also $Y_j \in C^\infty(\mathbb{R})$ is a cut–off function such that $0 \leq Y_j(t) \leq 1$ for all t, $Y_j(t) = 1$ for $-r_j/2 \leq t \leq r_j/2$ and $Y_j(t) = 0$ for $|t| \geq r_j$. Moreover, X_j and Y_j are chosen to satisfy, for $k = 1, 2$,

$$|\partial_{y_1}^k X_j(y_1)| \leq \frac{C}{y_1^k} \quad , \quad |\partial_{y_2}^k Y_j(y_2)| \leq C r_j^{-k} \tag{2.13}$$

We calculate some estimates for the functions (2.12), using (2.10), (2.11):

$$\|\Psi_j; L^2(\Lambda)\|^2 \leq a_j^2 \int\limits_{\Lambda \cap A_j} |U(y)|^2 dy \leq C a_j^2 \int\limits_{-r_j}^{r_j} \int\limits_{r_{j+1}}^{r_j} y_1^{-1} dy_1 dy_2$$

$$= 2C a_j^2 \Big[\log y_1 \Big]_{y_1 = r_{j+1}}^{y_1 = r_j} = C' a_j^2 r_j (2^{j+1} - 2^j) = C' a_j^2 r_j 2^j,$$

$$\|\Psi_j; L^2(\Lambda)\|^2 \geq a_j^2 \int\limits_{\Lambda \cap B_j \cap C_j} y_1^{-1} dy$$

$$\geq a_j^2 \int\limits_{-r_j/2}^{r_j/2} \int\limits_{2r_{j+1}}^{r_j/2} y_1^{-1} dy_1 dy_2 \geq c a_j^2 r_j 2^j \ , \ c > 0. \tag{2.14}$$

Moreover, using $\mathbf{h}(y) \leq C y_1^2$ on $\Omega \cap \mathbf{A}_j$ (see (*iii*) of Section 1) and (1.1), we get

$$\|\nabla_x \Psi_j; L^2(\Omega)\|^2$$

$$\leq a_j^2 \int\limits_{\Omega \cap \mathbf{A}_j} \Big(|\partial_{y_1} X_j|^2 |U(y)|^2 + |\partial_{y_2} Y_j|^2 |U(y)|^2 + |\nabla_y U(y)|^2 \Big) dy dz$$

$$\leq C a_j^2 \int\limits_{\mathbf{h}(y)}^{0} \int\limits_{-r_j}^{r_j} \int\limits_{r_{j+1}}^{r_j} (y_1^{-3} + y_1^{-1} r_j^{-2}) dy dz$$

$$\leq C' a_j^2 \int\limits_{-r_j}^{r_j} \int\limits_{r_j}^{r_{j+1}} y_1^{-1} dy_2 dy_1 \leq C'' a_j^2 r_j 2^j. \tag{2.15}$$

Hence, putting

$$a_j = 2^{-j/2} r_j^{-1/2} \tag{2.16}$$

we have

$$0 < c \leq \|\Psi_j; \mathcal{H}(\Omega, \Lambda)\| \leq C \tag{2.17}$$

Moreover,

$$\Psi_j \to 0 \text{ weakly in } \mathcal{H}(\Omega, \Lambda) \text{ as } \ j \to +\infty, \tag{2.18}$$

due to the choice of the supports of the functions Ψ_j.

Let us consider the expression (where $\Psi \in \mathcal{H}(\Omega, \Lambda)$)

$$\begin{aligned} & \| K_\Omega \Psi_j - (1+\lambda)^{-1}\Psi_j ; \mathcal{H}(\Omega,\Lambda)\| \\ = & \sup_{\|\Psi;\mathcal{H}(\Omega,\Lambda)\|=1} |\langle K_\Omega \Psi_j - (1+\lambda)^{-1}\Psi_j , \Psi\rangle_\Omega| \\ = & \sup_{\|\Psi;\mathcal{H}(\Omega,\Lambda)\|=1} \Big| (\Psi_j,\Psi)_\Lambda - (1+\lambda)^{-1}\big((\nabla_x\Psi_j,\nabla_x\Psi)_\Omega + (\Psi_j,\Psi)_\Lambda\big)\Big| \\ = & (1+\lambda)^{-1} \sup_{\|\Psi;\mathcal{H}(\Omega,\Lambda)\|=1} |(\nabla_x\Psi_j,\nabla_x\Psi)_\Omega - \lambda(\Psi_j,\Psi)_\Lambda| \qquad (2.19)\end{aligned}$$

Here we use the notation of (1.5) and (1.7).

Lemma 2.1. *The sequence* (Ψ_j) *has the property*

$$\| K_\Omega \Psi_j - (1+\lambda)^{-1}\Psi_j ; \mathcal{H}(\Omega,\Lambda)\| \to 0 \text{ as } j \to +\infty. \qquad (2.20)$$

Taking into account the bounds (2.17) and (2.18) and the remarks in the beginning of this section, Lemma 2.1 completes the proof of Theorem 1.1.

Let $\Psi \in \mathcal{H}(\Omega,\Lambda)$. For the proof of Lemma 2.1 it is necessary to consider the functions

$$\Psi(y,z) = \overline{\Psi}(y) + \Psi_\perp(y,z)\ ,\quad \overline{\Psi}(y) = \mathbf{h}(y)^{-1}\int_{-\mathbf{h}(y)}^{0} \Psi(y,z)dz. \qquad (2.21)$$

Recall the notation $\mathcal{U} \subset \Pi$ from Section 1. In the following we also denote $\mathcal{W} = \Lambda\cap\mathcal{U}$ and $\mathcal{V} = \{(y,z)\in\Omega \mid y \in \mathcal{W}\} \subset \Omega$.

Lemma 2.2. *Let* $\Psi \in \mathcal{H}(\Omega,\Lambda)$*. The following inequalities hold true:*

$$\|y_1\mathbf{h}^{-1/2}(\nabla_y\overline{\Psi} - \overline{\nabla_y\Psi}); L^2(\mathcal{W})\| \le c\|\partial_z\Psi; L^2(\mathcal{V})\|, \qquad (2.22)$$

$$\|\mathbf{h}^{-1/2}(\overline{\Psi} - \Psi(\cdot,0)); L^2(\mathcal{W})\| \le c\|\partial_z\Psi; L^2(\mathcal{V})\|, \qquad (2.23)$$

where $\Psi \in \mathcal{H}(\Omega,\Lambda)$ *and* $\Psi(y,0)$ *is the trace of* Ψ *on the surface* Λ.

Proof. Integrating over $\zeta \in (-\mathbf{h}(y),0)$ the identity

$$\Psi(y,\zeta) - \Psi(y,-\mathbf{h}(y)) = \int_{-\mathbf{h}(y)}^{\zeta} \partial_z\Psi(y,z)dz$$

we get

$$\overline{\Psi}(y) - \Psi(y,-\mathbf{h}(y)) = \frac{1}{\mathbf{h}(y)}\int_{-\mathbf{h}(y)}^{0}\int_{-\mathbf{h}(y)}^{\zeta} \partial_z\Psi(y,z)dzd\zeta = \frac{1}{\mathbf{h}(y)}\int_{-\mathbf{h}(y)}^{0} z\partial_z\Psi(y,z)dz.$$

Thus,

$$\begin{aligned} & \int_{\mathcal{W}} \mathbf{h}(y)^{-1}|\overline{\Psi}(y) - \Psi(y,-\mathbf{h}(y))|^2 dy \\ \le & \int_{\mathcal{W}} \mathbf{h}(y)^{-1}\Big|\int_{-\mathbf{h}(y)}^{0} \frac{z}{\mathbf{h}(y)}\partial_z\Psi(y,z)dz\Big|^2 dy\end{aligned}$$

$$\leq \int_{\mathcal{W}} \mathbf{h}(y)^{-1}\Big(\int_{-\mathbf{h}(y)}^{0} dz\Big)\Big(\int_{-\mathbf{h}(y)}^{0} \Big|\frac{z}{\mathbf{h}(y)}\partial_z\Psi(y,z)\Big|^2 dz\Big)dy$$

$$\leq \int_{\mathcal{W}}\int_{-\mathbf{h}(y)}^{0} |\partial_z\Psi(y,z)|^2 dydz. \tag{2.24}$$

An evident modification of the calculation yields the inequality (2.23).

Using (2.21), we continue by writing

$$\nabla_y\overline{\Psi}(y) - \overline{\nabla_y\Psi(y)} = -\mathbf{h}(y)^{-1}\nabla_y\mathbf{h}(y)(\overline{\Psi}(y) - \Psi(y,\mathbf{h}(y))).$$

Notice that $|\mathbf{h}(y)^{-1}\nabla_y\mathbf{h}(y)| \leq Cy_1^{-1}$, by (iii) of Section 1. Applying (2.24), we now obtain (2.22):

$$\int_{\mathcal{W}} y_1^2\mathbf{h}(y)^{-1}|\nabla_y\overline{\Psi}(y) - \overline{\nabla_y\Psi(y)}|^2 dy$$

$$\leq C\int_{\mathcal{W}} \mathbf{h}(y)^{-1}|\overline{\Psi}(y) - \Psi(y,-\mathbf{h}(y))|^2 dy \leq C\|\partial_z\Psi; L^2(\mathcal{V})\|^2. \quad \square$$

Proof of Lemma 2.1. Let $\Psi \in \mathcal{H}(\Omega,\Lambda)$ with $\|\Psi;\mathcal{H}(\Omega,\Lambda)\| \leq 1$. Since the function (2.12) does not depend on z, we have

$$(\nabla_x\Psi_j, \nabla_x\Psi)_\Omega - \lambda(\Psi_j,\Psi(\cdot,0))_\Lambda = (\mathbf{h}\nabla_y\Psi_j, \overline{\nabla_y\Psi})_{\mathcal{W}} - \lambda(\Psi_j,\Psi(\cdot,0))_{\mathcal{W}}$$

$$= \Big((\mathbf{h}\nabla_y\Psi_j, \nabla_y\overline{\Psi})_{\mathcal{W}} - \lambda(\Psi_j,\overline{\Psi})_{\mathcal{W}}\Big)$$

$$+ (\mathbf{h}\nabla_y\Psi_j, \overline{\nabla_y\Psi} - \nabla_y\overline{\Psi})_{\mathcal{W}} - \lambda(\Psi_j,\Psi(\cdot,0)-\overline{\Psi})_{\mathcal{W}}$$

$$=: I_1 + I_2 - I_3. \tag{2.25}$$

Estimating the addenda in (2.25), we start with I_3. The relation $\mathbf{h}(y) \geq cy_1^2$, (2.23) and the Cauchy–Schwartz inequality yields

$$|I_3| \leq ca_j\int_{A_j} y_1^{-1/2}|\Psi(y,0) - \overline{\Psi}(y)|dy$$

$$\leq ca_j\Big(\int_{A_j} dy\Big)^{1/2}\|y_1^{-1}(\overline{\Psi} - \Psi(\cdot,0)); L^2(\mathcal{W})\|$$

$$\leq c'a_j|A_j|^{1/2}\|\mathbf{h}^{-1/2}(\overline{\Psi} - \Psi(\cdot,0)); L^2(\mathcal{W})\|$$

$$\leq c2^{-j/2}r_j^{-1/2}r_j\|\nabla_x\Psi; L^2(\mathcal{V})\| \leq c2^{-j/2}r_j^{1/2} = c2^{-j/2}\exp(-2^{j-1}). \tag{2.26}$$

Using again (2.23) and (iii), Section 1, we get

$$|I_2| \leq ca_j\int_{A_j} \mathbf{h}(y)y_1^{-3/2}|\nabla_y\overline{\Psi}(y) - \overline{\nabla\Psi}(y)|dy$$

$$\leq ca_j\int_{A_j} y_1\mathbf{h}(y)^{-1/2}|\nabla_y\overline{\Psi}(y) - \overline{\nabla\Psi}(y)|dy$$

$$\leq c'a_j|A_j|^{1/2}\|y_1\mathbf{h}(y)^{-1/2}(\nabla_y\overline{\Psi}(y) - \overline{\nabla\Psi}(y)); L^2(\mathcal{W})\|$$

$$\leq c2^{-j/2}\exp(-2^{j-1}).$$

It suffices to examine the term I_1. It transforms into

$$\begin{aligned} I_1 &= \Big((\mathbf{h}\nabla_y\Psi_j, \nabla_y\overline{\Psi})_{\mathcal{W}} - \lambda(\Psi_j, \overline{\Psi})_{\mathcal{W}}\Big) \\ &= \int\limits_{A_j} \mathbf{h}(\nabla_y\Psi_j)\cdot\nabla_y\overline{\Psi}dy - (\lambda\Psi_j, \overline{\Psi})_{\mathcal{W}} \\ &= -\int\limits_{A_j} \big(\nabla_y\cdot\mathbf{h}\nabla_y\Psi_j + \lambda\Psi_j\big)\overline{\Psi}dy \end{aligned} \tag{2.27}$$

Since U solves (2.5), differentiation yields

$$\begin{aligned} &\nabla_y\cdot\mathbf{h}\nabla_y\Psi_j + \lambda\Psi_j \\ &= \mathbf{h}U(Y_j\Delta_yX_j + X_j\Delta_yY_j) + 2\mathbf{h}(Y_j\nabla_yX_j + X_j\nabla_yY_j)\cdot\nabla_yU \\ &+ U(Y_j\nabla_y\mathbf{h}\cdot\nabla_yX_j + X_j\nabla_y\mathbf{h}\cdot\nabla_yY_j) + \mathbf{h}U(\nabla_yX_j)\cdot\nabla_yY_j, \end{aligned} \tag{2.28}$$

where the last term vanishes since $\partial_{y_2}X_j = \partial_{y_1}Y_j = 0$.

We consider separately the terms

$$\begin{aligned} &\mathbf{h}UY_j\Delta_yX_j + 2Y_j\mathbf{h}\nabla_yX_j\cdot\nabla_yU + UY_j\nabla_y\mathbf{h}\cdot\nabla_yX_j \\ &= \mathbf{h}UY_j\partial_{y_1}^2X_j + 2Y_j\mathbf{h}(\partial_{y_1}X_j)\partial_{y_1}U + UY_j(\partial_{y_1}\mathbf{h})\partial_{y_1}X_j \end{aligned} \tag{2.29}$$

with derivatives of X_j only, and

$$\begin{aligned} &\mathbf{h}UX_j\Delta_yY_j + 2X_j\mathbf{h}\nabla_yY_j\cdot\nabla_yU + UX_j\nabla_y\mathbf{h}\cdot\nabla_yY_j \\ &= \mathbf{h}UX_j\partial_{y_2}^2Y_j + 2X_j\mathbf{h}(\partial_{y_2}Y_j)\partial_{y_2}U + UX_j(\partial_{y_2}\mathbf{h})\partial_{y_2}Y_j \end{aligned} \tag{2.30}$$

with derivatives of Y_j only.

Because of (iii) in Section 1, (2.10), and (2.13), the terms (2.29) are bounded by

$$\begin{aligned} &|\mathbf{h}UY_j\partial_{y_1}^2X_j| + |2Y_j\mathbf{h}(\partial_{y_1}X_j)\partial_{y_1}U| + |UY_j(\partial_{y_1}\mathbf{h})\partial_{y_1}X_j| \\ &\leq Ca_jy_1^2y_1^{-1/2}y_1^{-2} + Ca_jy_1^2y_1^{-1}y_1^{-3/2} + Ca_jy_1^{-1/2}y_1y_1^{-1} \leq C'a_jy_1^{-1/2} \end{aligned} \tag{2.31}$$

and similarly, for the terms of (2.30),

$$\begin{aligned} &|\mathbf{h}UX_j\partial_{y_2}^2Y_j| + |2X_j\mathbf{h}(\partial_{y_2}Y_j)\partial_{y_2}U| + |UX_j(\partial_{y_2}\mathbf{h})\partial_{y_2}Y_j| \\ &\leq Ca_jy_1^2y_1^{-1/2}r_j^{-2} + Ca_jy_1^2r_j^{-1}y_1^{-3/2} + Ca_jy_1^{-1/2}y_1r_j^{-1} \leq C'a_jy_1^{1/2}r_j^{-1}, \end{aligned} \tag{2.32}$$

since $r_j \geq y_1$ on A_j. Using the fact that the support of the derivatives of X_j is contained in $A_j \setminus B_j$ we estimate

$$\begin{aligned} |I_1| &\leq ca_j \int\limits_{A_j\setminus B_j} y_1^{-1/2}|\overline{\Psi}(y)|dy + ca_jr_j^{-1}\int\limits_{A_j} y_1^{1/2}|\overline{\Psi}(y)|dy \\ &\leq ca_j\Big(\Big(\int\limits_{A_j\setminus B_j} y_1^{-1}dy\Big)^{1/2} + r_j^{-1}\Big(\int\limits_{A_j} y_1dy\Big)^{1/2}\Big)\|\overline{\Psi}; L^2(\mathcal{W})\|. \end{aligned} \tag{2.33}$$

According to (2.23) and the definition (1.5) of the inner product in $\mathcal{H}(\Omega,\Lambda)$, we see that the last norm in (2.33) does not exceed

$$c\|\Psi(\cdot,0); L^2(\mathcal{W})\| + c\|\partial_z\Psi; L^2(\mathcal{V})\| \leq c\|\Psi; H^1(\Omega)\| \leq c.$$

Moreover,

$$\int_{A_j\setminus B_j} y_1^{-1}dy \le \int_{-r_j}^{r_j}\Big(\int_{r_j/2}^{r_j}\frac{1}{y_1}dy_1 + \int_{r_{j+1}}^{2r_{j+1}}\frac{1}{y_1}dy_1\Big)dy_2$$

$$= \int_{-r_j}^{r_j}\big(\log r_j - \log(r_j/2) + \log(2r_{j+1}) - \log r_{j+1}\big)dy_2 \le \int_{-r_j}^{r_j} Cr_j dy_2 \le Cr_j,$$

and

$$r_j^{-1}\Big(\int_{A_j} y_1 dy\Big)^{1/2} = r_j^{-1}\Big(\int_{-r_j}^{r_j}\int_{r_{j+1}}^{r_j} y_1 dy_1 dy_2\Big)^{1/2} \le Cr_j$$

hence, we find that $I_1 \le Ca_j r_j^{1/2} \le C'2^{-j/2}$, which is of the infinitesimal magnitude as $j \to +\infty$. This completes the proof of Lemma 2.1 and thus Theorem 1.1.

Acknowledgements. The authors want to thank Professor Sergei Nazarov for many discussion on the subject of this paper. This research was supported by the Academy of Finland project "Functional analysis and applications".

References

[1] Edwards, Penney, Elementary Differential Equations, 6th edition.
[2] Birman, M. S., Solomyak, M.Z., Spectral Theory of Self-Adjoint Operators in Hilbert Space. Reidel Publishing Company, Dordrecht, 1986.
[3] Johnson, R.S., A moderen introduction to the mahematical theory of water waves. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1997.
[4] Kuznetsov, N., Mazya, V., Vainberg, V., Linear water waves. A mathematical approach. Cambridge University Press, Cambridge, 2002.
[5] Ladyzhenskaya O.A., Boundary value problems of mathematical physics. Moscow: Nauka, 1973. 408 p.(English translation: The boundary value problems of mathematical physics. Translated from the Russian by Jack Lohwater. Applied Mathematical Sciences, 49. Springer-Verlag, New York, 1985.)
[6] Nazarov S.A. A general scheme for averaging self-adjoint elliptic systems in multidimensional domains, including thin domains, Algebra Analiz. 7,5 (1995), 1-92. (English transl.: St. Petersburg Math. 7,5 (1996), 681-748.)
[7] Nazarov S.A., Asymptotic Theory of Thin Plates and Rods. Vol.1. Dimension Reduction and Integral Estimates. Nauchnaya Kniga: Novosibirsk, 2001. (Russian)
[8] Nazarov, S.A., Concentration of the point spectrum on the continuous one in problems of linear water-wave theory, Zap. Nauchn. Sem. St.-Petersburg Otdel. Mat. Inst. Steklov. 348 (2007), 98–126. (English transl.: J. Math. Sci. 152 (2008), no.5, 674–689.)
[9] Nazarov S.A., On the essential spectrum of boundary value problems for systems of differential equations in a bounded domain with a cusp, Funkt. Anal. i Prilozhen 43 (2009), 55–67. (English transl.: Funct. Anal. Appl. 43 (2009), 44–54.)
[10] Nazarov, S.A., Taskinen, J., On the spectrum of the Steklov problem in a domain with a peak. Vestnik St.Petersburg Univ. 41 (2008), no. 4, 56–65.
[11] Nazarov, S.A., Taskinen, J., On essential and continuous spectra of the linearized water–wave problem in a finite pond. Math. Scand. 106, 1 (2010), 141–160.
[12] Nazarov, S.A., Taskinen, J., Radiation conditions at the top of a rotational cusp in the theory of water-waves. Math.Model.Numer.Anal. 45,4 (2011), 947–979.
[13] Nazarov, S.A., Sokolowski, J., Taskinen, J., Neumann Laplacian on a domain with tangential components in the boundary. Ann.Acad.Sci.Fenn. 34,1 (2009), 131–143.

[14] Stoker, J.J., Water waves. The mathematical theory with applications. Reprint of the 1957 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1992.

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