SPECTRUM OF THE LINEAR WATER MODEL FOR A TWO LAYER LIQUID WITH CUSPIDAL GEOMETRIES AT THE INTERFACE

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Abstract. We show that the linear water wave problem in a bounded liquid domain may have continuous spectrum, if the interface of a two-layer liquid touches the basin walls at zero angle. The reason for this phenomenon is the appearance of cuspidal geometries of the liquid phases. We calculate the exact position of the continuous spectrum. We also discuss the physical background of wave propagation processes, which are enabled by the continuous spectrum. Our approach and methods include constructions of a parametrix for the problem operator and singular Weyl sequences.

1. Introduction.

1.1. **Formulation of the problem and the main result.** In this paper we will consider water waves over a two-layer liquid and at its interface. For example, the setting could consist of warm/cold water or sweet/salted water in an infinitely long cylindrical container (pond or river) with a two-dimensional, bounded cross-section Ω as in Fig. 1.1. Studying waves which are constant in the direction of the cylinder axis, we pose in Ω the linear water-wave equation (1.5)–(1.9) with a spectral Steklov boundary condition. The geometric situation is such that the cross section Ω and the interface of the two liquid components bound a *cuspidal* subdomain of the water container. For comparison, in the case of a single layer liquid it is known [34] that cuspidal geometries may cause the appearance of continuous spectrum and wave processes even in bounded domains. In Theorem 1.1 we will show that under certain assumptions this also happens in the present, more complicated problem. In this case will also calculate the exact cut-off point of the continuous spectrum.

Let us describe the geometric setting, the equations and the main result in detail.

a) **Geometry of the liquid domain.** We assume that the cross-section $\Omega \subset \mathbb{R}^2$ of an infinitely long cylinder is bounded by the line segment (free water surface) $S = \{(y,z) \in \mathbb{R}^2 : z = 0, |y| < L\}$ and a smooth arc (walls and bottom) B connecting the points $(\pm L, 0)$ inside the lower half-plane $\mathbb{R}^2_-$. The domain Ω is divided into two parts

$$\Omega_0 = \{(y,z) \in \Omega : z > -d\} \text{ and } \Omega_1 = \{(y,z) \in \Omega : z < -d\} \tag{1.1}$$

at the level $z = -d \in (0, -b_0)$, where $-b_0$ is the smallest z-coordinate of the points of B. By rescaling, we reduce the half-length L to 1 and thus make the Cartesian coordinates (y,z) and all geometric parameters dimensionless. The subdomains Ω_0 and Ω_1 are filled by two immiscible liquids, which have densities ϱ_0 and ϱ_1, respectively, and we assume that $\varrho_1 > \varrho_0 > 0$ to obtain gravitational stability.

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FIGURE 1.1. Two-layer liquid with cuspidal geometries at the interface.

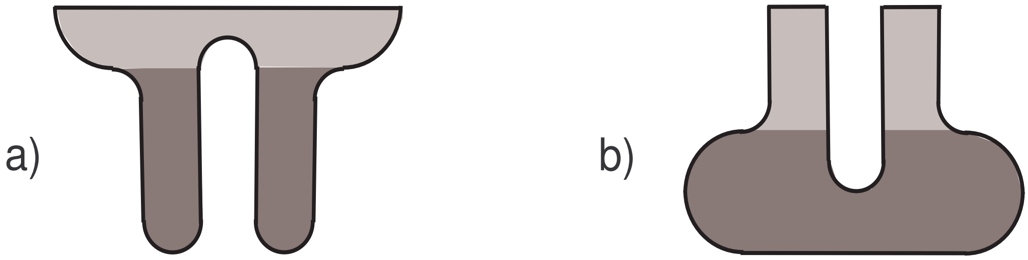

FIGURE 1.2. Alternative geometries.

The liquid interface $I = \partial\Omega_0 \cap \partial\Omega_1$ is assumed to be a segment $\{(y,z) : y \in [-l_-, l_+], z = -d\}$ of length $l = l_+ + l_- > 0$, however, our main results remain valid also in the situations of Fig. 1.2 a, b.

The cuspidal geometry is related to the *interface* I as follows: we assume that the curve B is given in the vicinity of the points $P^\pm = (\pm l_\pm, -d) \subset I$ by the equations

$$z = -d + h_\pm(y), \tag{1.2}$$

where

$$h_\pm(\pm l_\pm) = \partial_y h_\pm(\pm l_\pm) = 0 \ , \quad b_\pm := \partial_y^2 h_\pm(\pm l_\pm) \neq 0. \tag{1.3}$$

In other words, B is tangential with the segment I at its endpoints and therefore one of the domains Ω_j loses the Lipschitz property. However, no cuspidal geometry is supposed to appear in the vicinity of the free water surface: the curve B is defined to intersect the y-axis in the points $(\pm L, 0)$ at the angles $\theta_\pm \in (0, \pi)$, see Fig. 1.1 a and b. Hence, the domains Ω_0 in Fig. 1.1 a and Ω_1 in Fig. 1.1 b are Lipschitz while, respectively, Ω_1 and Ω_0 are not.

b) **Water-wave equations.** We deal with time harmonic liquid motion, more precicely, waves which are independent on the direction of the axis of the infinite cylinder. We assume that the liquid motion is irrotational and of small amplitude, and introduce the velocity potentials

$$\phi_j(y,z,t) = \mathrm{Re}\big(e^{-i\omega t}\varphi_j(y,z)\big) \ , \ j = 0, 1, \tag{1.4}$$

where φ_j satisfy the Laplace equation in their respective domains, that is,

$$-\varrho_j \Delta\varphi_j(y,z) = 0 \quad (y,z) \in \Omega_j, \ j = 0, 1. \tag{1.5}$$

This equation is supplied with the traditional spectral boundary condition of Steklov type at the free surface S,

$$\varrho_0 \partial_z \varphi_0(y, 0) = \varrho_0 \lambda \varphi_0(y, 0) \ , \quad |y| < L, \tag{1.6}$$

and with the spectral transmission condition at the interface I,

$$\begin{aligned} &\varrho_0\big(\partial_z\varphi_0(y,-d)-\lambda\varphi_0(y,-d)\big)\\ &=\varrho_1\big(\partial_z\varrho_1(y,-d)-\lambda\varphi_1(y,-d)\big)\ ,\quad y\in(-l_-,l_+). \end{aligned} \tag{1.7}$$

Here, $\lambda = g^{-1}\omega^2$, and the frequency ω is as in (1.4) and g is the acceleration of gravity. A physical interpretation of the conditions (1.6) and (1.7) can be found in [13] and, e.g., [11, 37].

The wetted surfaces $B_0 = \{(y,z) \in B : z \in (-d,0)\}$ and $B_1 = \{(y,z) \in B : z < -d\}$ are supplied with the Neumann boundary condition (no normal flow)

$$\partial_n\varphi_j(y,z) = 0\ ,\quad (y,z)\in B_j,\ j=0,1, \tag{1.8}$$

and the normal velocity is assumed continuous at the interface,

$$\partial_z\varphi_0(y,-d) = \partial_z\varphi_1(y,-d)\ ,\quad y\in(-l_-,l_+). \tag{1.9}$$

The outward normal derivative is denoted by ∂_n, and $\partial_n = \partial_z = \partial/\partial z$ on the horizontal surfaces S and I.

The main result of our paper reads as:

Theorem 1.1. *Assume that the water domain has the cuspidal geometries at the interface as described above. Then the water wave problem* (1.5)–(1.9) *for the two-layer liquid has the continuous spectrum*

$$\sigma_c = [\lambda_\dagger, +\infty) \tag{1.10}$$

with the cut-off value

$$\lambda_\dagger = \frac{1}{4}\min\{b_+,b_-\}\frac{1}{\varrho_j}(\varrho_1-\varrho_0), \tag{1.11}$$

where $j = 1$ (respectively, $j = 0$) in the situation of Fig. 1.1 a (resp. b). The interval $[0, \lambda_\dagger)$ contains the discrete spectrum, and in particular $\lambda = 0$ is an eigenvalue of multiplicity 2.

Theorem 1.1 will be proven in two steps. First, in Section 2 we introduce the trace/interface operator $\mathcal{T}$, which is a self-adjoint operator in Hilbert space. Using this operator we can give an exact meaning for the spectral concepts and use the machinery of operator theory in Sobolev spaces. In particular, we will use the statement of Proposition 2.2., which contains accurate information on the embedding constant of $H^1(\Omega_J) \subset L^2(I)$, to verify that $[0, \lambda_\dagger)$ belongs to the regularity field of the problem. Furthermore, this result will be used at the end of Section 2.3 to prove the statement about the discrete spectrum in Theorem 1.1.. In Section 3 we then investigate wave phenomena in the cusps and construct singular Weyl sequences for $\mathcal{T}$ in the case $\lambda \geq \lambda_\dagger$, thus showing that $\sigma_c = [\lambda_\dagger, +\infty)$. These steps will complete the proof of the main theorem.

1.2. **Review of the main result.** If the curve B touched the end points of the segments S and I at non-zero angles, then both domains Ω_j would be Lipschitz, and the embedding of the Sobolev space into Lebesgue trace spaces would be compact. In this case the spectrum of the problem (1.5)–(1.9) would be discrete and it would form an unbounded monotone sequence of (normal) eigenvalues

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots \to +\infty \tag{1.12}$$

FIGURE 1.3. Wedge domain and cuspidal domain.

(see Section 2.2 for details). However, under the conditions (1.2) and (1.3), the mentioned compactness of the Sobolev embedding is lost, and we will prove that the spectrum of the problem (1.5)–(1.9) is no longer discrete and has the continuous component $\sigma_c = [\lambda_\dagger, +\infty)$ with a positive cut-off $\lambda_\dagger$ dependent on the problem data ϱ_0, ϱ_1 and $b_\pm$. The appearance of the continuous spectrum (1.10) enables wave processes in a finite water volume. Describing wave propagation near the point $P^\pm$ and in particular imposing appropriate radiation conditions at these points becomes a challenge, which is postponed to a planned forthcoming paper.

Investigation of the linear water-wave problems for beaches and shore areas were initiated by Stokes [41], who observed that that the unbounded wedge, Fig. 1.3 a, supports a trapped surface wave. Later this classical result was improved in [38], where infinitely many trapped modes were detected in the same angular domain. Other geometric shapes of the underwater shore topography were considered in the paper [6], which also proves that in the cuspidal domain

$$\begin{aligned} &\{ \, (y,z) \, : \, |y| < L, 0 > z > -h(y)\}, \\ &h(y) = |y \mp L|^{\varkappa}\big(h_\pm + O(|y \mp L|)\big) \text{ for } y \to \pm L \mp 0, \end{aligned} \tag{1.13}$$

the conditions $h_\pm > 0$ and $\varkappa \in (1,2)$ imply that the Steklov spectrum is fully discrete. On the other hand, the papers [21, 22, 34] contain studies of the case $\varkappa \geq 2$ and proofs for the existence of a nontrivial continuous component of the spectrum in the case of a homogeneous liquid.

Other geometric shapes related to cuspidal boundary irregularities were examined in [8, 30, 34, 35]. Submerged bodies approaching the water surface may create cuspidal irregularities at the limit, and related asymptotic phenomena were described and studied rigorously in [9, 30]

In all of the above-mentioned geometric settings, cf. Fig. 1.1, 1.2, 1.3 b, there appear cuspidally thin water layers, and this contradicts in some sense with the standard assumptions, which are posed when deriving the linear water wave equations. The same problem appears for "black holes" for elastic and acoustic waves in solids with cuspidally irregular surfaces, see [3, 10, 25] for engineering aspects of the problem and [2, 28] for rigorous mathematical results. However, the effects of continuous spectrum to wave processes in bounded elastic cuspidal bodies have been justified experimentally, see Fig. 1.4. (The photo is given to the authors by M. A. Mironov and is published by his permission.) The authors thus expect that certain consequences of wave processes studied in this papers may be observed experimentally in finite water volumes, too. We mention that an underwater topography like in Fig. 1.1 a, 1.2 a, 1.3 b, can occur for example in Siberian rivers and Lapland lakes in summer, when the upper layer of the water is warmed up by the sunshine, but the lower layer is kept cold by the permafrost.

The first results about trapped modes in two layer fluids in two-dimensional channels of finite or infinite depth were published without proofs in [11, 12], where formal asymptotics of surface and interfacial waves was constructed in case $\varrho_0/\varrho_1 \to 1 - 0$.

FIGURE 1.4. Cuspidal solid metal body.

Later, in [18], the authors published computational results concerning trapped modes, which are supported by circular horizontal cylinders submerged either in the lower or upper layer. They made the very important observation that only interfacial waves may lead to trapped modes with frequencies below the continuous spectrum, i.e., frequencies belonging to the discrete spectrum. The uniqueness question for solutions in the case of a two layer liquid was studied in [14], and examples of two-dimensional obstacles supporting trapped modes were also given there. A simple sufficient condition, including a geometric integral, for the existence of trapped modes was derived in [37]. Rigorous asymptotic analysis in the cases $\varrho_0/\varrho_1 \to 1-0$ and $\varrho_0/\varrho_1 \to +0$ and corresponding error estimates are contained in [31, 36].

We also mention the papers [4, 17, 19, 40] which contain studies on wave scattering by submerged obstacles in two-layer fluids, and also the paper [7], where even a three-layer fluid was considered.

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2. Variational and operator formulation of the problem.

In this section we present the operator theoretic approach to the problem (1.5)–(1.9), which we also use to give an exact meaning for the spectral concepts of Theorem 1.1. At the end of the Section 2.3 we prove the result on the discrete spectrum as a consequence of Proposition 2.3.

2.1. **Variational formulation.**

We write the problem (1.5)-(1.9) in variational form in the sense of [16, 20]. According to [37], this formulation reads in the present setting as the equation

$$D(\varphi,\psi) = \lambda T(\varphi,\psi) \quad \forall \psi, \tag{2.1}$$

where

$$\begin{aligned} D(\varphi,\psi) &= \varrho_0(\nabla\varphi_0,\nabla\psi_0)_{\Omega_0} + \varrho_1(\nabla\varphi_1,\nabla\psi_1)_{\Omega_1} \\ T(\varphi,\psi) &= \varrho_0(\varphi_0,\psi_0)_S + \frac{1}{\varrho_1-\varrho_0}(\varrho_1\varphi_1-\varrho_0\varphi_0, \varrho_1\psi_1-\varrho_0\psi_0)_I. \end{aligned} \tag{2.2}$$

and the functions $\psi = (\psi_0,\psi_1)$ are test functions with components $\psi_j \in C_c^\infty(\overline{\Omega}_j \setminus \mathcal{P})$. (This is the space of functions that are smooth in the closure $\overline{\Omega}_j$ but vanish in a neighborhood of the set $\mathcal{P}$, and $\mathcal{P}$ is the set of the corner/cuspidal points marked with $\bullet$ in Fig. 1.1.) The relation (2.1) is obtained by multiplying the Laplace-equations (1.5) with a test function and using the Green formula, the boundary conditions (1.6), (1.8) and the transmission conditions (1.7), (1.9), and by rewriting (1.7), (1.9) as

$$\partial_z \varphi_0 = \partial_z \varphi_1 = \frac{\lambda}{\varrho_1 - \varrho_0}(\varrho_1 \varphi_1 - \varrho_0 \varphi_0) \text{ on } I.$$

To complete the formulation (2.1) we need to choose a Hilbert space $\mathcal{H}$ containing the functions φ and ψ. Since the bilinear forms (2.2) are positive, $\mathcal{H}$ is defined as the completion of $C_c^\infty(\overline{\Omega}_0 \setminus \mathcal{P}) \times C_c^\infty(\overline{\Omega}_1 \setminus \mathcal{P})$ with respect to the intrinsic norm

$$\|\varphi; \mathcal{H}\| = \big(D(\varphi, \varphi) + T(\varphi, \varphi)\big)^{1/2}. \tag{2.3}$$

Lemma 2.1. *The norm* (2.3) *is equivalent to the Sobolev norm*

$$\|\varphi; H^1(\Omega_0 \times \Omega_1)\| = \Big(\sum_{j=0,1} \|\nabla \varphi_j; L^2(\Omega_j)\|^2 + \|\varphi_j; L^2(\Omega_j)\|^2 \Big)^{1/2}.$$

Proof. We need to prove that the pair of inequalities

$$\|\varphi; \mathcal{H}\| \le c\|\varphi; H^1(\Omega_0 \times \Omega_1)\| \le C\|\varphi; \mathcal{H}\| \tag{2.4}$$

holds for some constants c, $C > 0$. In view of the trace inequality

$$\|\varphi_j; L^2(\partial\Omega_j)\| \le c_j \|\varphi_j; H^1(\Omega_j)\|, \tag{2.5}$$

the left inequality in (2.4) is evident. Notice that if the domain Ω_j is Lipschitz ($j = 0$ in Fig. 1.1 a and $j = 1$ in Fig. 1.1 b), the inequality (2.5) is classical (see e.g. [1]), but, for a domain with cuspidal boundaries ($j = 1$ in Fig. 1.1 a and $j = 0$ in Fig. 1.1 b), this is proved in [24]. The proof is based on the assumption (1.3), which means that B converges to the line $\{(y, z) : z = -d\}$ at a rate of order $|z + d|^2$ (see also [33] and Proposition 2.2 for details on the embedding constant).

To verify the right inequality in (2.4) we employ a theorem on equivalent norms, namely

$$\|\varphi_j; H^1(\Omega_j)\| \le C_j\big(\|\nabla\varphi_j; L^2(\Omega_j)\| + F_j(\varphi_j)\big), \tag{2.6}$$

where F_j can be any continuous functional such that

$$F_j(tv) = |t|F_j(v) \ge 0 \text{ and } F_j(1) > 0 \text{ for } t \in \mathbb{R},\ v \in H^1(\Omega_j), \tag{2.7}$$

since the seminorm $\|\nabla\varphi_j; L^2(\Omega_j)\|$ vanishes only on constant functions. (The inequality (2.6) is well-known for domains with, say, cone property, cf. [23], Section 1.1.16, but it holds also true in the present case, due to the continuous embedding $H^1(\Omega_j) \subset L^2(\partial\Omega_j)$, cf. [33], Section 2.)

We set $F_0(\varphi_0) = \|\varphi_0; L^2(S)\|$, and obtain from (2.2), (2.3)

$$\|\varphi_0; H^1(\Omega_0)\| \le C\|\varphi; \mathcal{H}\| \tag{2.8}$$

Next we observe that the triangle inequality implies

$$\begin{aligned}
\|\varphi_1; L^2(I)\| &\le \frac{1}{\varrho_1}\|\varrho_1\varphi_1 - \varrho_0\varphi_0; L^2(I)\| + \frac{1}{\varrho_1}\|\varrho_0\varphi_0; L^2(I)\| \\
&\le C\big(T(\varphi,\varphi)^{1/2} + \|\varphi_0; L^2(I)\|\big) \\
&\le C'\big(T(\varphi,\varphi)^{1/2} + \|\varphi_0; H^1(\Omega_0)\|\big) \le C''\|\varphi; \mathcal{H}\|, \qquad (2.9)
\end{aligned}$$

where also (2.5) with $j = 0$ and (2.8) were used. Now (2.6), with $F_1(\varphi_1) = \|\varphi_1; L^2(I)\|$, and (2.9) yield

$$\|\varphi_1; H^1(\Omega_1)\| \leq \|\nabla\varphi_1; L^2(\Omega_1)\| + \|\varphi_1; L^2(I)\| \leq C\|\varphi; \mathcal{H}\|,$$

hence, this and (2.8) complete the proof. ⊠

2.2. **Trace/interface operator.** The Hilbert space $\mathcal{H}$ can be endowed with the scalar product

$$\langle\varphi, \psi\rangle = D(\varphi, \psi) + T(\varphi, \psi). \tag{2.10}$$

We define the operator $\mathcal{T}$ in $\mathcal{H}$ by the formula

$$\langle\mathcal{T}\varphi, \psi\rangle = T(\varphi, \psi) \quad \forall\varphi, \psi \in \mathcal{H}. \tag{2.11}$$

It is continuous, positive and symmetric, hence, self-adjoint. Moreover, its norm is equal to 1 and $\mu = 0$ is an eigenvalue of infinite multiplicity, having the eigenspace

$$\{\varphi \in \mathcal{H} \,:\, \varphi = 0 \text{ on } S,\ \varrho_1\varphi_1 = \varrho_0\varphi_0 \text{ on } I\}. \tag{2.12}$$

By (2.10) and (2.11), the integral identity (2.1) can be written as the abstract equation

$$\mathcal{T}\varphi = \mu\varphi \ \text{ in } \mathcal{H} \tag{2.13}$$

with a spectral parameter

$$\mu = (1 + \lambda)^{-1}. \tag{2.14}$$

The operator $\mathcal{T}$ is called the *trace/interface operator*, cf. [29, 37]. If both Ω_0 and Ω_1 were Lipschitz (which is not the case in our paper), $\mathcal{T}$ would be compact due to the compact embedding $H^1(\Omega_j) \subset L^2(\partial\Omega_j)$, and hence Theorems 10.1.5 and 10.2.2 of [5] would ensure that its spectrum Σ would consist of the essential spectrum $\Sigma_e = \{0\}$ (see (2.12)) and the discrete spectrum Σ_d , which is the convergent sequence

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_k \geq \ldots \to +0. \tag{2.15}$$

The sequence (2.15) corresponds via the relation (2.14) to the eigenvalue sequence (1.12) for the problem (1.5)–(1.9), or for (2.1). However, $\mu = 0$ corresponds to the infinity point and does not influence the purely discrete spectrum σ. Note that the lowest eigenvalue λ_1 is null so that $\mu_1 = 1$ and, hence, the norm of $\mathcal{T}$ is indeed equal to 1.

In the case of a cuspidal boundary the operator $\mathcal{T}$ is no more compact, and by Theorem 10.1.5 of [5], its essential spectrum cannot consist of the point $\mu = 0$ only. Description of the sets Σ_e and σ_e becomes our main objective.

2.3. **Regularity field of $\mathcal{T}$.** Since $\mathcal{T}$ is self-adjoint and positive, it has a continuous resolvent $(\mathcal{T} - \mu)^{-1}$ for any $\mu \in \mathbb{C} \setminus [0, +\infty)$. In this section we will verify that the resolvent is also Fredholm for

$$\mu \in (\mu_\dagger, +\infty), \tag{2.16}$$

and we also compute the threshold $\mu_\dagger \in (0, 1)$. In this way we prove that $\Sigma_e \subset [0, \mu_\dagger]$ and, by (2.14), that $\sigma_e \subset [\lambda_\dagger, +\infty)$, where

$$\lambda_\dagger = \mu_\dagger^{-1} - 1 > 0. \tag{2.17}$$

In Section 3.3 we show that Σ_e coincides with $[0, \mu_\dagger]$ by constructing a singular Weyl sequence for $\mathcal{T}$ for any $\mu \in (0, \mu_\dagger]$. This last step completes the proof of the formulas (1.10), (1.11) and Theorem 1.1.

We need the following assertion, which was proven in [33].

Proposition 2.2. *Let Ω_j be as above, in particular assume that* (1.3) *holds in a neighbourhood of the points $P^{\pm}$. For all $u_j \in H^j(\Omega_j)$, $j = 0, 1$, and for all $\varepsilon > 0$ we have the inequality*

$$\Big(\frac{b}{4} - \varepsilon\Big)\|u_j; L^2(\partial\Omega_j)\|^2 \le \|\nabla u_j; L^2(\Omega_j)\|^2 + C_j(\varepsilon)\|u_j; L^2(\Omega_j)\|^2, \tag{2.18}$$

where $b = \min\{|b_-|, |b_+|\}$ and $0 < C_j(\varepsilon) \to +\infty$ as $\varepsilon \to +0$.

We modify the variational problem (2.1) as

$$D(\varphi, \psi) + \kappa(\varphi, \psi)_\Omega - \lambda T(\varphi, \psi) = f(\psi) \quad \forall \psi \in \mathcal{H}, \tag{2.19}$$

where $\lambda \in [0, \lambda_\dagger)$, κ is a large positive constant to be fixed later, and f belongs to the dual space $\mathcal{H}^*$, i.e., f is a continuous linear functional on $\mathcal{H}$. We consider the situation of Fig. 1.1 b and apply (2.18) with $j = 0$ as well as the inequalities

$$\|\varphi_0; L^2(S)\|^2 \le \delta_0\|\nabla\varphi_0; L^2(\Omega_0)\|^2 + c_0(\delta_0)\|\varphi_0; L^2(\Omega_0)\|^2, \tag{2.20}$$

$$\|\varphi_1; L^2(I)\|^2 \le \delta_1\|\nabla\varphi_1; L^2(\Omega_1)\|^2 + c_1(\delta_1)\|\varphi_1; L^2(\Omega_1)\|^2, \tag{2.21}$$

where $\delta_j > 0$ is arbitrary and $c_j(\delta_j) \to +\infty$ as $\delta_j \to +0$. These inequalities imply the compactness of the embeddings $H^1(\Omega_0) \subset L^2(S)$ and $H^1(\Omega_1) \subset L^2(I)$, as is well known e.g. by [1, 16]. We emphasize that the boundary of Ω_0 in Fig. 1.1 b is not Lipschitz, but (2.20) is still true, because $\partial\Omega_0$ is Lipschitz in a neighbourhood of the free surface $\overline{S}$, due to the assumptions below (1.3).

By the definition of T we obtain, for arbitrary positive τ and ε, δ_j,

$$\begin{aligned}
&T(\varphi, \varphi) \le \varrho_0\|\varphi_0; L^2(S)\|^2 \\
&\quad + \frac{1}{\varrho_1 - \varrho_0}\Big((1+\tau)\varrho_0^2\|\varphi_0; L^2(I)\|^2 + \Big(1 + \frac{4}{\tau}\Big)\varrho_1^2\|\varphi_1; L^2(I)\|^2\Big) \\
&\le \varrho_0\Big(\delta_0\|\nabla\varphi_0; L^2(\Omega_0)\|^2 + c_0(\delta_0)\|\varphi_0; L^2(\Omega_0)\|^2\Big) \\
&\quad + \frac{\varrho_0^2}{\varrho_1 - \varrho_0}(1+\tau)\Big(\frac{1}{4}b - \varepsilon\Big)^{-1}\big(\|\nabla\varphi_0; L^2(\Omega_0)\|^2 + C_0(\varepsilon)\|\varphi_0; L^2(\Omega_0)\|^2\big) \\
&\quad + \frac{\varrho_1^2}{\varrho_1 - \varrho_0}\Big(1 + \frac{4}{\tau}\Big)\Big(\delta_1\|\nabla\varphi_1; L^2(\Omega_1)\|^2 + c_1(\delta_1)\|\varphi_1; L^2(\Omega_1)\|^2\Big) \\
&\le K_0^1\|\nabla\varphi_0; L^2(\Omega_0)\|^2 + K_0^0\|\varphi_0; L^2(\Omega_0)\|^2 \\
&\quad + K_1^1\|\nabla\varphi_1; L^2(\Omega_1)\|^2 + K_1^0\|\varphi_1; L^2(\Omega_1)\|^2,
\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
K_0^1 &= \frac{\varrho_0^2}{\varrho_1 - \varrho_0}(1+\tau)\Big(\frac{1}{4}b - \varepsilon\Big)^{-1} + \varrho_0\delta_0, \\
K_0^0 &= \frac{\varrho_0^2}{\varrho_1 - \varrho_0}(1+\tau)C_0(\varepsilon)\Big(\frac{1}{4}b - \varepsilon\Big)^{-1} + \varrho_0 c_0(\delta_0), \\
K_1^1 &= \frac{\varrho_1^2\delta_1}{\varrho_1 - \varrho_0}\Big(1 + \frac{4}{\tau}\Big), \quad K_1^0 = \frac{\varrho_1}{\varrho_1 - \varrho_0}\Big(1 + \frac{4}{\tau}\Big)c_1(\delta_1).
\end{aligned} \tag{2.23}$$

Since $\lambda < \lambda_\dagger$ and

$$K_0^1 = \frac{4\varrho_0^2}{b(\varrho_1 - \varrho_0)} = \varrho_0\lambda_\dagger^{-1} \quad \text{for } \varepsilon = \tau = \delta_0 = 0,$$

we can select ε, τ and δ_0 so small that

$$\lambda K_0^1 = \varrho_0(1-a) \tag{2.24}$$

for some small positive a. After fixing τ we choose δ_1 such that

$$\lambda K_1^1 \leq \varrho_1(1-a). \tag{2.25}$$

Finally, we find a big number κ satisfying

$$\kappa \geq \lambda \max\{K_0^0, K_1^0\} + \max\{\varrho_0, \varrho_1\} \geq \lambda \max\{K_0^0, K_1^0\} + a. \tag{2.26}$$

From (2.22) and (2.24)–(2.26) we derive the inequality

$$\begin{aligned} &D(\varphi,\varphi) + \kappa(\varphi,\varphi)_\Omega - \lambda T(\varphi,\varphi) \\ &\geq a \sum_{j=0,1} \varrho_j \big(\|\nabla \varphi_j; L^2(\Omega_j)\|^2 + \|\varphi_j; L^2(\Omega_j)\|^2 \big). \end{aligned} \tag{2.27}$$

Hence, all assumptions of the Lax-Milgram lemma are met, and thus, for all $f \in \mathcal{H}^*$, there exists a unique solution $\varphi \in \mathcal{H}$ of the problem (2.19) satisfying

$$\|\varphi; \mathcal{H}\| \leq c(\lambda) \|f; \mathcal{H}^*\|. \tag{2.28}$$

The factor $c(\lambda)$ is of course independent of f, although it depends on the spectral parameter $\lambda \in (0, \lambda_\dagger)$, and it may happen that

$$c(\lambda) \to +\infty \quad \text{as } \lambda \to \lambda_\dagger - 0. \tag{2.29}$$

We will see that (2.29) indeed holds true.

The above consideration concerns the situation of Fig. 1.1 b, but it is valid also in the case of Fig. 1.1 a after obvious modifications. We formulate this as the following result, where the number κ is chosen in (2.23)–(2.26).

Proposition 2.3. *Assume that the geometry of the water domain is as described in Section 1.1, let $\lambda_\dagger$ be as in* (1.11) *and $\lambda \in [0, \lambda_\dagger)$. There exists $\kappa > 0$ such that the variational problem* (2.19) *is uniquely solvable for any $f \in \mathcal{H}^*$ and the solution satisfies the estimate* (2.28).

We finally show how the statement on the discrete spectrum in Theorem 1.1 follows from this proposition. Analogously to (2.10) and (2.11), we introduce a new scalar product

$$\langle \varphi, \psi \rangle_\kappa = \langle \varphi, \psi \rangle + \kappa(\varphi, \psi)_\Omega \tag{2.30}$$

in the Hilbert space $\mathcal{H}$, and the new continuous, positive and self-adjoint operator $\mathcal{T}_\kappa$,

$$\langle \mathcal{T}_\kappa \varphi, \psi \rangle_\kappa = T(\varphi, \psi) \quad \forall \varphi, \psi \in \mathcal{H}. \tag{2.31}$$

Proposition 2.3 means that for any $\mu \in (\mu_\dagger, 1]$ with $\mu_\dagger \in (0,1)$ and $\lambda_\dagger > 0$ as in (2.17) and (1.11), the operator $\mathcal{T}_\kappa - \mu$ is an isomorphism in $\mathcal{H}$. Since the scalar products (2.17) and (1.11) differ from each other by the L^2-scalar product only, the difference $\mathcal{T} - \mathcal{T}_\kappa$ is a compact operator. (Note that the embedding $\mathcal{H} \subset L^2(\Omega)$ is compact, since $\mathcal{H}$ is just the Sobolev space, by Lemma 2.1.) Thus, $\mathcal{T}$ is Fredholm and the half-open interval $(\mu_\dagger, 1]$ belongs to the regularity field of $\mathcal{T}$. By the relation (2.14) of the spectral parameters, the interval $[0, \lambda_\dagger)$ is free of the continuous spectrum of the problem (1.5)–(1.9) but includes its discrete spectrum. For example, $\lambda = 0$ is always an eigenvalue of multiplicity 2 and its eigenspace is spanned by piecewise constant functions $\varphi = \{c_0, c_1\}$ in $\Omega_0 \cup \Omega_1$.

3. Wave phenomena near the cuspidal irregularities.

We start this section by constructing formal asymptotics of solutions of the spectral problem. This asymptotic analysis is justified and the proof of Theorem 1.1 is completed in Section 3.3, while Section 3.2 contains a discussion of wave processes in the cuspidal domain.

3.1. **Formal asymptotic analysis.** We first consider the point P^- in the situation of Fig. 1.1 b. Other cases can be treated in the same way, and we will outline them at the end of this section. Placing the coordinate origin so that $-l_- = 0$, the curve B_0 is defined near P^- by the equation

$$z - d = h(y); \tag{3.1}$$

here and later in this section we omit the index "$-$" from h, thus, owing to (1.3) we have

$$h(0) = \partial_y h(0) = 0\ ,\quad h(y) = by^2 + O(|y|^3)\ ,\quad b > 0. \tag{3.2}$$

The cuspidal part of the domain Ω_0 is rapidly thinning when $y \to +0$, and hence we set for the solution $\varphi = \{\varphi_0, \varphi_1\}$ the asymptotic ansätze

$$\varphi_0(y,z) = \phi_0^0(y) + h(y)^2\phi_0^2(y,\zeta) + \ldots, \tag{3.3}$$

$$\varphi_1(y,z) = H(y,z)\phi_1^1(y,z) + \ldots. \tag{3.4}$$

These are suitable ansätze for solutions of elliptic transmission problems in domains with thin coatings, cf. [26, 27]. The functions ϕ_q^p are to be determined, dots stand for higher order terms inessential for our formal analysis, $H(y,z)$ is an extension of $h(y)$ to the cusp such that $\partial_z H(y,-d) = 0$, and

$$\zeta = h(y)^{-1}(z + d) \tag{3.5}$$

is a stretched coordinate (a rapid variable in the cusp of Ω_0).

We insert (3.3) and (3.4) into the problem (1.5)–(1.9) and collect the coefficients of the derivatives of h having equal powers. The procedure will be justified rigorously later.

First, since $\partial_z = h(y)\partial_\zeta$, we observe that

$$\partial_z\varphi_0(y-d) = h(y)\partial_\zeta\phi_0^2(y,0) + \ldots \tag{3.6}$$

and hence the transmission conditions (1.9) and (1.7) are converted into

$$\partial_\zeta\phi_0^2(y,0) = \partial_z\phi_1^1(y,0) \tag{3.7}$$

$$\varrho_0\big(h(y)\partial_\zeta\phi_0^2(y,0) - \lambda\phi_0^0(y)\big) = \varrho_1 h(y)\big(\partial_z\phi_1^1(y,-d) - \lambda\phi_1^1(y,-d)\big). \tag{3.8}$$

Assuming that $\phi_1^1(y,-d)$ is small in comparison with the derivative $\partial_z\phi_1^1(y,-d)$ (see (3.26) for explanation), we obtain from (3.7) and (3.8) by omitting the term $\lambda\phi_1^1(y,-d)$

$$-\partial_\zeta\phi_0^2(y,0) = G_0(y) := \varrho_0(\varrho_1 - \varrho_0)^{-1}h(y)^{-1}\lambda\phi_0^0(y,0). \tag{3.9}$$

On the curve (3.1) the normal derivative equals

$$\partial_n = \big(1 + |\partial_y h(y)|^2\big)^{-1/2}\big(\partial_z - \partial_y h(y)\partial_y\big) = h(y)^{-1}\partial_\zeta - \partial_y h(y)\partial_y + \ldots, \tag{3.10}$$

hence, the Neumann boundary condition (1.8) on B^0 turns into

$$\partial_\zeta\phi_0^2(y,1) = G_1(y) := h(y)^{-1}\partial_y h(y)\partial_y\phi_0^0(y). \tag{3.11}$$

Finally, the derivative $\partial_z^2 = h(y)^{-2}\partial_\zeta^2$ prevails over ∂_y^2 near the cusp tip, and hence, the equation (1.5), $j = 0$, means that the function ϕ_0^2 must satisfy the following ordinary differential equation in the rapid variable $\zeta \in (0,1)$:

$$-\partial_\zeta^2 \phi_0^2(y,\zeta) = F(y) := \partial_y^2 \phi_0^0(y). \tag{3.12}$$

The compatibility condition in the Neumann problem (3.12), (3.11), (3.9) reads as

$$0 = \int_0^1 F(y,\zeta)d\zeta + G_1(y) + G_0(y). \tag{3.13}$$

A simple calculation reduces (3.13) to an ordinary differential equation in the variable y,

$$-h(y)\partial_y^2\phi_0^0(y) - \partial_y h(y)\partial_y\phi_0^0(y) = \varrho_0(\varrho_1-\varrho_0)^{-1}\lambda\phi_0^0(y).$$

Replacing here $h(y)$ by its leading term by^2 as in (3.2) yields the Euler-type differential equation

$$-\partial_y\big(y^2\partial_y\phi_0^0(y)\big) = \frac{\lambda\varrho_0}{b(\varrho_1-\varrho_0)}\phi_0^0(y)\ , \quad y\in\mathbb{R}_+. \tag{3.14}$$

Setting

$$\lambda_\dagger = \frac{1}{4}b\Big(\frac{\varrho_1}{\varrho_0}-1\Big), \tag{3.15}$$

cf. (1.11), we can write the general solution of (3.14):

$$\phi_0^0(y) = C_+y^{-1/2+\varkappa} + C_-y^{-1/2-\varkappa} \quad \text{for } \lambda<\lambda_\dagger, \tag{3.16}$$

$$\phi_0^0(y) = C_+y^{-1/2+i\varkappa} + C_-y^{-1/2-i\varkappa} \quad \text{for } \lambda>\lambda_\dagger, \tag{3.17}$$

$$\phi_0^0(y) = y^{-1/2}(C_0 + C_1\ln y) \qquad \text{for } \lambda=\lambda_\dagger, \tag{3.18}$$

where

$$\varkappa = \Big|\frac{1}{4} - \frac{\lambda\varrho_0}{b(\varrho_1-\varrho_0)}\Big|^{1/2}. \tag{3.19}$$

We are now in the position to construct the main asymptotic term in (3.4),

$$\phi_1^0(y,z) = H(y,z)\phi_1^1(y,z). \tag{3.20}$$

Since the boundary of Ω_1 is flattening near the point P^-, we search for the function as a solution of the Laplace equation (cf. (1.5)) in the lower half-plane $\mathbb{R}^2_-(-d) = \{(y,z)\,:\,z<-d\}$,

$$-\Delta\phi_1^0(y,z) = 0\ , \quad (y,z)\in\mathbb{R}^2_-(-d). \tag{3.21}$$

The Neumann boundary conditions on $\partial\mathbb{R}^2_-(-d)$

$$\begin{aligned} &\partial_z\phi_1^0(y,-d) = 0\ ,\ y<0,\\ &\partial_z\phi_1^0(y,-d) = by^2\partial_\zeta\phi_2^0(y,0) = \frac{\varrho_0\lambda}{\varrho_1-\varrho_0}\phi_0^0(y)\ , \quad y>0, \end{aligned} \tag{3.22}$$

follow from (1.6), (3.10) and (3.7), (3.9), respectively. It is known by [32, Ch. 2] and can be verified directly, that in the case above the threshold $\lambda>\lambda_\dagger$ with ϕ_0^0 as in (3.17), there exists a unique solution of (3.21), (3.22) of the form

$$\phi_1^0(y,z) = C_+r^{1/2+i\varkappa}\Psi_+(\vartheta) + C_-r^{1/2-i\varkappa}\Psi_-(\vartheta), \tag{3.23}$$

where $(r,\vartheta) \in \mathbb{R}_+ \times (-\pi, 0)$ are the polar coordinates with center at P^- and $\Psi_\pm$ are some smooth functions in the angular variable $\vartheta \in [-\pi, 0]$. If $\lambda = \lambda_\dagger$ and (3.18) holds, we readily infer that

$$\begin{aligned} &\phi_1^0(y,z) \\ (3.24) \quad &= \frac{2\varrho_0\lambda}{\varrho_1 - \varrho_0} r^{1/2}\Big(C_1\big(\ln r \sin\frac{\vartheta}{2} + (\vartheta + \pi)\cos\frac{\vartheta}{2}\big) + (C_0 - 2C_1)\sin\frac{\vartheta}{2}\Big). \end{aligned}$$

In the case below threshold $\lambda \in [0, \lambda_\dagger)$ a solution of the problem (3.21), (3.22) can be found in the form

$$\phi_1^0(y,z) = C_+ r^{1/2+\varkappa}\Psi_+(\vartheta) + C_- r^{1/2-\varkappa}\Psi_-(\vartheta); \tag{3.25}$$

however, if $1/2 + \varkappa$ is an integer, the function $\Psi_\pm(\vartheta)$ must be replaced by $\Psi_\pm^0(\vartheta) + \ln r \Psi_\pm^1(\vartheta)$, where $\Psi_\pm^0$ and $\Psi_\pm^1$ are some smooth functions in the variable ϑ.

If $\lambda > \lambda_\dagger$, then (3.20), (3.2), and (3.23) yield

$$|\partial_z \phi_1^1(y,-d)| = O(r^{-5/2}) \quad \text{and} \quad |\phi_1^1(y,-d)| = O(r^{-3/2}), \tag{3.26}$$

which confirms that our previous conclusion on the last two terms in (3.8) was correct. One can easily verify the same for $\lambda \le \lambda_\dagger$, too.

Above, we used simplified notation concerning the point P^- in Fig. 1.1 b, and in principle two changes,

$$y \mapsto y + l_- \quad \text{and} \quad b \mapsto b_- \tag{3.27}$$

would be needed. To treat the point P^+ in Fig. 1.1 b we should apply instead of (3.27) the changes

$$y \mapsto -y + l_+ \quad \text{and} \quad b \mapsto b_+.$$

In addition, for the points $P^\pm$ in Fig. 1.1 a, the roles of the functions φ_0 and φ_1 must be changed, while (3.15) and (3.19) have to be replaced by

$$\lambda_\dagger = \frac{1}{4} b\Big(1 - \frac{\varrho_0}{\varrho_1}\Big) \quad \text{and} \quad \varkappa = \Big|\frac{1}{4} - \frac{\lambda\varrho_1}{b(\varrho_1 - \varrho_0)}\Big|^{1/2}.$$

3.2. **Cuspidal "black hole" for interfacial waves.** The cases below and above threshold are crucially different, and this will be explained here for the point P^- in Fig. 1.1 b (we assume that $b_- \le b_+$). The function $\phi_{0+}^0(y) := y^{-\frac{1}{2}+\varkappa}$ from (3.16), with $\varkappa > 0$ of (3.19), possesses finite energy in the cusp. This follows from the calculation

$$\int_0^\delta \int_0^{h(y)} \Big|\frac{\partial \phi_{0+}^0}{\partial y}(y)\Big|^2 dz dy \sim b\Big(\varkappa - \frac{1}{2}\Big)^2 \int_0^\delta y^2 y^{2(\varkappa - 3/2)} dy < +\infty.$$

On the contrary, the energy of $\phi_{0-}^0(y) := y^{-\frac{1}{2}-\varkappa}$ in the cusp is not finite, because the integral

$$Cb\Big(\varkappa + \frac{1}{2}\Big)^2 \int_0^\delta y^2 y^{-2(\varkappa+3/2)} dy$$

always diverges. This is why the choice of the Fredholm operator $\mathcal{T} - \lambda$ is exactly prescribed: one has to include the asymptotic expression $C_+\phi_{0+}^0(y)$ into the component φ_0 of the solution $\varphi \in \mathcal{H}$, but certainly exclude $C_-\phi_{0-}^0(y)$. Notice that both asymptotic expressions for the component φ_1 in (3.24) live in the Sobolev class H^1.

Above the threshold the functions $\phi^0_{0\pm}(y) = y^{-\frac{1}{2}\pm i\varkappa}$, see (3.17), possess infinite energy, since the integral

$$\int\limits_0^\delta \int\limits_0^{h(y)} \Big|\frac{\partial \phi^0_{0\pm}}{\partial y}(y)\Big|^2 dz dy \sim b\Big(\frac{1}{4}+\varkappa^2\Big)\int\limits_0^\delta y^2 y^{-3} dy$$

diverges at a "logarithmic" rate. Accordingly, the operator $\mathcal{T}-\lambda$ with $\lambda > \lambda_\dagger$ cannot be Fredholm, because on one hand the piston mode (positive in the cross-section) cannot be excluded from the asymptotic form of the solution $\varphi \in \mathcal{H}$, but on the other hand none of the functions $C_\pm \phi^0_{0\pm}(y)$ is good either since they do not belong to $\mathcal{H}$.

The above observation hints to imposing proper radiation conditions at the cuspidal points $P^\pm$ in order to supply the problem (1.5)–(1.9) with a Fredholm operator. A mathematically rigorous formulation of such conditions could be found following the scheme [35], which involves weighted spaces with detached asymptotics, that is, oscillating waves (3.17) of Section 3.1. However, we do not reproduce this scheme here, since the goal of our paper is just the description of the spectrum of the problem (1.5)–(1.9), while radiation conditions will be the subject of a planned forthcoming paper on three dimensional water domains with cuspidal edges on the boundary. Instead, we discuss a physical reasoning for radiation conditions and wave phenomenon in a finite water volume.

Since the time-dependent velocity potential is sought in the form (1.4), we obtain from (3.17) the following expressions:

$$W_0^\pm(y,z,t) = \mathrm{Re}\big(e^{-i\omega t} y^{-1/2\mp i\varkappa}\big) = y^{-1/2}\mathrm{Re}\big(e^{-i(\omega t \pm \varkappa \ln y)}\big). \tag{3.28}$$

Owing to the Sommerfeld radiation principle, the last factor in (3.28) shows that the wave W_0^+ moves to the direction of the cusp top P^-, because at two instances $t_2 > t_1$ there holds

$$e^{-i(\omega t_1 + \varkappa \ln y_1)} = e^{-i(\omega t_2 + \varkappa \ln y_2)}\ ,\ \text{if } e^{-\frac{\omega}{\varkappa}t_1} =: y_1 > y_2 := e^{-\frac{\omega}{\varkappa}t_2}.$$

On the contrary, the W_0^- propagates from the point P^- to the water massive, since

$$e^{-i(\omega t_1 - \varkappa \ln y_1)} = e^{-i(\omega t_2 - \varkappa \ln y_2)}\ ,\ \text{if } e^{-\frac{\omega}{\varkappa}t_1} =: y_1 < y_2 := e^{-\frac{\omega}{\varkappa}t_2}.$$

The first factor $y^{-1/2}$ in (3.28) leads to the following observation: the energy stored in the peak subdomain

$$\{(y,z)\,:\, -K-1 < \ln y < -K, 0 < z+d < h(y)\} \tag{3.29}$$

can be computed using (2.2) by

$$\int\limits_{e^{-K-1}}^{e^{-K}} \Big(\varrho_0 \int\limits_0^{h(y)} \big|\nabla W_0^\pm(y,z,t)\big|^2 dz + \frac{\varrho_0^2\lambda}{\varrho_1-\varrho_0}\big|W_0^\pm(y,-d,t)\big|^2\Big) dy$$

$$\sim \int\limits_{e^{-K-1}}^{e^{-K}} \Big(\varrho_0\Big(\frac{1}{4}+\varkappa^2\Big) b y^2 y^{-3} + \frac{\varrho_0^2\lambda}{\varrho_1-\varrho_0} y^{-1}\Big) dy$$

$$= \Big[\frac{2\varrho_0^2\lambda}{\varrho_1-\varrho_0}\ln y\Big]_{y=e^{-K-1}}^{y=e^{-K}} = \frac{2\varrho_0^2\lambda}{\varrho_1-\varrho_0}; \tag{3.30}$$

here we have used the identity $\varkappa^2 = -1/4 + (\lambda\varrho_0)/(\varrho_1 - \varrho_0)$ following from (3.19) and $\lambda > \lambda_\dagger$. In other words, the energy preserved in the K-dependent subdomain (3.29) does actually not depend on K. This indicates, by physical reasoning, the existence of a wave process, cf. a similar argument for a cylindrical wavequide in [15]. We will demonstrate in the next section that the waves (3.28) give rise to a singular Weyl sequence of the operator $\mathcal{T} - \mu$ with μ as in (2.14). This leads to the inclusion $\lambda \in \sigma_c$ and also indicates mathematically the appearance of the wave propagation phenomenon.

Similarity of wave processes in cuspidal and cylindrical waveguides follows much more evidently from the Euler transform

$$y \mapsto \eta = \ln y\ , \quad \phi_0^0 \mapsto \Psi_0^0(\eta) = e^{\eta/2}\phi_0^0(e^\eta),$$

which converts the differential equation (3.14) into

$$-\frac{d^2}{d\eta^2}\Psi_0^0(\eta) = \Big(-\frac{1}{4} + \frac{\lambda\varrho_0}{b(\varrho_1 - \varrho_0)}\Big)\Psi_0^0(\eta)\ , \quad \eta \in \mathbb{R}.$$

3.3. **Singular Weyl sequence and proof of Theorem 1.1.** Before proceeding with the construction of the Weyl sequence we prove an estimate for Sobolev functions in the peak $\Pi = \{(y,z) : y \in (0, y_0), z + d \in (0, h(y))\}$. For all $\psi_0 \in H^1(\Pi)$ we define the mean function $\overline{\psi}$ by

$$\overline{\psi}(y) = \frac{1}{h(y)} \int\limits_{-d}^{-d+h(y)} \psi_0(y,z)dz. \tag{3.31}$$

Lemma 3.1. *Denoting $s_1 = z$, $s_2 = -d$ and $s_3 = -d + h(y)$, there holds for all $\psi_0 \in H^1(\Pi)$ the estimate*

$$\int\limits_0^{y_0} h(y)^{-2} \int\limits_{-d}^{-d+h(y)} \Big(\sum_{k=1}^3 |\psi_0(y, s_k) - \overline{\psi}(y)|^2\Big)dzdy \le C\|\partial_z\psi_0; L^2(\Pi)\|^2. \tag{3.32}$$

Proof. For $s_1 = z$ we use the Poincaré inequality

$$\Big(\int\limits_{-d}^{-d+h(y)} |\psi_0(y,z) - \overline{\psi}(y)|^2 dz\Big)^{1/2} \le \frac{h(y)}{\pi}\Big(\int\limits_{-d}^{-d+h(y)} |\partial_z\psi_0(y,z)|^2 dz\Big)^{1/2},$$

which implies

$$\begin{aligned}\int\limits_0^{y_0} h(y)^{-2} \int\limits_{-d}^{-d+h(y)} |\psi_0(y,z) - \overline{\psi}(y)|^2 dzdy &\le \frac{1}{\pi^2}\int\limits_0^{y_0}\int\limits_{-d}^{-d+h(y)} |\partial_z\psi_0(y,z)|^2 dzdy \\ &= \frac{1}{\pi^2}\|\partial_z\psi_0; L^2(\Pi)\|^2. \end{aligned}\tag{3.33}$$

So we are left with the terms $k = 2$ and $k = 3$ in (3.32), which we denote by I_k. Recalling the definition (3.31) yields (here $\zeta \le s_k$ is assumed; the case $\zeta > s_k$ goes in the same way)

$$I_k = \int\limits_0^{y_0} h(y)^{-4} \int\limits_{-d}^{-d+h(y)} dz\Big|\int\limits_{-d}^{-d+h(y)} \big(\psi_0(y,s_k) - \psi_0(y,\zeta)\big)d\zeta\Big|^2 dy$$

$$= \int_0^{y_0} h(y)^{-3} \Big| \int_{-d}^{-d+h(y)} \Big(\int_{\zeta}^{s_k} \partial_\xi \psi_0(y,\xi) d\xi \Big) d\zeta \Big|^2 dy. \tag{3.34}$$

Here, the inclusion $[\zeta, s_k] \subset [-d, -d+h(y)]$ holds for the integration intervals. Thus, it follows from (3.34) that

$$\begin{aligned} I_k &\leq \int_0^{y_0} h(y)^{-3} \Big(\int_{-d}^{-d+h(y)} d\zeta \int_{-d}^{-d+h(y)} |\partial_\xi \psi_0(y,\xi)| d\xi \Big)^2 dy \\ &= \int_0^{y_0} h(y)^{-1} \Big(\int_{-d}^{-d+h(y)} |\partial_\xi \psi_0(y,\xi)| d\xi \Big)^2 dy \end{aligned} \tag{3.35}$$

and since

$$\begin{aligned} \Big(\int_{-d}^{-d+h(y)} |\partial_\xi \psi_0(y,\xi)| d\xi \Big)^2 &\leq \int_{-d}^{-d+h(y)} d\xi \int_{-d}^{-d+h(y)} |\partial_\xi \psi_0(y,\xi)|^2 d\xi \\ &= h(y) \int_{-d}^{-d+h(y)} |\partial_\xi \psi_0(y,\xi)|^2 d\xi \end{aligned}$$

by the Cauchy–Bunyakovski–Schwartz (CBS-)inequality, we see from (3.35) that

$$I_k \leq \int_0^{y_0} \int_{-d}^{-d+h(y)} |\partial_\xi \psi_0(y,\xi)|^2 d\xi dy = \|\partial_z \psi_0; L^2(\Pi)\|^2.$$

We obtain (3.32) from this and (3.33). ⊠

Proof of Theorem 1.1. Since the statement of Theorem 1.1 about the discrete spectrum was already proved in Section 2.3, it is enough to show that the interval $[\lambda_\dagger, \infty)$ is contained to the continuous spectrum of the water wave problem. To this end we assume for the rest of the section that $\lambda \geq \lambda_\dagger$ and that μ is given by (2.14), and we show that the operator $\mathcal{T} - \mu$ has a singular Weyl sequence in the sense of [5, §9.1]. i.e. a sequence $\{\varphi^{(N)}\}_{N=1}^\infty \subset \mathcal{H}$ with properties (3.40), (3.41) and (3.42) below. The reference [5, §9.1] then yields the claim about the continuous spectrum.

We introduce the functions $\varphi^{(N)} = \{\varphi_0^{(N)}, \varphi_1^{(N)}\}$, $N \in \mathbb{N} = \{1, 2, 3, \ldots\}$, by

$$\begin{aligned} &\varphi_1^{(N)}(y,z) = 0 \ \ , \ (y,z) \in \Omega_1, \\ &\varphi_0^{(N)}(y,z) = a_N X_N(-\ln y) Y(y) \ \ , \ (y,z) \in \Omega_0, \end{aligned} \tag{3.36}$$

where

$$a_N = \|\widehat{X}_N Y; \mathcal{H}\|^{-1}, \tag{3.37}$$

$Y(y) = y^{-1/2+i\varkappa}$ (cf. (3.17), (3.18)), and $\widehat{X}_N(y) = X_N(-\ln y)$; moreover, X_N is the plateau function

$$X_N(\tau) = \chi(\tau - 2^{N+1}) \chi(-\tau + 2^N) \ ,$$

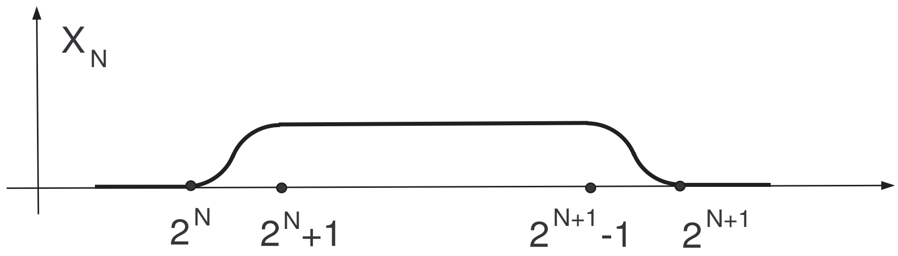

FIGURE 3.1. Graph of the plateau function.

where

$$\chi \in C^\infty(\mathbb{R}),\ 0 \le \chi \le 1,\ \chi(\tau) = 0 \text{ for } \tau \ge 0\ ,\ \chi(\tau) = 1 \text{ for } \tau \le -1. \tag{3.38}$$

The graph of X_N is depicted in Fig. 1.4. Obviously,

$$|\partial_y^k \varphi_0^{(N)}(y)| \le C a_N y^{-(1+2k)/2}, \quad \text{for } k = 0, 1, 2. \tag{3.39}$$

Notice that in (3.36) the traces of $\varphi_0^{(N)}$ and $\varphi_1^{(N)}$ do not need to coincide on I in order to make $\varphi^{(N)}$ belong to $\mathcal{H}$. This is why we could omit in (3.36) the higher order terms $h^2\phi_0^2$ and $h\phi_1^1$ from the previous ansätze (3.3) and (3.4).

Owing to (3.37) we have

$$\|\varphi^{(N)}; \mathcal{H}\| = 1. \tag{3.40}$$

Moreover, by the definition (3.38), $\operatorname{supp}\varphi^{(N)} \cap \operatorname{supp}\varphi^{(M)} = \emptyset$ for $N \ne M$, hence, (3.40) implies that

$$\varphi^{(N_k)} \rightharpoonup 0 \text{ weakly in } \mathcal{H} \tag{3.41}$$

for a subsequence $(N_k)_{k\in\mathbb{N}}$, as $k \to +\infty$. Two properties (3.40) and (3.41) of a singular Weyl sequence are thus at hand (see [5, §9.1]). The third one, namely

$$\|\mathcal{T}\varphi^{(N)} - \mu\varphi^{(N)}; \mathcal{H}\| \to 0 \text{ as } N \to \infty \tag{3.42}$$

requires much longer calculations. First of all, to estimate the coefficient (3.37) from below, we replace the integration domain $\{(y, z) : e^{-2^{N+1}} < y < e^{-2^N}, 0 < z < h(y)\}$ by subdomain where $\widehat{X}_N = 1$ and thus get

$$\begin{aligned}
\|\widehat{X}_N Y; \mathcal{H}\|^2 &\ge \int\limits_{e^{-2^{N+1}+1}}^{e^{-2^N-1}} \Big(\Big(\frac{1}{4} + \varkappa^2\Big) h(y) y^{-3} + y^{-1}\Big) dy \\
&= \Big(\Big(\frac{1}{4} + \varkappa^2\Big) b + 1\Big)\Big(\Big[\ln y\Big]_{y=e^{-2^{N+1}+1}}^{y=e^{-2^N-1}} + O(e^{-2^N})\Big) \\
&= \Big(\Big(\frac{1}{4} + \varkappa^2\Big) b + 1\Big)\big(2^{N+1} - 1 - 2^N - 1 + O(e^{-2^N})\big).
\end{aligned}$$

For $N \ge N_0$ the peak is included in the support of X_N, and we infer for these N from above that

$$a_N \le c 2^{-N/2} \tag{3.43}$$

for some constant $c>0$. Taking into account the formulas (2.1), (2.2), (2.14) and (3.36) yields

$$\begin{aligned}
&\|\mathcal{T}\varphi^{(N)}-\mu\varphi^{(N)};\mathcal{H}\| = \sup\big|\langle\mathcal{T}\varphi^{(N)},\psi\rangle-\mu\langle\varphi^{(N)},\psi\rangle\big|\\
&=\sup\Big|\varrho_0(\varphi_0^{(N)},\psi_0)_S+\frac{1}{\varrho_1-\varrho_0}(\varrho_1\varphi_1^{(N)}-\varrho_0\varphi_0^{(N)},\varrho_1\psi_1-\varrho_0\psi_0)_I\\
&\quad-\frac{1}{\lambda+1}\Big(\sum_{j=0,1}\varrho_j\big(\nabla\varphi_j^{(N)},\nabla\psi_j)_{\Omega_j}+\varrho_0(\varphi_0^{(N)},\psi_0)_S\\
&\quad+\frac{1}{\varrho_1-\varrho_0}(\varrho_1\varphi_1^{(N)}-\varrho_0\varphi_0^{(N)},\varrho_1\psi_1-\varrho_0\psi_0)_I\Big)\Big|\\
&=\frac{1}{\lambda+1}\sup\Big|\varrho_0\big(\nabla\varphi_0^{(N)},\nabla\psi_0)_{\Omega_0}+\lambda\varrho_0(\varphi_0^{(N)},\psi_0)_S\\
&\quad+\frac{\lambda\varrho_0}{\varrho_1-\varrho_0}(\varphi_0^{(N)},\varrho_1\psi_1-\varrho_0\psi_0)_I\Big|,
\end{aligned}\tag{3.44}$$

where the supremum is computed over all $\psi=\{\psi_0,\psi_1\}\in\mathcal{H}$ such that $\|\psi;\mathcal{H}\|=1$. Lemma 2.1 and (3.44) yield

$$\|\psi_0;H^1(\Omega_0)\|+\|\psi_1;H^1(\Omega_1)\|\le C.\tag{3.45}$$

Moreover, the trace inequality (2.18) and the normalization (3.37) also imply

$$\|\varphi_0^{(N)};L^2(I)\|\le C.\tag{3.46}$$

We observe using (3.38) that

$$\begin{aligned}
&\operatorname{supp}\varphi_0^{(N)}\subset\{(y,z)\,:\,y\in\overline{\Upsilon}_N,0\le z\le h(y)\},\\
&\Upsilon_N=\{y\,:\,e^{-2^{N+1}}\le y\le e^{-2^N}\}\ ,\ \operatorname{mes}_1\Upsilon_N=O(e^{-2^N}).
\end{aligned}\tag{3.47}$$

The relation (3.44) can be written as

$$\begin{aligned}
&(1+\lambda)\|\mathcal{T}\varphi^{(N)}-\mu\varphi^{(N)};\mathcal{H}\|\\
&=\sup\Big|\int\limits_{\Upsilon_N}\Big(\varrho_0\int\limits_{-d}^{-d+h(y)}\frac{\partial\varphi_0^{(N)}}{\partial y}(y)\frac{\partial\psi_0}{\partial y}(y,z)dz-\frac{\varrho_0^2\lambda}{\varrho_1-\varrho_0}\varphi_0^{(N)}(y)\psi_0(y,-d)\Big)dy\\
&+\frac{\varrho_1^2\lambda}{\varrho_1-\varrho_0}\int\limits_{\Upsilon_N}\varphi_0^{(N)}(y)\psi_1(y,-d)dy\Big|.
\end{aligned}\tag{3.48}$$

The last term satisfies the estimate

$$\frac{\varrho_1\lambda}{\varrho_1-\varrho_0}\Big|\int\limits_{\Upsilon_N}\varphi_0^{(N)}(y)\psi_1(y,-d)dy\Big|\le C\|\psi_1;L^2(\Upsilon_N)\|.\tag{3.49}$$

A scaling argument shows that the standard trace inequality in the stretched variables turns into

$$\|\psi_1;L^2(\Upsilon_N)\|^2\le ce^{-2^N}\big(\|\nabla\psi_1;L^2(\Omega_1)\|^2+e^{2^N}\|\psi_1;L^2(\Lambda_N)\|^2\big),$$

where $\Lambda_N:=\{(y,z)\,:\,2^{-N}\le|\ln y|^2+|z|^2\le 2^{-N+1},z>0\}$. The estimate

$$\|\rho^{-1}(1+|\ln\rho|)^{-1}\psi_1;L^2(\Omega_1)\|\le C\|\psi_1;H^1(\Omega_1)\|,\tag{3.50}$$

where ρ is the distance to the peak top, is a direct consequence of the one-dimensional Hardy inequality

$$\int_0^1 \rho^{-1}|\ln\rho|^{-2}|g(\rho)|^2 d\rho \le 4\int_0^1 \rho\Big|\frac{dg}{d\rho}(\rho)\Big|^2 d\rho \quad \forall g\in C_c^1([0,1)).$$

From (3.50) we conclude that

$$\begin{aligned}\|\psi_1; L^2(\Upsilon_N)\| &\le ce^{-2^N/2}\big(\|\nabla\psi_1; L^2(\Omega_1)\| + (1+|\ln e^{-2^N}|)\|\psi_1; L^2(\Omega_1)\|\\ &\le Ce^{-2^{N-1}}(1+N)\|\psi_1; H^1(\Omega_1)\|,\end{aligned}$$

therefore, the bound (3.49) is infinitesimal.

It suffices to consider the integral

$$\begin{aligned}&\int_{\Upsilon_N}\Big(\int_{-d}^{-d+h(y)} \frac{\partial\varphi_0^{(N)}}{\partial y}(y)\frac{\partial\psi_0}{\partial y}(y,z)dz - \frac{\varrho_0\lambda}{\varrho_1-\varrho_0}\varphi_0^{(N)}(y)\psi_0(y,-d)\Big)dy\\ &= -\int_{\Upsilon_N}\Big(\int_{-d}^{-d+h(y)} \frac{\partial^2\varphi_0^{(N)}}{\partial y^2}(y)\psi_0(y,z)dz\\ &\qquad + \frac{\partial_y h(y)}{(1+|\partial_y h(y)|^2)^{1/2}}\frac{\partial\varphi_0^{(N)}}{\partial y}(y)\psi_0(y,-d+h(y))\\ &\qquad + \frac{\varrho_0\lambda}{\varrho_1-\varrho_0}\varphi_0^{(N)}(y)\psi_0(y,-d)\Big)dy.\end{aligned}\tag{3.51}$$

We denote by J_N the right hand side of (3.51), and decompose it as

$$J_N = J_N{}^{\circ} + J_N{}^{\bullet},\tag{3.52}$$

where

$$\begin{aligned}J_N{}^{\circ} := -\int_{\Upsilon_N}\Big(&\int_{-d}^{-d+h(y)} \partial_y^2\varphi_0^{(N)}(y)\big(\psi_0(y,z)-\overline{\psi}(y)\big)dz\\ &+ \frac{\partial_y h(y)}{(1+|\partial_y h(y)|^2)^{1/2}}\partial_y\varphi_0^{(N)}(y)\big(\psi_0(y,-d+h(y))-\overline{\psi}(y)\big)\\ &+ \frac{\varrho_0\lambda}{\varrho_1-\varrho_0}\varphi_0^{(N)}(y)\big(\psi_0(y,-d)-\overline{\psi}(y)\big)\Big)dy\end{aligned}\tag{3.53}$$

and

$$\begin{aligned}J_N{}^{\bullet} := -\int_{\Upsilon_N}\Big(&\int_{-d}^{-d+h(y)} \partial_y^2\varphi_0^{(N)}(y)\overline{\psi}(y)dz + \frac{\partial_y h(y)}{(1+|\partial_y h(y)|^2)^{1/2}}\partial_y\varphi_0^{(N)}(y)\overline{\psi}(y)\\ &+ \frac{\varrho_0\lambda}{\varrho_1-\varrho_0}\varphi_0^{(N)}(y)\overline{\psi}(y)\Big)dy\end{aligned}\tag{3.54}$$

Our purpose is to show that $J_N\to 0$, as $N\to\infty$.

We first consider the term $J_N{}^{\circ}$. By (3.39) and (3.43) we have

$$|\partial_y^k\varphi_0^{(N)}(y)| \le C2^{-N/2}y^{-(1+2k)/2}, \quad \text{for} \quad k=0,1,2.\tag{3.55}$$

With the notation of Lemma 3.1, we have

$$|J_N{}^{\circ}| \le C2^{-N/2} \int\limits_{\Upsilon_N} \int\limits_{-d}^{-d+h(y)} y^{-5/2}\Big(\sum_{k=1}^{3} |\psi_0(y,s_k) - \overline{\psi}(y)|\Big) dz dy, \tag{3.56}$$

since

$$h(y) \sim y^2, \quad \frac{\partial_y h(y)}{(1+|\partial_y h(y)|^2)^{1/2}} \le Cy, \quad \frac{\varrho_0 \lambda}{\varrho_1 - \varrho_0} \le C. \tag{3.57}$$

Using again the relation $h(y) \sim y^2$, CBS-inequality, and Lemma 3.1, we get

$$\begin{aligned} |J_N{}^{\circ}| &\le C2^{-N/2}\Big(\int\limits_{\Upsilon_N} \int\limits_{-d}^{-d+h(y)} y^{-1} dy dz\Big)^{1/2} \|\partial_z \psi_0; L^2(\Pi)\| \\ &\le C' 2^{-N/2}\Big(\int\limits_{\Upsilon_N} y dy\Big)^{1/2} \le C'' 2^{-N/2} \exp(-2^N), \end{aligned} \tag{3.58}$$

since $\|\partial_z \psi_0; L^2(\Pi)\| \le \|\psi; \mathcal{H}\| = 1$. Thus $J_N{}^{\circ} \to 0$, as $j \to \infty$.

For the estimate of the term $J_N{}^{\bullet}$ we need the observation that for all $\psi \in H^1(\Pi)$,

$$\begin{aligned} \|\overline{\psi}; L^2(0,y_0)\|^2 &\le \int\limits_0^{y_0} \frac{1}{h(y)^2}\Big|\int\limits_{-d}^{-d+h(y)} \psi_0(y,z)dz\Big|^2 dy \le \int\limits_0^{y_0} \frac{1}{h(y)} \int\limits_{-d}^{-d+h(y)} |\psi_0(y,z)|^2 dz dy \\ &\le C\int\limits_{\Pi} y^{-2} |\psi_0(y,z)|^2 dy dz \le C\|\psi_0; H^1(\Pi)\|^2; \end{aligned} \tag{3.59}$$

the last inequality is proven in [33], Proposition of Section 2.

We decompose $J_N{}^{\bullet}$ into two parts;

$$J_N{}^{\bullet} = J_N{}' + J_N{}'', \tag{3.60}$$

where (see (3.2))

$$\begin{aligned} J_N{}' := -\int\limits_{\Upsilon_N} \Big(&\big(h(y) - by^2\big)\partial_y^2 \varphi_0^{(N)}(y) \\ &+\Big(\frac{\partial_y h(y)}{(1+|\partial_y h(y)|^2)^{1/2}} - 2by\Big)\partial_y \varphi_0^{(N)}(y)\Big)\overline{\psi}(y) dy \end{aligned} \tag{3.61}$$

and

$$J_N{}'' := -\int\limits_{\Upsilon_N} \Big(by^2 \partial_y^2 \varphi_0^{(N)}(y) + 2by\partial_y \varphi_0^{(N)}(y) + \frac{\varrho_0 \lambda}{\varrho_1 - \varrho_0} \varphi_0^{(N)}\Big)\overline{\psi}(y) dy. \tag{3.62}$$

By (3.2) and the Taylor expansion $(1+y)^{-1/2} = 1 - y/2 + O(y^2)$ for $|y| < 1$, we have

$$h(y) - by^2 = O(y^3) \ \text{ and } \ \frac{\partial_y h(y)}{(1+|\partial_y h(y)|^2)^{1/2}} - 2by = O(y^2). \tag{3.63}$$

Using the estimates (3.63), (3.55), and (3.59) and the CBS-inequality we see that, for large enough N

$$\begin{aligned} |J_N{}'| &\le C2^{-N/2}\int\limits_{\Upsilon_N} y^{1/2}|\overline{\psi}(y)|dy \\ &\le C2^{-N/2}\Big(\int\limits_{\Upsilon_N} ydy\Big)^{1/2}\|\overline{\psi};L^2(0,y_0)\| \le C'2^{-N/2}\exp(-2^N), \end{aligned} \tag{3.64}$$

since $\|\psi_0;H^1(\Pi)\| \le C\|\psi;\mathcal{H}\| = C$. Thus $J_N{}' \to 0$, as $N\to\infty$.

Finally, since Y in (3.36) is a solution of the equation (3.14), we have

$$\begin{aligned} -\partial_y\big(by^2\partial_y\varphi_0^{(N)}\big) - \frac{\varrho_0\lambda}{\varrho_1-\varrho_0}\varphi_0^{(N)} = a_N\big(&by^2Y\partial_y^2\hat{X}_N + 2by^2\partial_y\hat{X}_N\partial_yY \\ &+2byY\partial_y\hat{X}_N\big), \end{aligned} \tag{3.65}$$

where $\hat{X}_N(y) = X_N(-\ln y)$. Now

$$|\partial_y^kY(y)| \le Cy^{-(1+2k)/2}, \quad k=0,1,2 \tag{3.66}$$

and

$$|\partial_y^kX_N(-\ln y)| \le Cy^{-k}, \quad k=0,1,2 \tag{3.67}$$

for all $N\ge N_0$. Hence, the inequality

$$|-\partial_y\big(by^2\partial_y\varphi_0^{(N)}(y)\big) - \frac{\varrho_0\lambda}{\varrho_1-\varrho_0}\varphi_0^{(N)}| \le Cy^{-1/2} \tag{3.68}$$

holds for the integrand of $J_N{}''$, and it is supported in the set

$$\Theta_N := \Upsilon_N\setminus(e^{-2^{N+1}+1}, e^{-2^N-1}), \tag{3.69}$$

since $\partial_yX_N(-\ln y) = 0$ in the complement of Θ_N. We also notice that

$$\begin{aligned} \int\limits_{\Theta_N} y^{-1}dy &= \Big(\int\limits_{\exp(-2^{N+1})}^{\exp(-2^{N+1}+1)} y^{-1}dy + \int\limits_{\exp(-2^N-1)}^{\exp(-2^N)} y^{-1}dy\Big) \\ &= -2^{N+1}+1+2^{N+1}-2^N+2^N+1 = 2. \end{aligned} \tag{3.70}$$

Hence, by applying the CBS-inequality, we see that

$$\begin{aligned} |J_N{}''| &\le Ca_N\int\limits_{\Theta_N} y^{-1/2}|\overline{\psi}(y)|dy \\ &\le C'2^{-N/2}\Big(\int\limits_{\Theta_N} y^{-1}dy\Big)^{1/2}\|\overline{\psi};L^2(\Theta_N)\| \le C''2^{-N/2}. \end{aligned} \tag{3.71}$$

Here we used the equality $\|\psi;\mathcal{H}\| = 1$ and the inequalities

$$\|\overline{\psi};L^2(\Theta_N)\| \le \|\psi_0;H^1(\Pi)\| \le C\|\psi;\mathcal{H}\|, \tag{3.72}$$

which are valid for large enough values of N, since then $\Theta_N\subset(0,y_0)$ and the inequality (3.59) can be applied. Thus $J_N{}''\to 0$, as $N\to\infty$.

The Weyl sequence is thus found, and this completes the proof of the main result Theorem 1.1. ⊠

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