Localization estimates for eigenfrequencies of waves trapped by freely floating body in channel

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The linear water-wave problem of F. John is studied in the case of a freely floating body in an unbounded cylindrical channel, and a localization estimate for an eigenfrequency is obtained. This eigenfrequency is compared with an eigenfrequency of a reference problem involving an integro-differential boundary condition on the wetted surface of the body. The localization estimate is derived by reducing the John problem to a standard spectral problem for a self-adjoint operator in a complicated Hilbert space, and by an application of spectral measure theory. Sufficient conditions for the existence of trapped modes are found for the reference problem and then, using the localization estimate, for the original problem. Applications of these conditions to concrete cases are given.

Key words: linear water-wave, freely floating body, spectral measure, comparison principle, trapped modes, localization estimate.

1. Introduction

The linearized equations describing the small amplitude motion of the system "liquid/object" under gravity were derived in John 1950. The mathematical model covers both the case of a freely floating body and a fixed obstacle. As an illuminating physical example one may think about a ship in a pond or channel, a submarine in the ocean, or an underwater ridge. During more than 50 years of reaserch, most of the attention has been paid on the second case, and many remarkable results have been obtained, see for example the review articles Ben Dhia & Joly 1993, Linton & McIver 2007, the monograph Kuznetsov et al. 2002, and the citation lists therein. Starting from the classical works John 1950, Ursell 1951, Jones 1953, Garipov 1967, linear water-wave problems have been studied as spectral boundary value problems, and typical results include detection of eigenfrequences inside the discrete spectrum or embedded into the continuous spectrum. The corresponding eigenfunctions are recognized as trapped modes or guided waves (localized solutions) in mechanics. We again refer to the cited items for an extensive list of examples of trapped surface waves.

Apart from calculation of explicit solutions and various computational methods, a tool for finding trapped modes arises from so-called comparison principles. A succesful application of this method is presented in the paper Ursell 1987, where the first comparison principle for surface waves over a submerged

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obstacle was discovered. Namely, Ursell 1951 contains a construction of guided surface waves along submerged circular cylinders, while in Ursell 1987 it was shown that enlarging the cross-section of the cylinder leads to diminishing of an eigenvalue, which therefore remains inside the discrete spectrum of the corresponding two-dimensional problem. This observation ensures the existence of trapped modes for a cylinder of any cross-section with positive area. We also mention the paper Motygin 2008, where the above principle was adopted to surface-piercing obstacles. Based on the notion Nazarov 2008 of trace operators and basic results for self-adjoint operators in Hilbert spaces, a new approach was proposed in Nazarov 2009. This has produced elementary proofs of known and new comparison principles, as well as simple sufficient conditions for the existence of trapped modes in various situations, see Nazarov & Videman 2009, 2010, Nazarov 2011a and others.

All of the above mentioned results are only related to fixed, immovable objects, like submerged or surface-piercing obstacles and straight or periodic coastlines. However, the John problem for freely floating objects differs in two essential respects from the fixed obstacle case. First, of minor importance is the fact that in addition to the velocity potential φ in the liquid, a column vector a of height 6 is involved. This descibes the motions, three translations and three rotations, of the body. The second difference brings in serious difficulties and makes many traditional approaches and methods useless. This is the intrinsic convertion of the boundary value problem into an abstract equation involving the quadratic pencil

$$\mathbb{R} \ni \omega \mapsto A(\omega) = A_0 + \omega A_1 + \omega^2 A_2 \tag{1.1}$$

with self-adjoint operators A_q, a special case of a holomorphic spectral family. Of course, also the problem with a fixed obstacle involves the square ω^2 of the frequency, but since it has no linear term of type ωA_1, redenoting $\lambda = g^{-1}\omega^2$ with an appropriate factor g^{-1}, the pencil (1.1) turns into a standard linear spectral family.

During the last decade the interest in the John problem has been recovered. Numerical schemes were worked out and some particular instances, mainly related to the two-dimensional case and the heaving motion of a float, were considered in a series of publications McIver & McIver 2006, Evans & Porter 2007, McIver & McIver 2007, Porter & Evans 2008, 2009. Furthermore, as observed in Kuznetsov 2010, the hydrodynamic forces acting on two identical surface-piercing obstacles of specific shape (see §4.2.2.3 in Kuznetsov et al. 2002) have null principal vector and torque, and thus the velocity potential φ for fixed obstacles becomes a solution to the John problem, because the body stays motionless.

General tools to compare spectra of quadratic pencils like (1.1) are missing (see Gohberg & Krein 1969 and cf. the partial ordering of semi-bounded self-adjoint operators in Chapter 10 of Birman & Solomyak 1987), and this makes it difficult to derive a comparison principle and to find sufficient conditions for the existence of surface waves trapped by freely floating objects. To overcome this difficulty, in Nazarov & Videman (submitted) the authors developed a reduction scheme to replace the quadratic pencil (1.1) with a continuous self-adjoint operator $\mathcal{A}$ in a special Hilbert space $\mathcal{H}$. As a result, several sufficient conditions for trapping surface waves were derived, and in the environment of symmetric channels, concrete couples of identical floating bodies meeting these conditions were found.

In Section 3 we repeat the reduction scheme, but in order to avoid its buffing complexity we only deal with a unique symmetric body. Using the idea of Evans et al. 1994, we impose an artificial Dirichlet condition on the mirror symmetry plane. This requires three a priori restrictions on the rigid motion column a (see (2.22)), and it results into a positive lower bound $\omega_\dagger$ for the positive part of the continuous spectrum. Under these assumptions we are able to derive in Theorem 2 a new sufficient condition for the existence of trapped modes.

However, the main goal of the paper is to employ a new idea of a so-called localization estimate. We consider a reference problem with nonempty discrete spectrum and show that a neighbourhood of its eigenfrequency certainly contains a spectral point of the original John problem. If it happens that this neighbourhood does not touch the continuous spectrum, then an eigenfrequency is found, and making the neighbourhood as short as possible becomes a natural task. This "localization estimate" somehow substitutes the comparison principle, which does not hold true for freely floating objects. We shall use the theory of spectral measures in this approach. Let us still mention that Nazarov 2011b contains an incomplete comparison principle for freely floating objects in a symmetric channel. This principle requires additional restrictions on the eigenfrequency of the reference problem, and our present localization estimate explains why those restrictions are needed.

The choice of the proper reference problem becomes a major challenge. As discussed in Nazarov 2011b, the problem with a fixed obstacle cannot give a good approximation of spectrum for the freely floating case. The coincidence of the eigenfrequences observed in Kuznetsov 2010 is accidental; we continue this discussion in Section 6. In Section 4 we impose integro-differential boundary conditions on the immersed part of the body surface. Taking into account the artificial Dirichlet conditions due to symmetry (2.20), we investigate the discrete spectrum of the corresponding problem in variational formulation. In the same way as in Nazarov 2008 and 2009 we derive two different sufficient conditions for trapping surface waves. The integro-differential boundary condition, introduced heuristically in Nazarov 2011b, is derived by solving the algebraic part of the John problem while assuming formally that the buoyancy matrix, see (2.7) and (3.5), vanishes completely. The latter may happen only for a submerged body whose mass centre coincides with the buoyancy centre. This coincidence of course destroys the requirement of John 1950 on the floating body to have a stable equilibrium. Hence, a physical meaning of the reference problem becomes doubtful, but being a mathematical tool, one should not expect it to have a fair physical interpretation. The important point is that the smaller the buoyancy matrix is, the better is our localization estimate. In Section 6 we continue the argument for the choice of the reference problem.

The paper is organized as follows. In Section 2 we state the John problem in a channel and introduce our symmetry assumptions. In Section 3 we reduce the pencil (1.1) to a self-adjoint operator and examine its spectrum. The reference problem is formulated and studied in Section 4 so that in Section 5 it will become possible to make the comparison and obtain the localization estimate. Concluding remarks are collected in Section 6.

2. Equations of motion.

Let $\Pi = \mathbb{R} \times \varpi$ be a cylindrical channel such that its cross-section ϖ is bounded by the line segment $\gamma = \{(y, z) : |y| < l, z = 0\}$ and a Lipschitz curve σ connecting the points $(\pm l, 0)$ in the lower half-plane $\mathbb{R}^2_- = \{(y, z) : z < 0\}$. We rescale l to 1 so that all coordinates and geometrical parameters become dimensionless. A body Θ with two dimensional Lipschitz boundary $\partial\Theta$ and compact closure $\overline{\Theta} = \Theta \cup \partial\Theta$ floats freely in the channel, and its non-empty submerged part and wetted surface are denoted, respectively, by

$$\Xi = \{\mathbf{x} = (x, y, z) \in \Theta : z < 0\} \ , \ \xi = \{\mathbf{x} \in \partial\Theta : z < 0\}. \tag{2.1}$$

We assume that the liquid (water) is homogeneous, incompressible, and inviscid, while its motion is irrotational and of small amplitude. Under these conditions, the velocity potential $\phi(\mathbf{x}, t)$ satisfies the Laplace equation in the water filled, connected domain $\Omega = \Pi \setminus \overline{\Theta}$, and also the linearized kinematic boundary condition

$$\partial_t^2 \phi = -g \partial_z \phi$$

on the free water surface $\Upsilon = \Gamma \setminus \overline{\theta}$, as well as the Neumann boundary condition

$$\partial_n \phi = 0$$

on the bottom and walls $\Sigma = \mathbb{R} \times \sigma$. Here, $\Gamma = \mathbb{R} \times \gamma = \partial\Pi \setminus \overline{\Sigma}$ and $\theta = \{\mathbf{x} \in \Theta : z = 0\}$ is the cross-section of Θ by Γ, $g > 0$ is the acceleration due to gravity, $\partial_t = \partial/\partial t$ denotes the time derivative and $\partial_n = \mathbf{n}^\top \nabla$ the normal derivative; $\top$ stands for the transposition, ∇ is the gradient and $\mathbf{n}$ is the outward unit normal vector defined almost everywhere on the Lipschitz surfaces Σ and $\partial\Theta$. Notice that the sets (2.1) may consist of several connected components but Θ is supposed to be a domain. Moreover, we have to assume that Ω constitutes a Lipschitz domain as well (see the comment to (3.3) below).

The motion of the body is of small amplitute, too, and it is coupled with the motion of water through the kinematic condition John 1950

$$\partial_n \phi(\mathbf{x}, t) = \mathbf{n}(\mathbf{x})^\top D(\mathbf{x} - \mathbf{x}^\bullet) \partial_t \mathbf{a}(t), \quad x \in \xi.$$

Here,

$$\mathbf{a}(t) = \big(\mathbf{a}_1(t), \mathbf{a}_2(t), \mathbf{a}_3(t), \mathbf{a}_4(t), \mathbf{a}_5(t), \mathbf{a}_6(t)\big)^\top \tag{2.2}$$

is a column describing the position of the mass centre $\mathbf{x}^\bullet$ of Θ at time t,

$$\mathbf{x}^\bullet = \frac{1}{m} \int_\Theta \mathbf{x} \varrho(\mathbf{x}) d\mathbf{x} \ , \ m = \int_\Theta \varrho(\mathbf{x}) d\mathbf{x}, \tag{2.3}$$

while $\varrho(\mathbf{x})$ and m denote the density and total mass of the body, respectively. Furthermore,

$$D(\mathbf{x}) = \begin{bmatrix} 1 & 0 & y & 0 & 0 & -z \\ 0 & 1 & -x & 0 & z & 0 \\ 0 & 0 & 0 & 1 & -y & x \end{bmatrix} \tag{2.4}$$

is the matrix of rigid motions so that the translations of Θ are given by the components $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_4$, while $\partial_t \mathbf{a}_3$ and $\partial_t \mathbf{a}_5$, $\partial_t \mathbf{a}_6$ express the angular velocities of Θ about the point $\mathbf{x}^\bullet$. Note that the matrix (2.4) is composed in such a way that the first three components of (2.2) correspond to horizontal motions of the body, which are not influenced by the buoyancy forces.

Since external forces on the system "water/body" are absent, the linearized equations of motion of the body arise from the conservation of linear and angular momenta, and they read (cf. John 1950) as

$$M\partial_t^2\mathbf{a} = -S\partial_t\phi - gK\mathbf{a},$$

where M and K are symmetric 6×6-matrices and S is the integral vector functional

$$S\varphi = \int_\xi \varphi(\mathbf{x})D(\mathbf{x}-\mathbf{x}^\bullet)^\top \mathbf{n}(\mathbf{x})ds_{\mathbf{x}}. \tag{2.5}$$

Being a Gram matrix, the inertia matrix

$$M = \int_\Theta D(\mathbf{x}-\mathbf{x}^\bullet)^\top D(\mathbf{x}-\mathbf{x}^\bullet)\varrho(\mathbf{x})d\mathbf{x} \tag{2.6}$$

is positive definite, but the matrix K related to the buoyancy is degenerate and looks as follows:

$$K = \begin{bmatrix} \mathbb{O}_3 & \mathbb{O}_3 \\ \mathbb{O}_3 & K' \end{bmatrix}\ ,\ K' = K^\theta + K^\Xi,$$

$$K^\theta = \int_\theta d(x-x^\bullet, y-y^\bullet)^\top d(x-x^\bullet, y-y^\bullet)dxdy\ ,\ d(x,y) = (1,-y,x) \tag{2.7}$$

$$K^\Xi = \mathrm{diag}\{0, I^\Xi, I^\Xi\}\ ,\ I^\Xi = \int_\Xi (z-z^\bullet)d\mathbf{x}.$$

Here, $\mathbb{O}_3$ is the null matrix of size 3×3, $d(x,y)$ denotes a fragment of the matrix (2.4), and in the integrals we write $\mathbf{x}^\bullet = (x^\bullet, y^\bullet, z^\bullet)$.

In the case the body Θ is surface-piercing, i.e. $\theta \neq \varnothing$, K^θ is a Gram matrix constructed from linearly independent functions and hence positive definite. If the body is submerged and $\theta = \varnothing$, then K^θ vanishes, the rank of K' becomes 2, and K is still positive provided

$$z^\ominus > \mathbf{z}^\bullet, \tag{2.8}$$

where the buoyancy centre is denoted by

$$\mathbf{x}^\ominus = (x^\ominus, y^\ominus, z^\ominus) = \int_\Xi \mathbf{x}d\mathbf{x}. \tag{2.9}$$

The condition (2.8), together with

$$x^\ominus = x^\bullet\ ,\ y^\ominus = y^\bullet \tag{2.10}$$

is known to maintain the stable equilibrium of a floating submerged body (see John 1950, Mei et al. 2005). For a surface-piercing body, the classical stability condition of Euler 1773 (see also the previous references) becomes a bit more cumbersome but also leads to positivity of the matrix K with $\operatorname{rank} K = 3$. Referring to the above cited publications, we write the stable equilibrium condition in the condensed form

$$\theta \neq \varnothing \Rightarrow k = \operatorname{rank} K = 3 \ , \ K \geq 0,$$
$$\theta = \varnothing \Rightarrow k = \operatorname{rank} K = 2 \ , \ K \geq 0, \tag{2.11}$$

and, finally, we recall the Archimedean law

$$m = v = \int_{\Xi} d\mathbf{x}. \tag{2.12}$$

We assume that the motion of the "water/body"-system is time-harmonic,

$$(\phi(\mathbf{x}, t), \mathbf{a}(t)) = \operatorname{Re}(e^{-i\omega t}(\varphi(\mathbf{x}), a)), \tag{2.13}$$

and write down the boundary value problem, the John problem, for the spectral triple $\{\omega, \varphi, a\}$, consisting of the eigenfrequency ω and compound eigenvector $\{\varphi, a\}$ as follows:

$$-\Delta\varphi(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega, \tag{2.14}$$

$$\partial_z \varphi(\mathbf{x}) = g^{-1}\omega^2 \varphi(\mathbf{x}), \qquad \mathbf{x} \in \Upsilon, \tag{2.15}$$

$$\partial_n \varphi(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Sigma, \tag{2.16}$$

$$\partial_n \varphi(\mathbf{x}) = -i\omega \mathbf{n}(\mathbf{x})^\top D(\mathbf{x} - \mathbf{x}^\bullet) a, \qquad x \in \xi, \tag{2.17}$$

$$gKa - i\omega S\varphi = \omega^2 Ma. \tag{2.18}$$

We are interested in detecting trapped modes, which are solutions to the problem (2.14)–(2.16) decaying at infinity; the rate is exponential due to the cylindrical structure of the channel, cf. Mazya et al. 1991, Ch.1,3, and Nazarov 2009. Thus, we do not need to supply the boundary value problem with radiation conditions but just to seek for the velocity potential φ in an appropriate Sobolev space. However, in view of the possible irregularities of the Lipschitz boundary $\partial\Omega$ the useful space is nothing but $H^1(\Omega)$ so that the problem must be reformulated as a variational spectral problem, cf. Nazarov & Videman (submitted). This is done in detail in Section 3, after imposing the artificial boundary conditions of Evans et al. 1994. Let us next proceed with that and a discussion on the spectrum.

As verified in Nazarov & Videman (submitted), the continuous spectrum of the John problem (in variational formulation) covers the whole real axis $\mathbb{R}$ of the complex plane. In the sequel we employ the idea of Evans et al. 1994 about artificial boundary conditions to create a non-empty interval

$$(-\omega_\dagger, \omega_\dagger) \ , \tag{2.19}$$

which may only contain discrete spectrum. Namely, we assume that both the channel Π and the body Θ are symmetric with respect to the central plane $\{\mathbf{x}:$

$y = 0\}$, which formally means

$$\Omega = \{\mathbf{x} : (x, -y, z) \in \Omega\} \ , \ \varrho(x, y, z) = \varrho(x, -y, z) \tag{2.20}$$

and $y^\bullet = y^\ominus = 0$. We also fix the origin such that $x^\bullet = 0 = x^\ominus$, cf. (2.10). The symmetry requirement (2.20) permits to impose the artificial Dirichlet condition Evans et al. 1994

$$\varphi(x, 0, z) = 0, \tag{2.21}$$

and this, together with the restriction,

$$a_1 = a_4 = a_5 = 0 \tag{2.22}$$

forbids surging, heaving and swaying motions of the body. We emphasize that due to its definition (2.5), the functional S acts as follows:

$$S : H^1_{\rm odd} \to \mathbb{C}^6_{\rm odd}, \tag{2.23}$$

where

$$H^1_{\rm odd}(\Omega) \ = \ \{\varphi \in H^1(\Omega) : \varphi \text{ satisfies (2.21)}\}, \tag{2.24}$$

$$\mathbb{C}^6_{\rm odd} \ = \ \{a = (a_1, \dots, a_6)^\top \in \mathbb{C}^6 : \text{(2.22) is fulfilled}\} \tag{2.25}$$

and $H^1(\Omega)$ stands for the Sobolev space. In its subspace (2.24), an equivalent norm is given by the Dirichlet norm $\|\nabla\varphi; L^2(\Omega)\| = (\nabla\varphi, \nabla\varphi)^{1/2}_\Omega$, where $(\cdot, \cdot)_\Omega$ is the natural scalar product of the Lebesgue space $L^2(\Omega)$.

As shown in Nazarov & Videman (submitted), Nazarov 2011b, the continuous ω-spectrum of the John problem, with additional restrictions (2.21), (2.22), coincides with the set

$$(-\infty, -\omega_\dagger] \cup [\omega_\dagger, +\infty), \tag{2.26}$$

where $\omega_\dagger = \sqrt{g\lambda_\dagger}$ and $\lambda_\dagger > 0$ is the principal eigenvalue of the following model Steklov problem

$$\begin{array}{rcll} -(\partial_y^2 + \partial_z^2)\varphi(y, z) & = & 0 \ , & (y, z) \in \varpi_+ \\ \partial_z\varphi(y, 0) & = & \lambda\varphi(y, 0) \ , & y \in (0, l), \\ \partial_n\varphi(y, z) & = & 0, & (y, z) \in \sigma \ , \ y > 0, \\ \varphi(0, z) & = & 0, & (0, z) \in \varpi, \end{array} \tag{2.27}$$

in the half $\varpi_+ = \{(y, z) \in \varpi : y > 0\}$ of the cylinder cross-section. The corresponding eigenfunction is denoted by $\varphi_\dagger$.

Thus, the interval (2.19) indeed remains available for the discrete spectrum. Any eigentriple

$$\{\omega, \varphi, a\} \in (-\omega_\dagger, \omega_\dagger) \times H^1_{\rm odd}(\Omega) \times \mathbb{C}^6_{\rm odd} \tag{2.28}$$

obviously remains as an eigentriple for the original problem in Ω, i.e., ω is an eigenfrequency embedded in the continuous spectrum, and $\{\varphi, a\}$ is a trapped mode with velocity potential having exponential decay at infinity.

3. Variational and operator formulations of the problem.

Following Nazarov & Videman (submitted) we formulate the boundary value problem (2.14)–(2.18) as the integral identity

$$\begin{aligned}&(\nabla\varphi,\nabla\psi)_\Omega+g(Ka,b)_{\mathbb{C}}+i\omega(a,S\psi)_{\mathbb{C}}-i\omega(S\varphi,b)_{\mathbb{C}}\\ =\;&\omega^2g^{-1}(\varphi,\psi)_\Upsilon+\omega^2(Ma,b)_{\mathbb{C}}\quad\forall\ \{\psi,b\}\in H^1(\Omega)\times\mathbb{C}^6,\end{aligned}\tag{3.1}$$

where $(\cdot,\cdot)_\Upsilon$ and $(\cdot,\cdot)_{\mathbb{C}}$ are the natural scalar products in $L^2(\Upsilon)$ and $\mathbb{C}^6$, respectively. We still call the variational problem (3.1) the John problem. It can be restricted onto the product space $H^1_{\rm odd}(\Omega)\times\mathbb{C}^6_{\rm odd}$. We then introduce the Dirichlet scalar product in $H^1_{\rm odd}(\Omega)$,

$$\langle\varphi,\psi\rangle=(\nabla\varphi,\nabla\psi)_\Omega\tag{3.2}$$

and the trace operator T, Nazarov 2008, 2009,

$$\langle T\varphi,\psi\rangle=(\varphi,\psi)_\Upsilon\quad\forall\ \varphi,\psi\in H^1_{\rm odd}(\Omega);\tag{3.3}$$

this is positive, continuous, and self-adjoint with the essential spectrum $[0,\lambda_\dagger^{-1}]$. Note that $\mu=0$ is an eigenvalue of T with infinite multiplicity. Moreover, the continuous spectrum $(0,\lambda_\dagger^{-1}]$ emerges from surface wave processes at infinity; such processes may also appear in a finite volume water domain, if the boundary is not Lipschitz (see Nazarov & Taskinen 2010 and Nazarov & Taskinen (in press)), however, our assumption on the Lipschitz property of $\partial\Omega$ prevents the latter in the present work.

The restricted problem can now be rewritten equivalently as an abstract spectral equation in the Hilbert space $H^1_{\rm odd}(\Omega)\times\mathbb{C}^6_{\rm odd}$

$$\begin{pmatrix}\mathbb{I}&0\\0&gK\end{pmatrix}\begin{pmatrix}\varphi\\a\end{pmatrix}+\omega\begin{pmatrix}0&iS^*\\-iS&0\end{pmatrix}\begin{pmatrix}\varphi\\a\end{pmatrix}=\omega^2\begin{pmatrix}g^{-1}T&0\\0&M\end{pmatrix}\begin{pmatrix}\varphi\\a\end{pmatrix},\tag{3.4}$$

where $\mathbb{I}$ is the identity operator in $H^1(\omega)$ and S^* is the adjoint of (2.23), (2.5).

The equation (3.4) gives rise to the quadratic operator pencil $\omega\mapsto A(\omega)$ (a polynomial spectral family), and, aiming to use general results for self-adjoint operators in Hilbert space (cf. Birman & Solomyak 1987), we further process it following the scheme developed in Nazarov & Videman (submitted). The symmetry assumption (2.20) will lead to substantial simplifications.

To simplify the equation (3.4), we take into account the requirement (2.22) and the structure (2.7) of the matrix K and write

$$\begin{aligned}&M^\natural=\begin{pmatrix}M_{\flat\flat}&M_{\flat5}\\M_{5\flat}&M_{55}\end{pmatrix}=\int_\Theta D^\natural(x,y,z-z^\bullet)^\top D^\natural(x,y,z-z^\bullet)\varrho(\mathbf{x})d\mathbf{x},\\ &K^\natural=\begin{pmatrix}0&0&0\\0&0&0\\0&0&k\end{pmatrix},\quad k=\int_\theta y^2dxdy+\int_\Xi(z-z^\bullet)d\mathbf{x}>0.\end{aligned}\tag{3.5}$$

Here, M_{55} is a scalar, $M_{5\flat}=M_{\flat5}^\top$ is a row of length 2, and $M_{\flat\flat}$ is a positive definite 2×2-block of the 3×3-matrix $M^\natural$ constructed similarly to (2.6) from the fragment

$$D^{\natural}(x,y,z)=\begin{pmatrix} 0 & y & 0 \\ 1 & -x & z \\ 0 & 0 & -y \end{pmatrix} \tag{3.6}$$

of the rigid motion matrix (2.4). Denoting $a_\flat=(a_2,a_3)^\top$ and $S_\flat\varphi=(S_2\varphi,S_3\varphi)^\top$ we omit the empty lines in the algebraic part of the system (3.4) and reduce the latter to

$$-i\omega S_\flat\varphi \;=\; \omega^2(M_{\flat\flat}a_\flat+M_{\flat 5}a_5), \tag{3.7}$$

$$gka_5-i\omega S_5\varphi \;=\; \omega^2(M_{5\flat}a_\flat+M_{55}a_5). \tag{3.8}$$

We still introduce the function and the column

$$\eta=\omega g^{-1/2}T^{1/2}\varphi\ ,\ f^{\natural}=(f_2,f_3,f_5)^\top=\omega M^{\natural}a^{\natural}, \tag{3.9}$$

where $T^{1/2}$ is the positive square root of the positive, continuous, self-adjoint operator T in (3.3) (see for example Th. 12.33 in Rudin 1982 or §10.3 in Birman & Solomyak 1987). We also split the (symmetric and positive definite) inverse matrix $N^{\natural}=(M^{\natural})^{-1}$ into the blocks $N_{\flat\flat}$, $N_{\flat 5}=N_{5\flat}^\top$ and N_{55} as in (3.5). From (3.7) and (3.9) we derive

$$\begin{aligned} &-iS_\flat\varphi=\omega M_{\flat\flat}a_\flat+\omega M_{\flat 5}a_5=f_\flat\ , \\ &-iN_{5\flat}S_\flat\varphi+N_{55}f_5=\omega a_5\ , \\ &-iS_\flat^*a_\flat=-iS_\flat^*N_{\flat\flat}f_\flat-iS_\flat^*N_{\flat 5}f_5=S_\flat^*N_{\flat\flat}S_\flat\varphi-iS_\flat^*N_{\flat 5}f_5. \end{aligned} \tag{3.10}$$

These relations reduce the equations (3.7) and (3.9) to the spectral equation for a linear pencil

$$\mathcal{B}\mathcal{X}=\omega\mathcal{D}\mathcal{X}\ \text{ in }\ \mathcal{H}=H^1_{\mathrm{odd}}(\Omega)\times H^1_{\mathrm{odd}}(\Omega)\times\mathbb{C}\times\mathbb{C}, \tag{3.11}$$

where $\mathcal{X}=(\varphi,\eta,a_5,f_5)^\top$ and the matrix operators $\mathcal{B}$ and $\mathcal{D}$, respectively, take the form

$$\begin{bmatrix} \mathbb{I}+S_\flat^*N_{\flat\flat}S_\flat & 0 & 0 & iS_\flat^*N_{\flat 5} \\ 0 & \mathbb{I} & 0 & 0 \\ 0 & 0 & gk & 0 \\ -iN_{5\flat}S_\flat & 0 & 0 & N_{55} \end{bmatrix}, \quad \begin{bmatrix} 0 & g^{-1/2}T^{1/2} & -iS_5^* & 0 \\ g^{-1/2}T^{1/2} & 0 & 0 & 0 \\ iS_5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.12}$$

Both operators are clearly self-adjoint, and $\mathcal{B}$ is also positive definite due to the calculation

$$\begin{aligned} (\mathcal{B}\mathcal{X},\mathcal{X})_{\mathcal{H}}&=\langle\varphi+S_\flat^*N_{\flat\flat}S_\flat\varphi,\varphi\rangle+i\langle S_\flat^*N_{\flat 5}f_5,\varphi\rangle+\langle\eta,\eta\rangle \\ &+gk|a_5|^2-i(N_{5\flat}S_\flat\varphi,f_5)_{\mathbb{C}}+N_{55}|f_5|^2 \\ &=\langle\varphi,\varphi\rangle+\langle\eta,\eta\rangle+gk|a_5|^2+\left(N\begin{pmatrix}-iS_\flat\varphi\\ f_5\end{pmatrix},\begin{pmatrix}-iS_\flat\varphi\\ f_5\end{pmatrix}\right)_{\mathbb{C}}. \end{aligned} \tag{3.13}$$

We now set

$$\mathcal{Y}=\mathcal{B}^{1/2}\mathcal{X}\ ,\ \mathcal{A}=\mathcal{B}^{-1/2}\mathcal{D}\mathcal{B}^{-1/2}\ ,\ \alpha=1/\omega \tag{3.14}$$

and write (3.11) as

$$\mathcal{A}\mathcal{Y} = \alpha\mathcal{Y} \quad , \mathcal{Y} \in \mathcal{H}. \tag{3.15}$$

The operator $\mathcal{A}$ is continuous, self-adjoint but not positive: due to the relation of the spectral parameters α and ω in (3.14), the essential spectrum of $\mathcal{A}$ equals the segment

$$[-\omega_\dagger^{-1}, \omega_\dagger^{-1}], \tag{3.16}$$

which is just the inversion of the set (2.26).

The spectral problems (3.15) with $\alpha > 0$, and (3.4) or (3.10) with $\omega > 0$ are evidently equivalent with each other. As a conclusion, we have obtained the standard spectral problem (3.15) using the reduction scheme Nazarov & Videman (submitted).

4. Reference problem.

Setting $K = \mathbb{O}_6$ formally in (2.18) yields

$$a = -i\omega^{-1}M^{-1}S\varphi \tag{4.1}$$

and therefore the boundary condition (2.17) on the surface ξ of the immersed body turns into

$$\partial_n\varphi(\mathbf{x}) = -\mathbf{n}(\mathbf{x})^\top D(\mathbf{x} - \mathbf{x}^\bullet)M^{-1}S\varphi \ , \ \mathbf{x} \in \xi. \tag{4.2}$$

To write the variational formulation (cf. Nazarov 2011b) of the boundary value problem (2.14)–(2.16), (4.2) we again employ the artificial Dirichlet condition (2.21) and take into account the evident formula

$$\int_\xi \overline{\psi(\mathbf{x})}\mathbf{n}(\mathbf{x})^\top D(\mathbf{x} - \mathbf{x}^\bullet)M^{-1}S\Phi ds_{\mathbf{x}} = (M^{-1}S\Phi, S\psi)_{\mathbb{C}}, \tag{4.3}$$

thus obtaining the integral identity

$$(\nabla\Phi, \nabla\psi)_\Omega + (M^{-1}S\Phi, S\psi)_{\mathbb{C}} = \Lambda(\Phi, \psi)_\Gamma \quad \forall\ \psi \in H^1_{\text{odd}}(\Omega). \tag{4.4}$$

The notation Φ, Λ is used instead of φ, λ in order to distinguish (4.4) from the problems discussed above. Since M^{-1} is positive definite, the left hand side of (4.4) can be taken as the new scalar product $\langle\Phi, \psi\rangle_M$ in the Sobolev space (2.24). Moreover, similarly to (3.3), the formula

$$\langle T_M\Phi, \psi\rangle_M = (\Phi, \psi)_\Upsilon \quad \forall\ \Phi, \psi \in H^1_{\text{odd}}(\Omega) \tag{4.5}$$

generates the modified trace operator T_M in $H^1_{\text{odd}}(\Omega)$. It is still positive, continuous and self-adjoint. The scalar product $\langle\Phi, \psi\rangle_M$ differs from the "old" one, (3.2), by the term $(M^{-1}S\varphi, S\psi)_{\mathbb{C}}$, but this only gives rise to a compact perturbation operator; recall that the integration surface in (4.3) is bounded and hence the trace embedding $H^1(\Omega) \subset L^2(\xi)$ is compact. As a consequence, T_M inherits the essential spectrum $[0, \lambda_\dagger^{-1}]$ from T.

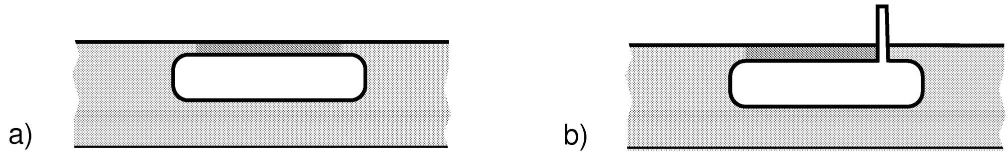

Figure 1. The longitudinal cross-section of the channel with bodies.

Problem (4.4) converts into the abstract equation

$$T_M \Phi = \mu \Phi \tag{4.6}$$

in $\mathcal{H}$, containing the spectral parameter $\mu = 1/\Lambda$. The variational method of Evans et al. 1994, Kamotskii & Nazarov 2003 was adapted in Nazarov 2009 and also Nazarov & Videman (submitted) to water-wave problems, and it was used in Nazarov 2011b to prove the following sufficient condition for the existence of trapped modes in the problem (4.4):

THEOREM 1. *Assume that*

$$(\nabla\varphi_\dagger, \nabla\varphi_\dagger)_\Xi \geq \lambda_\dagger(\varphi_\dagger, \varphi_\dagger)_\theta + (M^{-1}S\varphi_\dagger, S\varphi_\dagger)_\mathbb{C}, \tag{4.7}$$

where $\{\lambda_\dagger, \varphi_\dagger\}$ *is the principal eigenpair of the model Steklov problem* (2.27). *Then the dicrete spectrum of* T_M *is not empty, and thus the problem* (4.4) *has an eigenvalue* $\Lambda_1 \in (0, \lambda_\dagger)$, *and the corresponding eigenfunction* $\Phi_1 \in H^1_{\rm odd}(\Omega)$ *constitutes a trapped mode.*

The condition (4.7) is stronger than the sufficient condition in Nazarov 2009,

$$\int\limits_\Xi |\nabla\varphi_\dagger(y,z)|^2 d\mathbf{x} \geq \lambda_\dagger \int\limits_\theta |\varphi_\dagger(y,0)|^2 dydz, \tag{4.8}$$

which ensures a trapped surface wave in the case of a fixed obstacle Θ, corresponding to the boundary value problem (2.14)–(2.16), (4.2), with $M^{-1} = \mathbb{O}_6$. For example, (4.8) is always satisfied when the body Θ is submerged, because $\theta = \varnothing$ and the right hand side of (4.8) vanishes. (Recall the classical works Ursell 1951, 1987 and Garipov 1967 on trapped modes for submerged obstacles.) However, the inequality (4.7) is not evident at all even in case $(\varphi_\dagger, \varphi_\dagger)_\theta = 0$. Note that passing to the limit $M \to \infty$ leads to a fixed obstacle Θ, and this surely contradicts with the Archimedean law (2.12), because the volume v is prescribed. This is the very reason why the the fixed obstacle problem cannot form a good approximation of the John problem.

We next employ the ideas of Nazarov 2008 to give an example of a trapped mode in the problem (4.4). We denote, for some positive numbers b, d and h,

$$G_h = \{(x,y,z) : |x| < b, y \in (0,d), z \in (-h,0)\} \subset \Omega_+ = \Omega \cap \{y > 0\}, \tag{4.9}$$

which is a water layer near the surface surrounding the submerged part (2.1) of Θ (overshadowed in Fig. 4.1).

The lateral surface of G_h is denoted by L_h and the horizontal surfaces by R_0 and R_h. Considering the auxiliary Steklov problem

$$\begin{array}{rcll} \Delta v(\mathbf{x}) &=& 0, & \mathbf{x}\in G_h, \\ \partial_z v(x,y,0) &=& \beta v(x,y,0), & \mathbf{x}\in R_0, \\ -\partial_z v(x,y,-h) &=& 0, & \mathbf{x}\in R_h, \\ v(\mathbf{x}) &=& 0, & \mathbf{x}\in L_h, \end{array} \tag{4.10}$$

we readily compute one of its eigenpairs, which is not the principal one:

$$V(\mathbf{x}) = \sin\left(\frac{\pi}{b}x\right)\sin\left(\frac{\pi}{d}y\right)\left(e^{zF}+e^{-(z+2h)F}\right)\;,\;\; B = F\frac{1-e^{-2hF}}{1+e^{-2hF}}, \tag{4.11}$$

with $F = \pi\sqrt{b^{-2}+d^{-2}}$. Because of the homogeneous Dirichlet condition on L_h we can extend V as null to Ω_+, and then to Ω as an odd function in y. Since V is odd in both variables x and y, we obtain the equality

$$Sv = \int\limits_{R_0} V(x,y,-h)D(x,y,z-z^\bullet)^\top(0,0,-1)^\top\,dxdy = 0\in\mathbb{C}^6, \tag{4.12}$$

which is a crucial property of (4.11). We insert this trial function to the very definition of the operator norm $\|T_M\|$, obtaining

$$\begin{aligned} \|T_M\| &= \sup_{v\in H^1_{\rm odd}(\Omega)} \frac{\langle T_M v, v\rangle_M}{\langle v,v\rangle_M} \geq \frac{\langle T_M V, V\rangle_M}{\langle V,V\rangle_M} \\ &= \frac{\|V;L^2(\Upsilon)\|^2}{\|\nabla V;L^2(\Omega)\|^2+(M^{-1}SV,SV)_{\mathbb{C}}} = \frac{\|V;L^2(R_0)\|^2}{\|\nabla V;L^2(G_h)\|^2} \\ &= \frac{1}{B} = \frac{1}{F}\frac{1+e^{-2hF}}{1-e^{-2hF}}. \end{aligned}$$

From (4.11) it follows that B^{-1} and $\|T_M\|$ can be made arbitrarily large by preserving the sizes b, d and diminishing h. In particular, for a certain $h_\dagger > 0$ and $h < h_\dagger$ there holds the inequality $\|T_M\| > \mu_\dagger$, where $\mu_\dagger = \lambda_\dagger^{-1}$ is the end point of the continuous spectrum $[0,\mu_\dagger]$ of T_M. Since the norm of a continuous self-adjoint operator always belongs to its spectrum, we conclude that $\mu_1 := \|T_M\|$ is an eigenvalue of T_M. Thus, $\lambda_1 := \mu_1^{-1}\in(0,\lambda_\dagger)$ is an eigenvalue of the auxiliary problem (4.4), or, the problem (2.14)–(2.16), (4.2). We found a trapped mode $\Phi_1\in H^1_{\rm odd}(\Omega)$, and the desired example is completed.

5. Localization estimate for an eigenfrequency.

According to the spectral theorem (see Rudin 1982, Thm. 12.21, 12.22, or Birman & Solomyak, Thm. 6.1.1) any continuous self-adjoint operator $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ can be associated with an operator valued spectral measure $E_{\mathcal{A}}$, which in turn defines the scalar valued positive measure $\mu_{\mathcal{Y},\mathcal{Y}} = (E_{\mathcal{A}}\mathcal{Y},\mathcal{Y})_{\mathcal{H}}$ for any element $\mathcal{Y}\in\mathcal{H}$. We need the following formulas, which can be found for example in Theorem 12.21 of

Rudin 1982 or in the proof of Theorem 6.1.3 of Birman & Solomyak 1987:

$$\begin{aligned} \|\mathcal{Y};\mathcal{H}\|^2 &= \int_{\mathbb{R}} d\mu_{\mathcal{Y},\mathcal{Y}}(t) \quad \text{for } \mathcal{Y}\in\mathcal{H}, \\ \|\mathcal{A}\mathcal{Y}-\alpha\mathcal{Y};\mathcal{H}\|^2 &= \int_{\mathbb{R}} (t-\alpha)^2 d\mu_{\mathcal{Y},\mathcal{Y}}(t) \quad \text{for } \mathcal{Y}\in\mathcal{H} \text{ and } \alpha\in\mathbb{R}. \end{aligned} \tag{5.1}$$

Let $\{\Lambda,\Phi\}$ be an eigenpair of the problem (4.5). Recalling the formulas (4.1) and (3.9), (3.14), we set

$$\begin{aligned} &\alpha=\omega^{-1}=(g\Lambda)^{-1/2}\ ,\ \mathcal{X}=(\varphi,\eta,a_5,f_5)^\top\ ,\ \mathcal{Y}=\mathcal{B}^{1/2}\mathcal{X}\ , \\ &\varphi=\Phi\ ,\ \eta=\omega g^{-1/2}T^{1/2}\Phi\ ,\ a=-i\omega^{-1}M^{-1}S\Phi\ ,\ f=\omega Ma=-iS\Phi. \end{aligned} \tag{5.2}$$

Since the components (2.22) of the comlumn a vanish, (3.13) and (4.5) imply

$$\begin{aligned} \|\mathcal{Y};\mathcal{H}\|^2 &= (\mathcal{B}\mathcal{X},\mathcal{X})_{\mathcal{H}}=\langle\Phi,\Phi\rangle+g^1\omega^2\langle T\Phi,\Phi\rangle+gk|a_5|^2+(M^{-1}S\Phi,S\Phi)_{\mathbb{C}} \\ &= 2\Lambda\|\Phi;L^2(\Upsilon)\|^2+gk|a_5|^2. \end{aligned}$$

Then

$$\|\mathcal{A}\mathcal{Y}-\alpha\mathcal{Y};\mathcal{H}\|=\sup_{\mathcal{W}\in\mathcal{H},\,\|\mathcal{W};\mathcal{H}\|=1}|(\mathcal{A}\mathcal{Y}-\alpha\mathcal{Y},\mathcal{W})_{\mathcal{H}}|$$

$$=\ \frac{1}{\omega}\sup_{\mathcal{U}\in\mathcal{H},\,\|\mathcal{B}^{1/2}\mathcal{U};\mathcal{H}\|=1}|(\omega\mathcal{D}\mathcal{X}-\mathcal{B}\mathcal{X},\mathcal{U})_{\mathcal{H}}|\ ;$$

here we have changed $\mathcal{W}$ to $\mathcal{U}=\mathcal{B}^{-1/2}\mathcal{W}$ and used (3.14). Let us calculate the components of the column

$$\mathcal{B}\mathcal{X}-\omega\mathcal{D}\mathcal{X}=(\psi,Y,A,F)^\top.$$

In view of the structure of the matrix operators (3.12) we obtain

$$\begin{aligned} \psi &= \varphi+S_b^*N_{\flat\flat}S_b\varphi+iS_b^*N_{\flat 5}f_5-\omega g^{-1/2}T^{1/2}\eta+i\omega S_5^*a_5 \\ &= \Phi+S_b^*N_{\flat\flat}S_b\Phi+S_b^*N_{\flat 5}S_5\Phi-g^{-1}\omega^2T\Phi+S_5^*(M^{-1}S\Phi)_5 \\ &= \Phi+S^*M^{-1}S\Phi-\Lambda T\Phi=0, \\ Y &= \eta-\omega g^{-1/2}T^{1/2}\varphi=\omega g^{-1/2}T^{1/2}\Phi-\omega gT^{1/2}\Phi=0, \\ A &= gka_5-i\omega S_5\varphi-\omega f_5=gka_5-i\omega S_5\Phi+i\omega(S\Phi)_5=gka_5, \\ F &= -iN_{5\flat}S_b\varphi+N_{55}f_5-\omega a_5=-iN_{5\flat}S_b\Phi-iN_{55}S_5\Phi+i(M^{-1}S\Phi)_5=0. \end{aligned}$$

We now see that

$$\|\mathcal{A}\mathcal{Y}-\alpha\mathcal{Y};\mathcal{H}\|=\omega^{-1}\sup|gka_5\bar{b}_5|=\omega^{-1}\sqrt{gk}|a_5|^2. \tag{5.3}$$

Indeed, the supremum is computed over all $\mathcal{U}=(\psi,\xi,b,h)^\top\in\mathcal{H}$ such that $(\mathcal{B}\mathcal{U},\mathcal{U})_{\mathcal{H}}=1$, while the latter formula together with (3.13) yields the inequality $gk|b_5|^2\le 1$, and this readily leads to (5.3).

The above calculations show that

$$\delta(\Lambda,\Phi) := \frac{\|\mathcal{A}\mathcal{Y}-\alpha\mathcal{Y};\mathcal{H}\|}{\|\mathcal{Y};\mathcal{H}\|} = \frac{1}{\omega}\sqrt{\frac{gk|a_5|^2}{2\Lambda\|\Phi;L^2(\Upsilon)\|^2+gk|a_5|^2}}\ . \tag{5.4}$$

Taking some $\delta > \delta(\Lambda,\Phi)$ we assume that the segment $\upsilon(\delta) = [\alpha-\delta,\alpha+\delta]$ does not contain a point of the spectrum of $\mathcal{A}$. Using (5.1) we then write

$$\|\mathcal{A}\mathcal{Y}-\alpha\mathcal{Y};\mathcal{H}\|^2 = \int\limits_{\mathbb{R}\setminus\upsilon(\delta)} (t-\alpha)^2 d\mu_{\mathcal{Y},\mathcal{Y}}(t)$$

$$\geq \ \delta^2 \int\limits_{\mathbb{R}\setminus\upsilon(\delta)} d\mu_{\mathcal{Y},\mathcal{Y}}(t) = \delta^2\int\limits_{\mathbb{R}} d\mu_{\mathcal{Y},\mathcal{Y}}(t) = \delta^2\|\mathcal{Y};\mathcal{H}\|^2,$$

which is absurd due to the choice of δ and (5.4). Hence, the segment $\upsilon(\delta)$ must have a non-empty intersection with the spectrum of $\mathcal{A}$. If in addition

$$\frac{1}{\omega} - \delta > \frac{1}{\omega_\dagger}, \tag{5.5}$$

then the intersection becomes free of the continuous spectrum (3.16) of $\mathcal{A}$ and thus must contain an eigenvalue $\alpha_1 = (\omega_1)^{-1}$ of the discrete spectrum of $\mathcal{A}$. This means that the problem (3.1) gets an eigenfrequency $\omega_1 > 0$ with a trapped mode $\{\varphi, a\} \in H^1_{\rm odd}(\Omega) \times \mathbb{C}^6_{\rm odd}$.

Let us formulate the result obtained above.

THEOREM 2. *Let* $\{\Lambda_1,\Phi_1\} \in (0,\lambda_\dagger) \times H^1_{\rm odd}(\Omega)$ *be an eigenpair of the reference problem* (4.4)*,* Φ_1 *normalized in* $L^2(\Upsilon)$*, and let* s_5 *be the fifth component of the column*

$$s = M^{-1}S\Phi_1, \tag{5.6}$$

where M *is the inertia matrix* (2.6) *and* S *is the functional* (2.23)*. If*

$$1 > \sqrt{\frac{\Lambda_1}{\lambda_\dagger}} + \sqrt{\frac{k|s_5|^2}{2\Lambda^2+k|s_5|^2}}, \tag{5.7}$$

then the problem (3.1) *has an eigenfrequency* ω_1 *such that*

$$\Big|\frac{\sqrt{g\Lambda_1}}{\omega_1} - 1\Big| \leq \sqrt{\frac{k|s_5|^2}{2\Lambda^2+k|s_5|^2}}. \tag{5.8}$$

Note that, first, $\lambda_1 < \lambda_\dagger$ and the first term on the right of (5.7) is strictly less than 1, and, second, the condition (5.7) ensures the existence of an eigenfrequency embedded into the continuous spectrum of the problem (3.1). It is different from the sufficient conditions of Nazarov 2011b and Nazarov & Videman (submitted), which also concern modes trapped by freely floating bodies.

Applications of the method, which we have developed above for localization estimates, will not be restricted to Theorem 2.

6. Concluding remarks.

Since the factor k, from (3.5), does not appear in the reference problem (4.5), it is straightforward to give an example of a trapped mode, based on our study in Section 4 on wave processes in the near-surface water-layer (4.9). Indeed, after finding an eigenpair $\{\Lambda_1, \Phi_1\} \in (0, \lambda_\dagger) \times H^1_{\rm odd}(\omega)$ we may choose k so small that (5.7) is valid, since the first term on the right hand side of it is smaller than 1. We emphasize that for a submerged body with $\theta = \varnothing$ one has $k \to +0$, if $z^\bullet \to z^\ominus - 0$ (see (2.9) and (3.5)). The relations $z^\bullet \approx z^\ominus$ and hence $k \ll 1$ can be achieved by redistributing the mass inside the fixed volume Θ, that is, by varying the density $\varrho(\mathbf{x})$ while keeping the matrix (2.6) unchanged and the conditions (2.8), (2.20) satisfied. If the body Θ is surface-piercing, then to make k small we need to assume that the cross-section $\theta = \Theta \cap \Gamma$ is small, cf. Fig. 4.1,b. It should be stressed here that the analysis at the end of Section 4 is *local*, and the existence of an eigenvalue $\Lambda_1 \in (0, \lambda_\dagger)$ is proven without any *global* assumptions on the geometry of Θ or Π. Moreover, a simple asymptotic analysis of singular perturbations, Maz'ya et al. 1991, Ch. 2,5, would demonstrate that varying θ at some distance from G_h has only small influence on the eigenvalue.

The inequality (5.8) must be regarded as a *localization estimate*, since it contains an estimate for the deviation of ω_1 from $\sqrt{g\Lambda_1}$. We emphasize that in the case $s_5 = 0$ the quantities ω_1 and $\sqrt{g\Lambda_1}$ coincide. This curious observation has a clear physical backround. The heaving and pitching motions are forbidden by the symmetry restriction (2.22), and the rolling is cancelled, too, by the condition $s_5 = 0$, see (5.6) and (4.1), hence, we have $Ka = 0$, and the formal derivation of the integro-differential boundary condition (4.2) becomes fully rigorous. In other words, the equality $s_5 = 0$ turns $\{\Lambda_1, \Phi_1, -i(g\Lambda_1)^{-1/2} M^{-1} S\Phi\} \in (0, \lambda_\dagger) \times H^1_{\rm odd}(\omega) \times C^6_{\rm odd}$ into a spectral triple of the John problem (3.1).

It is worthwhile to point out the difference between the above conclusion and a result in Kuznetsov 2010. First, we do not need a complete annulation of the motion column a (yawing and swaying may appear in $M^{-1}S\phi$), and, second, the localization estimate allows for a perturbation, because the sufficient condition (5.7) still remains valid, if s_5 is small.

We conlude by a remark that the present approach gives a straightforward way to find motionless freely floating bodies supporting trapped modes. To this end, we just assume additional symmetry with respect to the plane $\{x = 0\}$,

$$\Omega = \{\mathbf{x} : (-x, y, z) \in \Omega\} \ , \quad \varrho(x, y, z) = \varrho(-x, y, z), \tag{6.1}$$

and impose the duplicate of the artificial boundary condition

$$\varphi(0, y, z) = 0 \tag{6.2}$$

with the concomitant restrictions

$$a_2 = a_3 = a_6 = 0, \tag{6.3}$$

cf. (2.20), (2.21) and (2.22), respectively. Since the velocity potential φ is now odd both in x and y, we have $S\varphi = 0$, cf. (4.12). Hence, if

$$\Omega_{++} := \{\mathbf{x} \in \Omega : x, y > 0\} \ , \quad \Theta_{++} := \{\mathbf{x} \in \Theta : x, y > 0\}$$

and φ_{++} is a trapped mode of the water-wave problem in Ω_{++} with the fixed "quarter" obstacle Θ_{++}, then extending φ to Ω as an odd function of x and y and augmenting with the null column $a = 0$ give a trapped mode for the John problem with the same eigenfrequency. The analysis of trapped modes arising from the near-surface layer is local (in Nazarov 2008 and also in Section 4), hence, the appearance of the additional Dirichlet condition (6.2) does not affect the existence of trapped modes for the problem in Ω_{++}, Θ_{++}. Of course, we did not use the additional symmetry assumption (6.1) in Section 4.

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