EMBEDDED EIGENVALUES FOR WATER-WAVES IN A THREE DIMENSIONAL CHANNEL WITH A THIN SCREEN

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Abstract. We construct asymptotic expansions as $\varepsilon \to +0$ for an eigenvalue embedded into the continuous spectrum of water-wave problem in a cylindrical three dimensional channel with a thin screen of thickness $O(\varepsilon)$. The screen may be either submerged or surface-piercing. The channel and the screen are mirror symmetric so that imposing the Dirichlet condition in the middle plane creates an artificial positive cut-off-value $\Lambda_\dagger$ of the modified spectrum. The wetted part of the screen has a sharp edge. Depending on a certain integral characteristics I of the screen profiles, we find two types of asymptotics, $\Lambda_\dagger - O(\varepsilon^2)$ and $\Lambda_\dagger - O(\varepsilon^4)$ in the cases $I > 0$ and $I = 0$, respectively. We prove that in the case $I < 0$ there are no embedded eigenvalues in the interval $[0, \Lambda_\dagger]$, while this interval contains exactly one eigenvalue, if $I \geq 0$. For the justification of these result, the main tools are a reduction to an abstract spectral equation and the use of the max-min-principle.

1. Introduction

1.1. **Formulation of the problem.** We investigate the interaction of water-waves with a thin screen, which is submerged or surface piercing in a cylindrical three dimensional channel. The channel is infinite and invariant along the x_1-direction, moreover, it and the screen are assumed to be mirror symmetric. The wave motion is supposed to take place in an incompressible and inviscid fluid.

We consider the linear-water wave equation, where the spectral parameter, related to wave motion, appears in the Steklov boundary condition on the free water surface. Our aim is to discuss the existence and uniqueness of an eigenvalue embedded in the continuous spectrum. Our main results state that such an eigenvalue exists depending on the behaviour of a certain integral characteristics $I(h)$ to be defined later. Indeed we shall show that, in the case $I(h) < 0$, no eigenvalues exist in the interval $(0, \Lambda^\dagger)$, where $\Lambda_\dagger$ is a positive, artificial cut-off point, and that for $I(h) \geq 0$ an eigenvalue does exist in $(0, \Lambda_\dagger)$. However, in the cases $I(h) > 0$ and $I(h) = 0$ the eigenvalues have different asymptotic behaviour. For a sufficiently thin screen an eigenvalue is shown to be unique in $(0, \Lambda_\dagger)$ so that the inequality $I(h) \geq 0$ becomes a criterion for a trapped mode. The edge of the screen is assumed to be sharp, which simplifies our justification scheme but on the other hand requires an elaborate analysis of singularities of solutions on the edge, see Section 3.3.

Let us start by describing the water domain under consideration and formulating the spectral problem. The cylindrical three dimensional channel (Fig. 1.1.a) is

Key words and phrases. linear water wave system, cylindrical channel, Steklov condition, asymptotic analysis, artificial Dirichlet condition, continuous spectrum, embedded eigenvalue.

The second named author was supported by RFFI, grant 15-01-02175 and by the Academy of Finland Project 289706. The third named author was partially supported by grants from the group GNAMPA of INDAM, the Magnus Ehrnrooth Foundation and Väisälä Foundation of the Finnish Academy of Science and Letters.

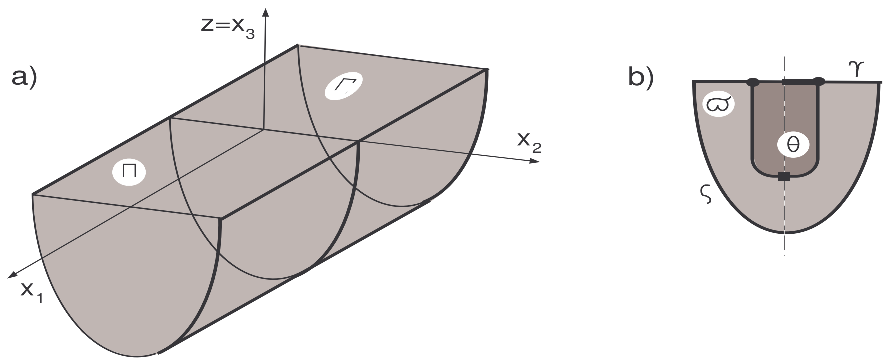

FIGURE 1.1. Channel and its cross-section.

defined by

$$\Pi = \{x = (x_1, x_2, x_3) = (x_1, x') : x_1 \in \mathbb{R}, x' \in \varpi\} = \mathbb{R} \times \varpi, \tag{1.1}$$

where the cross-section $\varpi \subset \mathbb{R}^2$ is a bounded domain, the boundary $\partial\varpi$ of which consists of the line segment

$$\gamma = \{x' = (x_2, x_3) : z = x_3 = 0, |x_2| < l\} \ , \quad l > 0, \tag{1.2}$$

and of a smooth arc $\varsigma \subset \mathbb{R}^2_- = \{(x_2, x_3) : x_3 < 0\}$ connecting the points $P^\pm = (\pm l, 0)$.

The thin screen Θ^ε depending on the small parameter $\varepsilon > 0$ is described as follows. Let $\theta \subset \mathbb{R}^2$ an open subset of $\overline{\varpi}$, such that $P^\pm \notin \overline{\theta} = \theta \cup \partial\theta$. Assuming that two, not identically zero profile functions $h_\pm \in \mathcal{C}^2(\theta)$ are given such that $h = h_+ + h_- \geq 0$, we define the thin screen, flat screen and the profile boundary, respectively, by

$$\Theta^\varepsilon = \{x : x' \in \overline{\theta}, -\varepsilon h_-(x') \leq x_1 \leq \varepsilon h_+(x')\} \ , \tag{1.3}$$

$$\Theta^0 = \{x : x' \in \overline{\theta}, x_1 = 0\}, \tag{1.4}$$

$$\theta^\varepsilon_\pm = \{x : x' \in \theta, x_1 = \pm\varepsilon h_\pm(x')\} \ . \tag{1.5}$$

By rescaling we reduce the characteristic size of the cross section ϖ to one and, therefore, make the Cartesian coordinates x and all geometric parameters dimensionless. To avoid many inessential technical difficulties we assume that the curve $\psi = \partial\theta \cap \varpi$ is smooth and that ψ and ς both intersect γ at right angle $\alpha = \pi/2$. Note in particular that in this case the boundary is non-cuspidal and there does not appear wave processes in finite volume, contrary to cases considered in [27, 28].

We denote by $\Omega^\varepsilon = \Pi \setminus \Theta^\varepsilon$, $\varepsilon \geq 0$, the channel (1.1) with the thin or flat vertical screen (1.3), Fig. 1.2.a, and consider the Steklov spectral problem describing the propagation of water-waves along the horizontal free surface

$$\Gamma^\varepsilon = \Gamma \setminus \Theta^\varepsilon, \tag{1.6}$$

where $\Gamma = \gamma \times \mathbb{R}$ is the intact channel surface . Notice that Γ can be pierced by the screen, but in the case $\partial\theta \cap \gamma = \varnothing$ the obstacle Θ^ε is submerged, and, therefore $\Gamma^\varepsilon = \Gamma$. The bottom and walls $\Sigma = \varsigma \times \mathbb{R}$ of the channel Π can be touched by the obstacle, too, and we denote

$$\Sigma^\varepsilon = \Sigma \setminus \Theta^\varepsilon. \tag{1.7}$$

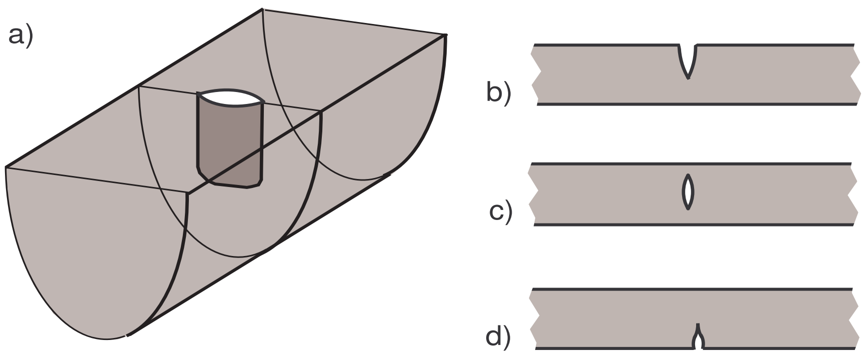

FIGURE 1.2. Channel with screen in different positions.

Let us formulate the problem. For any $\varepsilon > 0$, the velocity potential u^ε satisfies the Laplace equation

$$-\Delta u^\varepsilon(x) = 0,\ x \in \Omega^\varepsilon, \tag{1.8}$$

the Neumann (no-flow) boundary condition on the wetted surfaces (1.7) and (1.5),

$$\partial_\nu u^\varepsilon(x) = 0,\ x \in \Sigma^\varepsilon \cup \theta_+^\varepsilon \cup \theta_-^\varepsilon, \tag{1.9}$$

and the kinematic condition on the free surface (1.6)

$$\partial_z u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\ x \in \Gamma^\varepsilon. \tag{1.10}$$

We denote the gradient and Laplacian with respect to the variable x by ∇ and Δ, while ∂_z and ∂_ν stand for the partial derivative with respect to $z = x_3$ and the outer unit normal, respectively. Moreover, $\lambda^\varepsilon = g^{-1}(\omega^\varepsilon)^2$ is a spectral parameter, where $g > 0$ is the acceleration of gravity and $\omega^\varepsilon > 0$ is the frequency of time harmonic oscillations.

We make the following assumptions on symmetry and shape of the screen, the role of which will be discussed in Section 1.3.

1°. Both ϖ and θ are symmetric with respect to the axis $\{x' : x_2 = 0\}$.
2°. Both profile functions $h_\pm$ in (1.3), (1.5) are even in x_2.
3°. We have $h_\pm(x') = 0$ for $x' \in \psi = \partial\theta \setminus \gamma$.

1.2. **Main results and plan of the paper.** It is known that the spectrum of the problem (1.8)–(1.10) is continuous and coincides with the intact closed positive semi-axis $\overline{\mathbb{R}_+} = [0, +\infty) \subset \mathbb{C}$, see [13]. However, it may contain embedded eigenvalues associated with exponentially decaying eigenfunctions. The main purpose of our paper is to derive and justify an asymptotic formula for such eigenvalues as well as to prove a uniqueness result. To this end we shall use in Section 1.3 the symmetry assumptions $\mathbf{1}^\circ - \mathbf{3}^\circ$ to introduce a problem (1.15)–(1.18) with an artificial Dirichlet condition on the symmetry plane. The continuous spectrum of this problem is known to be the interval $[\Lambda_\dagger, +\infty)$, where the threshold $\Lambda_\dagger$ is positive. In Sections 2 and 3 we construct formal asymptotics for an eigenvalue

$$\lambda_\bullet^\varepsilon = \Lambda_\dagger - \widehat{\lambda}^\varepsilon\ ,\quad \widehat{\lambda}^\varepsilon \to +0 \text{ as } \varepsilon \to +0. \tag{1.11}$$

of the problem (1.15)–(1.18); $\lambda_\bullet^\varepsilon$ is also an eigenvalue of the problem (1.8)–(1.10). We shall apply asymptotic analysis, which involves rectifying the screen Θ^ε and transferring the Neumann boundary conditions onto the faces of the flat screen Θ^0.

Moreover, in Section 2 we introduce an integral characteristics (2.28), denoted by $I(h)$, such that for $I(h) > 0$ the correction term $\widehat{\lambda}^\varepsilon \approx \lambda_0 \varepsilon^2$ is positive, but for $I(h) < 0$ it is not. We can now formulate the first main result of our paper.

Theorem 1.1. *Assume that the conditions* $\mathbf{1}^\circ$–$\mathbf{3}^\circ$ *hold true. Then, there exists* $\varepsilon_1 = \varepsilon_1(\theta, h_\pm) > 0$ *such that*
1) if $I(h) < 0$, *the problem* (1.8)–(1.10) *has no eigenvalue in the segment* $[0, \Lambda_\dagger]$, *when* $\varepsilon \in (0, \varepsilon_1]$,
2) if $I(h) > 0$, *the problem* (1.8)–(1.10) *has for every* $\varepsilon \in (0, \varepsilon_1]$ *a unique eigenvalue* (1.11) *inside the segment* $[0, \Lambda_\dagger]$. *The coefficient* $\lambda_0 > 0$ *is given by* (2.31), (2.28) *and the asymptotic remainder* $\widetilde{\lambda}^\varepsilon = \lambda_\bullet - (\Lambda_\dagger - \lambda_0 \varepsilon_0^2) = \lambda_0 \varepsilon_0^2 - \widehat{\lambda}^\varepsilon$ *satisfies the estimate*

$$|\widetilde{\lambda}^\varepsilon| \leq c_1 \varepsilon^{5/2}, \tag{1.12}$$

where c_1 *is independent of the small parameter* ε.

The case $I(h) = 0$ will be examined in Section 3, where a new characteristics $J(h) > 0$, (3.22), as well as the formula $\widehat{\lambda}^\varepsilon \approx \lambda_1 \varepsilon^4$ will be derived. The related calculations become much more complicated, and they crucially rely on the assumption $\mathbf{3}^\circ$. The corresponding result is formulated as Theorem 3.1 below.

The asymptotic procedure will be justified in the last two sections. In Section 4 we prove uniqueness assertions, namely, we verify that in the case $I(h) < 0$ the interval $(0, \Lambda_\dagger)$ does not contain eigenvalues at all, but in the case $I(h) \geq 0$ the eigenvalue $\lambda^\varepsilon \in (0, \Lambda_\dagger)$ is unique. In Section 5 we show that indeed, the eigenvalue λ^ε exists and has the asymptotic form claimed in Theorems 1.1 and 3.1. Moreover, we give estimates for the asymptotic remainders. All these results are based on the reduction of water-wave problem (1.15)–(1.18) to the abstract spectral equation (4.18) and the application of basic theory of self-adjoint Hilbert space operators, cf. [2, 31].

We finish the paper with several particular conclusions, possible generalisations and open questions.

1.3. **Role of symmetry restrictions.** Operator theoretic methods, crucial in our paper, work only for the discrete spectrum, but the problem (1.8)–(1.10) cannot have isolated eigenvalues, since the continuous spectrum is $\overline{\mathbb{R}_+} = [0, +\infty)$. To create an artificial positive cut-off value $\Lambda_\dagger$ we borrow an elegant idea [6] of the Dirichlet boundary condition on the midplane of the the waveguide Π^ε, for which we need the symmetry assumptions $\mathbf{1}^\circ$, $\mathbf{2}^\circ$. These requirements allow us to restrict the problem (1.8)–(1.10) to the half of the channel Ω^ε,

$$\Omega^\varepsilon_\flat = \{x \in \Omega^\varepsilon \,:\, x_2 > 0\}, \tag{1.13}$$

and to impose the artificial Dirichlet condition on the middle plane

$$\begin{aligned} &\Upsilon^\varepsilon = \{x \in \Omega^\varepsilon \,:\, x_2 = 0\} = \Upsilon \setminus \Theta^\varepsilon \,, \\ &\Upsilon = \upsilon \times \mathbb{R} \,, \quad \upsilon = \{x' \in \omega \,:\, x_2 = 0\}. \end{aligned} \tag{1.14}$$

All objects restricted to the domain (1.13) are supplied with the subscript $\flat$ so that the new problem reads as

$$-\Delta u^\varepsilon(x) = 0, \qquad x \in \Omega^\varepsilon_\flat, \tag{1.15}$$

$$\partial_\nu u^\varepsilon(x) = 0, \qquad x \in \Sigma_\flat \cup \theta^\varepsilon_{+,\flat} \cup \theta^\varepsilon_{-,\flat} \tag{1.16}$$

$$\partial_z u^\varepsilon(x) = \lambda^\varepsilon_\flat u(x), \quad x \in \Gamma_\flat, \tag{1.17}$$

$$u^\varepsilon(x) = 0, \qquad x \in \Upsilon^\varepsilon. \tag{1.18}$$

A reason for the artificial boundary condition (1.18) is that, owing to general results in [11], [26, §3.1, §5.1], the continuous spectrum of the problem (1.15)–(1.18) coincides with the ray $[\Lambda_\dagger, +\infty)$, where $\Lambda_\dagger$ is nothing but the first eigenvalue of the model problem on the half $\varpi_\flat$ of the cross-section ϖ:

$$-\Delta' U(x') = 0, \qquad x' \in \varpi_\flat, \tag{1.19}$$

$$\partial_\nu U(x') = 0, \qquad x' \in \varsigma_\flat \tag{1.20}$$

$$\partial_z U(x') = \Lambda U(x'), \; x' \in \gamma_\flat, \tag{1.21}$$

$$U(x') = 0, \qquad x' \in \upsilon. \tag{1.22}$$

Here, Δ' is the Laplacian in the coordinates x'. Due to the Dirichlet condition (1.22), the first eigenvalue $\Lambda = \Lambda_\dagger$ is positive and can be computed from the max-min-principle

$$\Lambda_\dagger = \inf_V \frac{\|\nabla' V; L^2(\varpi_\flat)\|^2}{\|V; L^2(\gamma_\flat)\|^2}, \tag{1.23}$$

where the infimum is taken over all $V \in H^1_0(\varpi_\flat, \upsilon) \setminus H^1_0(\varpi_\flat, \gamma_\flat)$ and $H^1_0(\varpi_\flat, \omega)$ is the Sobolev space of functions vanishing in the subdomain $\omega \subset \varpi_\flat$. According to the strong maximum principle the corresponding eigenfunction $U_\dagger$ can be chosen positive in $\varpi_\flat$ and subject to the normalization condition

$$\int_0^l |U_\dagger(x_2, 0)|^2 dx_2 = 1. \tag{1.24}$$

Since the cut-off value (1.23) is positive, the problem (1.15)–(1.18) may still have discrete spectrum in the interval $(0, \Lambda_\dagger)$. Moreover, the odd extension of an eigenfunction $u^\varepsilon \in H^1(\Omega^\varepsilon_\flat; \Upsilon^\varepsilon)$ of (1.15)–(1.18) is smooth and harmonic, and therefore it becomes an eigenfunction of the original problem (1.8)–(1.10). In this way, eigenvalues embedded into the interval $(0, \Lambda_\dagger)$ can be examined using operator theory.

Remark 1.2. It was observed in [22], in connection with a different spectral problem, that the existence of the eigenvalue (1.11) for (1.8)–(1.10) implies the existence of a solution, stabilizing at infinity, for the limit problem corresponding to $\varepsilon = 0$,

$$-\Delta u^0(x) = 0, \quad x \in \Omega^0_\flat = \Pi_\flat \setminus \Theta^0_\flat, \tag{1.25}$$

$$\partial_\nu u^0(x) = 0, \quad x \in \Sigma_\flat \cup \theta^0_{+,\flat} \cup \theta^0_{-,\flat} \tag{1.26}$$

$$\partial_z u^0(x) = \Lambda_\dagger u^0(x), x \in \Gamma_\flat, \tag{1.27}$$

$$u^0(x) = 0, \quad x \in \Upsilon^0. \tag{1.28}$$

In our case this stabilizing at infinity-solution can be readily found: it is

$$\mathbf{u}^0(x) = U_\dagger(x'), \tag{1.29}$$

where $U_\dagger$ is the eigenfunction of (1.19)–(1.22) associated with the eigenvalue $\Lambda_\dagger$. Indeed, on the surfaces $\theta^0_\pm = \{x : x' \in \theta, x_1 = 0\}$ of the flat screen (1.4), the derivative ∂_ν equals $\mp\partial_1 = \mp\partial_{x_1}$, while the stable wave (1.29) does not depend on the longitudinal coordinate x_1.

1.4. **Literature review.** There exist quite many interesting results about eigenfrequencies of water-waves in different open geometries, like the half-space of an infinite channel with obstacles, obtained with miscellaneous methods. We refer to the review papers [14, 5] and the monograph [13] for exhaustive expositions, and mention here only a few cases.

The first example of an eigenvalue belonging to the discrete spectrum of a problem on oblique waves for a submerged circular cylinder was proposed in [32]. An eigenvalue embedded in the continuous spectrum was constructed in [12] by means of the semi-inverse method. The results in these pioneering papers were obtained by analytic calculations, and they have inspired many other publications with analytic, operator theoretic or numerical methods. In particular, the existence of eigenvalues below the continuous spectrum has been verified with the help of a comparison principle in the paper [33], which also extends the results of [32] to a cylinder with an arbitrary cross-section with positive area.

The papers [19, 21] and also [29] contain an approach relying upon a reformulation of the water-wave problem as a self-adjoint operator in a specific Hilbert space and an application of the max-min-principle, see e.g. [2, Thm. 10.2.2.] and [31]. This method has given rather simple proofs of known facts and also new results. In particular, assuming the symmetry property $\mathbf{1}^\circ$–$\mathbf{2}^\circ$ and taking a screen Θ^ε, $\varepsilon > 0$, which is either submerged or of null thickness $h = 0$ (the case of an absolutely flat screen (1.4) excluded), it follows from [19] that the problem (1.8)–(1.10) has an eigenvalue and the corresponding eigenfunction decays exponentially at infinity; see also [35]. This result cannot be obtained by the approach of [33], because $\mathrm{mes}_3\Theta^\varepsilon = 0$ in the case $h = 0$. Moreover, the paper [19] gives a simple sufficient condition for the existence of trapped modes supported by surface-piercing obstacles, but it does not provide any uniqueness result. In Section 6.1 we shall show that in the case of small ε this condition becomes a necessary condition for a unique eigenvalue.

In addition to the above described approach of [6], which requires the symmetry conditions $\mathbf{1}^\circ$ and $\mathbf{2}^\circ$, there exists another method [22, 25] to detect embedded eigenvalues. This is based on the asymptotic analysis of the so-called augmented scattering matrix, which provides a criterion for the existence of trapped modes. This approach does not require the symmetry of the domains ϖ, θ, or the evenness of the profile functions $h_\pm$. Instead, it uses the natural instability of embedded eigenvalues and performs a very fine tuning of several geometric parameters of the screen shape in order to keep an eigenvalue in the continuous spectrum. We plan to return to this later. We emphasize that the eigenvalue $\lambda^\varepsilon_\bullet$, to be found in the sequel, is stable, when $h_\pm$ are perturbed with functions even in x_2, but asymmetric perturbations may lead $\lambda^\varepsilon_\bullet$ out of the spectrum and turn it into a point of complex resonance, cf. [1, 23].

The method of matched asymptotic expansions, cf. [34, 9] will be employed in Sections 2 and 3 by applying the interpretation of [20, 22]. Related asymptotic procedures have been used in [8, 7, 20, 3, 4] etc. to describe asymptotic behaviour of eigenvalues of different physical systems in cylindrical waveguides with small regular and singular perturbations. However, the present work is quite different in several aspects. Let us conclude this section by discussing that.

First of all, the reference waveguide $\Omega^0 = \Pi \setminus \Theta^0$ has originally a large defect, the flat screen (1.4), but it is known not to possess an eigenvalue. The two papers [35, 24] treat problems for two-dimensional water-wave and acoustic waveguides with similar screens. Asymptotic analysis is much simpler in dimension 2, and in the citations it is thus possible to investigate the boundary layer phenomenon in the vicinity of rounded, chamfered and biased ends of the linear screen. Such boundary layers have not been investigated yet in dimension 3, although a few particular results can be found in [15, Part IV]; see also Section 6.3 of the present paper. Thus, we unfortunately have to accept the restriction **3**°: this makes the edge of the screen

$$\Psi = \{x \,:\, x_1 = 0, x' \in \psi\} \tag{1.30}$$

dihedral or cuspidal, see Fig. 1.2.b-d, but it avoids the boundary layer effect. See Section 6 for a further discussion.

Second, we deal with screens which pierce the free surface, Fig. 1.2.b, and abut the walls and bottom, Fig. 1.2.d, while in [35] the screen is situated inside the channel, Fig. 1.2.c. Note that in the case of a surface-piercing screen Θ^ε we are able to single out shapes, which do not support trapped modes for any $\lambda^\varepsilon \in (0, \Lambda_\dagger)$, while screens which always trap a wave are outlined in Fig. 1.2.c,d.

Third, although the sharp edge (1.30) of the screen causes singular behaviour of the velocity potential u^ε, the assumption **3**° enables the use of asymptotic methods generated by regular perturbations of the boundary.

Finally, we shall find two different types of asymptotic expansions of the eigenvalue (1.11), which depend on some integral characteristics of the screen and which are in full agreement with the sufficient condition for the existence of trapped modes, see Section 6.1. In this way the sufficient condition of [19] becomes also a necessary one for a small ε.

2. Asymptotic analysis. Non-degenerate case

2.1. **Outer expansions.** In this section we search for an eigenvalue of (1.15)–(1.18) in the form

$$\lambda^\varepsilon_\bullet = \Lambda_\dagger - \varepsilon^2\lambda_0 + \widetilde{\lambda}^\varepsilon, \tag{2.1}$$

where λ_0 is a *positive* number to be computed. We shall obtain an estimate for the remainder $\widetilde{\lambda}^\varepsilon$ in Section 5.4.

At a long distance from the screen Θ^ε we assume the following asymptotic ansatz for a trapped wave corresponding to (2.1),

$$u^\varepsilon_\bullet(x) = c_\pm(\varepsilon)e^{\mp\mu(\varepsilon)x_1}V(\varepsilon; x') + \dots\ , \ \pm x_1 \gg 1, \tag{2.2}$$

where the dots stand for higher order terms and the couple $\{\mu(\varepsilon), V(\varepsilon; x')\}$ is a solution of the following problem in a two-dimensional domain,

$$\begin{aligned} -\Delta' V(\varepsilon; x') &= \mu(\varepsilon)^2 V(\varepsilon; x'), \quad x' \in \varpi_\flat, \\ \partial_\nu V(\varepsilon; x') &= 0,\ x' \in \varsigma_\flat, \quad V(\varepsilon; x') = 0,\ x' \in \upsilon, \\ \partial_z V(\varepsilon; x') &= \lambda^\varepsilon V(\varepsilon; x'), \quad x' \in \gamma_\flat. \end{aligned} \tag{2.3}$$

Perturbation theory of linear operators, see e.g. [10, Ch. 6], yields the representations

$$\mu(\varepsilon) = 0 + \varepsilon\mu_0 + \widetilde{\mu}(\varepsilon)\ ,\ V(\varepsilon; x') = U_\dagger(x') + \varepsilon^2 V_0(x') + \widetilde{V}(\varepsilon; x'), \tag{2.4}$$

the following problem for the correction terms in (2.4),

$$-\Delta' V_0(x') = \mu_0^2 U_\dagger(x'), \quad x' \in \varpi_\flat, \tag{2.5}$$

$$\partial_\nu V_0(x') = 0,\ x' \in \varsigma_\flat\ , \quad V_0(x') = 0,\ x' \in \upsilon, \tag{2.6}$$

$$\partial_z V_0(x') = \Lambda_\dagger V_0(x') - \lambda_0 U_\dagger(x'), x' \in \gamma_\flat, \tag{2.7}$$

as well as the error estimates

$$|\widetilde{\mu}(\varepsilon)| \leq c\varepsilon^2 \ , \quad \|\widetilde{V}(\varepsilon,\cdot); H^1(\varpi_\flat)\| \leq c\varepsilon^3. \tag{2.8}$$

We mention that (2.3) is obtained by inserting the exponential waves $e^{\pm\mu(\varepsilon)x_1}V(\varepsilon,x')$ into the problem (1.15)–(1.18), while (2.5)–(2.7) follows by substituting (2.1), (2.4) into (2.3) and extracting terms of order ε^2.

Since $\Lambda_\dagger$ is a simple eigenvalue of the problem (1.19)–(1.22), the Fredholm alternative yields only one compatibility condition, which by the Green formula turns into

$$\mu_0^2 \int\limits_{\varpi_\flat} |U_\dagger(x')|^2 dx' = - \int\limits_{\varpi_\flat} \big(U_\dagger(x')\Delta' V_0(x') - V_0(x')\Delta' U_\dagger(x')\big) dx'$$

$$= \int\limits_{\partial\varpi_\flat} \big(V_0(x')\partial_\nu U_\dagger(x') - U_\dagger(x')\partial_\nu V_0(x')\big) ds_{x'} = \lambda_0 \int\limits_0^l |U_\dagger(x_2,0)|^2 dx_2.$$

This was obtained by taking into account the differential equations (1.19) and (2.5) as well as the boundary conditions (1.20)-(1.22) and (2.6), (2.7). Moreover, according to the normalization condition (1.24) we have

$$\mu_0 = \|U_\dagger; L^2(\varpi_\flat)\|^{-1}\lambda_0^{1/2}. \tag{2.9}$$

As a consequence, the outer expansions (2.2) looks as follows:

$$u^\varepsilon(x) = c_\pm(0)U_\dagger(x') + \varepsilon\big(c'_\pm(0)U_\dagger(x') \mp c_\pm(0)\mu_0 x_1 U_\dagger(x')\big) + \ldots, \ \pm x_1 \gg 1 \tag{2.10}$$

Note that $c_\pm(\varepsilon) = c_\pm(0) + \varepsilon c'_\pm(0) + O(\varepsilon^2)$ is just the Taylor formula for the coefficients in (2.2).

2.2. **Inner expansion.** In a bounded part of the channel $\Omega^\varepsilon_\flat$, e.g. near the screen $\Theta^\varepsilon_\flat$, we can take a traditional expansion for a trapped mode:

$$u^\varepsilon_\bullet(x) = v_0(x) + \varepsilon v_1(x) + \ldots \ . \tag{2.11}$$

The matching procedure, cf. [34, 9, 20, 22], requires that the behaviour of $v_0(x)$ and $v_1(x)$ as $x_1 \to \pm\infty$ is given by the similar terms in (2.10). Thus, as the first step we notice that

$$v_0(x) = c_\pm(0)U_\dagger(x) + \ldots \text{ for } x_1 \to \pm\infty.$$

Recalling the stabilizing solution (1.29) of the limit problem (1.25)–(1.28), we set

$$c_\pm(0) = 1 \text{ and } v_0(x) = U_\dagger(x'). \tag{2.12}$$

To derive a problem for the correction term v_1 in (2.11) we first observe that passing to the limit $\varepsilon \to 0^+$ flattens the curved screen Θ^ε into the planar one Θ^0, cf. formulas (1.3) and (1.4). Hence, the equation (1.15) in Ω and the Neumann condition (1.16) on $\Sigma^\varepsilon_\flat$ yield

$$-\Delta v_1(x) = 0 \ , \ x \in \Omega \ , \tag{2.13}$$

$$-\partial_\nu v_1(x) = 0 \ , \ x \in \Sigma^0_\flat, \ , \tag{2.14}$$

In the same way, the artificial Dirichlet condition (1.18) turns into

$$v_1(x) = 0 \ , \ x \in \Upsilon^0, \tag{2.15}$$

while the spectral condition (1.17) gains the threshold parameter because of the relation $\lambda_\bullet = \Lambda_\dagger + O(\varepsilon^2)$, so that

$$-\partial_\nu v_1(x) = \Lambda_\dagger v_1(x)\ ,\ \ x \in \Gamma^0_\flat,\ , \tag{2.16}$$

It remains to transfer the Neumann condition (1.16) from the curved surfaces $\theta^\varepsilon_{\pm,\flat}$ onto the flat ones $\theta^0_{\pm,\flat}$. To do so, we recall definition (1.3) and write the representation

$$\nu^\varepsilon_\pm(x') = \big(1 + \varepsilon|\nabla' h_\pm(x')\big)^{-1/2}\big(\pm 1, \varepsilon\nabla' h_\pm(x')\big)$$

for the unit normal vector. Hence,

$$\big(1 + \varepsilon^2|\nabla' h_\pm(x')|^2\big)^{1/2}\partial_{\nu^\varepsilon_\pm} = \mp\partial_1 + \varepsilon\nabla' h_\pm(x')\cdot\nabla', \tag{2.17}$$

where $\nabla' = (\partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$ and the central dot stands for the scalar product in $\mathbb{R}^2$. This and the Taylor formula with respect to x_1 yield

$$\begin{aligned}
&\big(1 + \varepsilon^2|\nabla' h_\pm(x')|^2\big)^{1/2}\partial_{\nu^\varepsilon_\pm} v(\pm\varepsilon h_\pm(x'), x') \\
&= \pm\partial_1 v(\pm\varepsilon h_\pm(x'), x') + \varepsilon\nabla' h(x')\cdot v(\pm\varepsilon h_\pm(x'), x') \\
&= \pm\partial_1 v(\pm 0, x') - \varepsilon h_\pm(x')\partial_1^2 v(\pm 0, x') \\
&\quad + \varepsilon\nabla' h_\pm(x')\cdot\nabla' v(\pm 0, x') + \dots\ ,\ \ x' \in \theta.
\end{aligned} \tag{2.18}$$

Finally, inserting (2.11), (2.12) into (2.18) and extracting terms $O(\varepsilon)$ yield the following Neumann conditions on the faces $\theta^0_{\pm\flat}$ of the planar incision $\Theta^0_\flat$:

$$\mp\partial_1 v_1(\pm 0, x') = -\nabla' h_\pm(x')\cdot\nabla' U_\dagger(x')\ ,\ \ x' \in \theta_\flat. \tag{2.19}$$

2.3. **Solutions of the limit problem at threshold and matching procedure.** In addition to the solution (1.29), even in x_1, the problem (1.25)–(1.28) has a solution, which is odd in x_1 and has the representation

$$\mathbf{u}^1(x) = \widetilde{\mathbf{u}}(x) + \sum_\pm \chi_\pm(x_1)(x_1 \pm \mathbf{b})U_\dagger(x'), \tag{2.20}$$

where the remainder $\widetilde{\mathbf{u}}^1(x)$ decays exponentially as $x_1 \to \pm\infty$, $\mathbf{b}$ is a constant depending on ϖ, θ, and $\chi_\pm$ are smooth cut-off functions such that

$$\chi_\pm(x_1) = 1 \text{ for } \pm x_1 > 2\ ,\ \ \chi_\pm(x_1) = 0 \text{ for } \pm x_1 < 1\ ,\ \ 0 \le \chi_\pm \le 1. \tag{2.21}$$

There is no other solution with at most polynomial growth at infinity.

These facts are based on the information on the model problem (1.19)–(1.22) in Section 1.3 and follow from general results of the elliptic theory in domains with cylindrical outlets to infinity, see e.g. [26, Ch. 5], [17, § 3], [18]. They can also be obtained using the Fourier method by reducing the problem (1.25)–(1.28) to the quarter

$$\Pi_{\flat,+} = \{x \in \Pi\ :\ x_1 > 0, x_2 > 0\} \tag{2.22}$$

of Π and imposing either the Neumann condition (even case) or the Dirichlet condition (odd case) on the subset $\{x\ :\ x_1 = 0, x' \in \varpi_\flat \setminus \overline{\theta}_\flat\}$ of the end of the semi-infinite cylinder (2.22). We also mention that in Section 4.1 we shall verify the absence of trapped modes in the problem (1.25)–(1.28).

By similar arguments we can find out that the problem (2.13)–(2.16), (2.19) has a solution v_1 with linear growth at infinity. Since it is only defined up to a linear combination $c_0\mathbf{u}^0 + c_1\mathbf{u}^1$, we may choose the coeffients c_0, c_1 such that

$$v_1(x) = \widetilde{v}_1(x) + \sum_{\pm} \chi_{\pm}(x_1)\big(b_1^1|x_1| \pm b_1^0\big)U_\dagger(x'), \tag{2.23}$$

see (1.29), (2.20). The remainder $\widetilde{v}_1(x)$ decays exponentially and the coefficients b_1^1, b_1^0 are now uniquely defined. We do not need an explicit expression for b_1^0 and thus we only note that, if $h_+ = h_-$ in (2.19), the function (2.23) is even in x_1 and therefore $b_1^0 = 0$. Let us compute b_1^1.

We insert v_1 and $\mathbf{u}^0$ into the Green formula on the truncated channel $\Omega_\flat^0(R) = \{x \in \Omega_\flat : |x_1| < R\}$ and obtain

$$\begin{aligned} 0 &= \int\limits_{\partial\Omega_\flat^0(R)} \big(U_\dagger(x')\partial_\nu v_1(x) - v_1(x)\partial_\nu U_+(x')\big)ds_x \\ &= \sum_{\pm} - \int\limits_{\theta_\flat} U_\dagger(x')\nabla' h_\pm(x') \cdot \nabla' U_\dagger(x')dx' + \sum_{\pm} \pm \int\limits_{\varpi_\flat} U_\dagger(x')\partial_\nu v_1(\pm R, x')dx' \\ &= - \int\limits_{\theta_\flat} U_\dagger(x')\nabla' h(x') \cdot \nabla' U_\dagger(x')dx' + 2b_1^1 \int\limits_{\varpi_\flat} |U_\dagger(x')|^2 dx' + o(1) \end{aligned} \tag{2.24}$$

as $R \to +\infty$. Here we have used the boundary conditions on $\partial\Omega_\flat^0$, in particular (2.19), and the asymptotic expansion (2.23) at $x_1 = \pm R$. Passing to the limit $R \to +\infty$ in (2.24) yields

$$b_1^1 = -\frac{1}{2}\|U_\dagger; L^2(\varpi_\flat\|^{-2} I(h), \tag{2.25}$$

where we have after integration by parts

$$\begin{aligned} I(h) = &\int\limits_{\theta_\flat} h(x')U_\dagger(x')\Delta' U_\dagger(x')dx' + \int\limits_{\theta_\flat} h(x')|\nabla' U_\dagger(x')|^2 dx' \\ &- \int\limits_{\partial\theta_\flat} h(x')U_\dagger(x')\partial_\nu U_\dagger(x')ds_{x'}. \end{aligned} \tag{2.26}$$

The first integral on the right vanishes due to (1.19), and our assumption **3°** reduces the last integral to the set

$$\phi_\flat = \gamma_\flat \cap \partial\theta_\flat \tag{2.27}$$

(the bold segment in Fig. 1.1.b), where $\partial_\nu U_\dagger = \partial_z U_\dagger = \Lambda_\dagger U_\dagger$ according to (1.21). Thus,

$$I(h) = \int\limits_{\theta_\flat} h(x')|\nabla' U_\dagger(x')|^2 dx' - \Lambda_\dagger \int\limits_{\phi_\flat} h(x')|\nabla' U_\dagger(x')|^2 dx_2. \tag{2.28}$$

Notice that $I(h) > 0$ for sure, if h does not vanish identically and the set (2.27) is empty, i.e., the screen is submerged.

We can now deduce the formula for the coefficient λ_0 in the asymptotic formula for the eigenvalue $\lambda_\bullet$, see Theorem 1.1. As mentioned in Section 2.2, the behaviour of the correction term $v_1(x)$ as $x_1 \to \pm\infty$ in the inner expansion (2.11) is described by the coefficients of ε in the outer expansion. Comparing the linear functions in (2.10) and (2.23) we see that

$$\mp\mu_0 = \pm b_1^1 \;\text{ and }\; c'_\pm(0) = \pm b_0^0. \tag{2.29}$$

Hence, the relations (2.9) and (2.25) lead us to the formula

$$\|U_\dagger; L^2(\varpi_\flat)\|^{-1}\lambda_0^{1/2} = \mu_0 = -b_1^1 = \frac{1}{2}\|U_\dagger; L^2(\varpi_\flat)\|^{-2}I(h), \tag{2.30}$$

which makes sense only, if $I(h) > 0$, whence

$$\lambda_0 = \frac{1}{4}\|U_\dagger; L^2(\varpi_\flat)\|^{-2}I(h)^2. \tag{2.31}$$

On the other hand, the inequality $I(h) < 0$ makes it impossible to obtain $\lambda_0 > 0$ from (2.30). We shall consider the degenerate case $I(h) = 0$ in the next section and now only recall that the specification (2.31) of λ_0 completes the formulation of Theorem 1.1; the justification will be presented in Sections 4 and 5.

3. Asymptotic analysis. Degenerate case.

3.1. **Updating the asymptotic ansätze.** Throughout this section we assume $I(h) = 0$. Then, by (2.25), (2.31), and (2.9), we have

$$b_1^1 = 0 \;\text{ and }\; \lambda_0 = 0, \mu_0 = 0. \tag{3.1}$$

The solution (2.20) of the problem (2.13)–(2.16), (2.19) becomes bounded. Moreover, to ensure the inclusion $\lambda_\bullet^\varepsilon \in (0, \Lambda_\dagger)$, we must amend the ansatz (2.1) by setting

$$\lambda_\bullet^\varepsilon = \Lambda_\dagger - \varepsilon^4\lambda_1 + \widetilde{\lambda}^\varepsilon \;, \lambda_1 > 0. \tag{3.2}$$

We also have to modify the ansätze (2.4) as follows:

$$\mu(\varepsilon) = 0 + \varepsilon^2\mu_1 + \widetilde{\mu}(\varepsilon) \;,\; V(\varepsilon, x') = U_\dagger(x') + \varepsilon^4 V_1(x') + \widetilde{V}(\varepsilon; x') \tag{3.3}$$

Accordingly, estimates (2.8) turn into

$$|\widetilde{\mu}(\varepsilon)| \le c\varepsilon^4 \;,\; \|\widetilde{V}(\varepsilon, \cdot); H^1(\varpi_\flat)\| \le c\varepsilon^6. \tag{3.4}$$

The pair $\{\mu_1, V_1\}$ in (3.3) satisfies the problem (2.5)–(2.7), which is again derived from (2.9) with evident changes. The compatibility condition in this problem is converted into the relation

$$\mu_1 = \|U_\dagger; L^2(\varpi_\flat)\|^{-1}\lambda_1^{1/2}. \tag{3.5}$$

Finally, applying the above mentioned modifications to the outer expansions (2.2) results into the following ansatz,

$$\begin{aligned} u_0^\varepsilon &= c_\pm(0)U_\dagger(x') + \varepsilon c'_\pm(0)U_\dagger(x') \\ &+\varepsilon^2\big(c''_\pm(0)U_\dagger(x') \mp c_\pm(0)\mu_1 x_1 U_\dagger(x')\big) + \ldots,\; \pm x_1 >> 1. \end{aligned} \tag{3.6}$$

Then, the inner expansion (2.11) becomes

$$u_0^\varepsilon(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \ldots. \tag{3.7}$$

3.2. **First asymptotic terms.** Formulas (2.12) can be obtained in the same way as in Section 2.2. Moreover, the coefficient (2.25) in the decomposition (2.23) vanishes, so that matching the multiplier of ε in (3.6) with the corresponding term in (3.7), namely

$$v_1(x) = \widetilde{v}_1(x) + \sum_{\pm} \pm\chi_\pm(x_1) b_1^0 U_\dagger(x'), \tag{3.8}$$

gives the relation (2.29).

Let us compose a boundary value problem in $\Omega_\flat^\varepsilon$ for the term v_2 in (3.7). Of course this function satisfies the differential equation (2.13) and the boundary conditions (2.14)–(2.16), when the subscripts are changed from 1 to 2. To derive the boundary conditions on the faces $\theta^0_{\pm,\flat}$, we refine the decomposition (2.18) and write

$$\begin{aligned}
&\big(1+\varepsilon^2|\nabla' h_\pm(x')|^2\big)^{1/2}\partial_{\nu_\pm^\varepsilon}\big(v_0(x')+\varepsilon v_1(\pm\varepsilon h_\pm(x'),x')+\varepsilon^2 v_2(\pm\varepsilon h_\pm(x'),x')\big)\\
&= 0+\varepsilon\big(\pm\partial_1 v_1(\pm 0,x')+\nabla' h(x')\cdot\nabla' v_0(x')\big)\\
&+\varepsilon^2\big(\pm\partial_1 v_2(\pm 0,x')+\nabla' h_\pm(x')\cdot\nabla' v_1(\pm 0,x')-h_\pm(x')\partial_1^2 v_1(\pm 0,x')\big)+\dots. \qquad (3.9)
\end{aligned}$$

We recall the Laplace equation (2.13) and delete the coefficient of ε^2 in (3.9) by imposing the Neumann conditions

$$\begin{aligned}
\pm\partial_1 v_2(\pm 0,x') &= -\nabla' h_\pm(x')\cdot\nabla' v_1(\pm 0,x') - h_\pm(x')\Delta' v_1(\pm 0,x')\\
&= -\nabla'\cdot\big(h_\pm(x')\nabla' v_1(\pm 0,x')\big)\ ,\ x'\in\theta_\flat. \qquad (3.10)
\end{aligned}$$

3.3. **Remarks on singularities.** The boundary value problems under consideration have been posed on domains with corner points and edges, which may cause singular behaviour for their solutions. Actually, some of our geometric assumptions in Section 1 were made in order to reduce the influence of the singularities to the asymptotic procedure.

First of all we mention that the eigenfunction $U_\dagger$ of the problem (1.19)–(1.22) is infinitely differentiable everywhere in $\overline{\varpi_\flat}$, because the arc $\varsigma_\flat$ is smooth and meets the x_2- and x_3-axis at right angle. A reason for the exclusion of the singularities can be found, e.g., in [26, § 2.4].

The behaviour of the solution v_1 of the problem (2.13)–(2.16), (2.19) near the edge $\Psi_\flat$ of the screen $\Theta_\flat$ may be quite complicated because of the endpoints of the arc $\psi_\flat$, which are tops of polyhedral angles. As known e.g. by [26, Ch. 10, Ch. 11], the behaviour of v_1 in the interior of Ψ is determined by the functions

$$K_j(s) r^{j/2}\cos(j\varphi/2)\ ,\ j=0,1,2,\dots, \tag{3.11}$$

where $s\in(0,L)$ is the arc length along ψ and $(r,\varphi)\in\mathbb{R}_+\times(0,2\pi)$ is the polar coordinate system in planes, which are perpendicular to Ψ. The function (3.11) with $j=0$ is smooth so that the main singularities of the derivatives of v_1 are produced by $K_1(s)r^{1/2}\cos(\varphi/2)$. The coefficient function K_1 is called the intensity factor in mechanics, and since the data in (2.19) is infinitely differentiable, it belongs to $C^\infty(0,L)$. However, K_1 may become singular at the tops $s=0$ and $s=L$ of the polyhedral angles.

As for the point $s=0$, which is marked by ■ in Fig. 1.1.b, the function K_1 is smooth there, since v_1 can be extended as an odd function with respect to x_2 from $\Omega_\flat^0$ onto Ω^0 (recall the artificial Dirichlet condition): such an extension preserves the differentiability properties of the data and renders the point in the middle of the

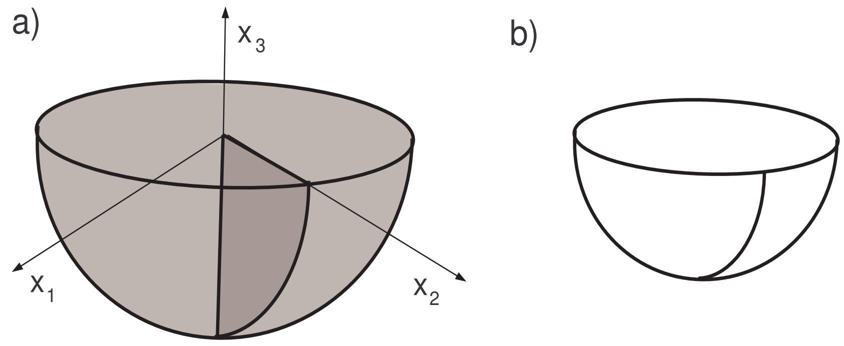

FIGURE 3.1. Hemisphere with incision.

smooth edge Ψ. However, K_1 may be only Hölder-continuous at the point $s = L$ which is marked by $\bullet$ in Fig. 1.1.b; that is, $K_1 \in C^{0,\delta}[0, L]$ for any $\delta \in (0, 1)$, while

$$|\partial_s^p K_1(s)| \le C(L - s)^{1-p} \ , \quad p = 0, 1, 2, \dots . \tag{3.12}$$

Let us explain this last fact. According to the general procedure, e.g. [26, Ch. 10, Ch. 11], the asymptotic expansion of v_1 near the endpoint of $s = L$ of the edge Ψ includes the power-law solutions

$$\varrho^\beta \phi(\vartheta) \tag{3.13}$$

of the Laplace-Neumann problem in the polyhedron $\mathbb{K}$, which is the complement of the quadrant $\{x \,:\, x_1 = 0, \ x_2 > x_2^0, \ x_3 < 0\}$ in the lower half-space. In (3.13), (ϱ, ϑ) are the polar coordinates, β is a number, and ϕ is a function in the lower hemisphere without half of the meridian, Fig. 3.1.b. We extend this model problem evenly with respect to x_1 through the horizontal plane, and this turns it into the Neumann problem in the domain, which is the full space $\mathbb{R}^3$ with unbounded incision of the shape of the half-plane. The power-law solutions (3.13) with non-negative exponents β of this problem look like

$$\varrho^{k/2} \phi_k(\vartheta) = r^{k/2} \cos(\pi k/2) \ , \quad k = 0, 1, 2, \dots, \tag{3.14}$$

cf. (3.11). In this way the extension turns the endpoint $s = L$ into an interior point of a smooth edge. At the same time the Neumann boundary condition on the plane $\{x \,:\, z = 0\}$ (the horizontal one in Fig. 3.1.a) was obtained by neglecting the term $\Lambda_\dagger v_1$ in the Steklov condition (2.16). Thus, there emerges a discrepancy, the main term of which is $\Lambda_\dagger K_1(L) r^{1/2} \cos(\varphi/2)$, and this has to compensated by a solution of the following model problem in $\mathbb{K}$:

$$\begin{aligned} &\varrho^{3/2} \big(C_0 \phi_3(\vartheta) \ln \varrho + \phi_3'(\vartheta) \big) \\ &= \frac{1}{2} C_0 r^{3/2} \ln(r^2 + z^2) \cos\left(\frac{3}{2}\varphi\right) + (r^2 + z^2)^{3/4} \phi_3'(\vartheta). \end{aligned} \tag{3.15}$$

The first term on the right (with $\cos(3\varphi/2)$) does not affect the singularity of (3.11) (which is $\cos(\varphi/2)$ anyway), but the second term may cause a peculiar behaviour of $K_1(s)$ as $s \to L - 0$, and this is apparent in the estimates (3.12).

The asymptotic expansion of v_1 could be studied further, in particular it would be possible to verify that the derivative $\partial_s K_1$ is Hölder continuous. However, this would require a much more elaborate analysis, while the information contained in (3.12) suffices in order to conclude that the problem (2.13)–(2.16), (3.10) has a solution which belongs to $H^1(\Omega^0_\flat(R))$ for any $R > 0$. The inclusion $v_2 \in H^1_{\rm loc}(\Omega^0_\flat)$ is obtained from the Hardy-type inequality

$$\int\limits_{\theta^0_{\pm,\flat}} r^{-1}(1+|\ln r|)^{-2}|w(0,x')|^2 dx' \le c \int\limits_{\Omega^0_\flat(R)} \big(|\nabla w(x)|^2 + |w(x)|^2\big)dx, \tag{3.16}$$

and the weak formulation of the problem in a weighted space with detached asymptotics, cf. [18, 22]. Instead of using these involved techniques one may directly observe that the right hand sides $g_\pm(x')$ in (3.10) satisfy the bound $|g_\pm(x')| \le cr^{-1/2}(1+r/\varrho)$ as a consequence of the assumption $\mathbf{3}^\circ$.

We shall return to a discussion on the singularities in Section 6.4 and now finalize our consideration by writing down the following expansion near the edge:

$$v_1(x) = \widehat{v}_1(x) + K_0(s) + K_1(s)r^{1/2}\cos(\varphi/2). \tag{3.17}$$

Here, $K_0 \in C^\infty[0,L]$, K_1 belongs to $C^\infty[0,L)$ and satisfies (4.8), and the remainder satisfies the estimates

$$\begin{aligned} & r^{-1}|\widehat{v}_1(x)| + |\nabla \widehat{v}_1(x)| \le C\ , \\ & |\nabla^p \widehat{v}_1(x)| \le cr^{-p+3/2}(1+|\ln(L-s)|)\ ,\ p = 2,3,\dots \end{aligned} \tag{3.18}$$

for small $r > 0$. Notice that the first of these estimates follow from the smooth term $K_2(s)r^1\cos(\varphi)$, see (3.11) with $j = 2$, but the last one indicates the singularities $K_3(s)r^{3/2}\cos(3\varphi/2)$ and (3.15).

3.4. **Asymptotics of v_2 at infinity.** Since the data in the problem (2.13)–(2.16), (3.10) is compact, its solution v_2 admits the same representation (2.23) as v_1:

$$v_2(x) = \widetilde{v}_2(x) + \sum_\pm \chi_\pm(x_1)\big(b^1_2|x_1| \pm b^0_2\big)U_\dagger(x'). \tag{3.19}$$

Let us compute the coefficient b^1_2. Using integration by parts inside $\Omega^0_\flat(R)$ and along $\theta^0_{\pm,\flat}$ we obtain, similarly to (1.12),

$$\begin{aligned} 2b^1_2 \int\limits_{\varpi_\flat} |U_\dagger(x')|^2 dx' &= \lim_{R\to+\infty} \sum_\pm \pm \int\limits_{\varpi_\flat} U_\dagger(x')\partial_1 v_2(\pm R, x')dx' \\ &= \sum_\pm \pm \int\limits_{\theta_\flat} U_\dagger(x')\partial_1 v_2(\pm R, x')dx' \\ &= \sum_\pm \int\limits_{\theta_\flat} U_\dagger(x')\nabla' \cdot \big(h_\pm(x')\nabla' v_1(\pm 0, x')\big)dx' \\ &= \sum_\pm \Big(- \int\limits_{\theta_\flat} h_\pm(x')\nabla' U_\dagger(x') \cdot \nabla' v_1(\pm 0, x')\big)dx' \\ &\quad + \int\limits_{\phi_\flat} h_\pm(x_2,0)U_\dagger(x_2,0)\partial_z v_1(\pm 0, x_2, 0)\Big)dx_2 \\ &= \sum_\pm \Big(\int\limits_{\theta_\flat} v_1(\pm 0, x')\nabla' h_\pm(x') \cdot \nabla' U_\dagger(x')dx' \end{aligned}$$

$$+\int_{\theta_\flat} v_1(\pm 0,x')h_\pm(x')\Delta' U_\dagger(x')dx'$$

$$+\int_{\phi_\flat} h_\pm(x_2,0)\big(U_\dagger(x_2,0)\partial_z v_1(\pm 0,x_2,0)-v_1(\pm 0,x_2,0)\partial_z U_\dagger(x_2,0)\big)dx_2\Big). \tag{3.20}$$

The last and second but last integrals vanish, due to the Steklov conditions (1.21), (2.16) and the Laplace equation (1.19), respectively. Hence, similarly to (2.25), we have

$$b_2^1 = -\frac{1}{2}\|U_\dagger; L^2(\varpi_\flat)\|^{-2}J(h), \tag{3.21}$$

where $J(h)$ is obtained by taking (2.19) into account and integrating by parts in $\Omega_\flat^0$:

$$J(h) = \sum_\pm \mp \int_{\theta_\flat} v_1(\pm 0,x')\partial_1 v_1(\pm 0,x')dx'$$

$$= \int_{-\infty}^{\infty}\Big(\int_{\varpi_\flat}|\nabla v_1(x)|^2dx' - \Lambda_\dagger\int_0^l |v_1(x_1,x_2,0)|^2dx_2\Big)dx_1. \tag{3.22}$$

We emphasize that the representation (3.8) of the bounded solution v_a guarantees that the integrand

$$j(v_1;x_1) = \int_{\varpi_\flat}|\nabla v_1(x)|^2dx' - \Lambda_\dagger\int_0^l |v_1(x_1,x_2,0)|^2dx_2 \tag{3.23}$$

decays exponentially at infinity: in view of (1.19)–(1.22), we have $j(U_\dagger)=0$ and hence, constant terms become null in the asymptotics of (3.23) as $x_1\to\pm\infty$. It is worth mentioning that the convergence of all integrals in (3.20) follows from the material in Section 3.3.

3.5. **Asymptotics of the eigenvalue.** In Section 4.1 we shall verify the inequality

$$J(h) > 0. \tag{3.24}$$

We are now in position to complete the matching procedure and to derive a formula for the correction term in (3.2). Recalling the conclusions (2.12) and (2.29) we compare linear terms in the coefficients of ε^2 in (3.6) and (3.7). According to (3.5), (3.19), and (3.21) we see that, first, $\mp\mu_1 = \pm b_2^1$, and, second,

$$\|U_\dagger; L^2(\varpi_\flat)\|^{-1}\lambda_1^{1/2} = \mu_1 = -b_2^1 = \frac{1}{2}\|U_\dagger; L^2(\varpi_\flat)\|^{-2}J(h).$$

Because of (3.24) we can write

$$\lambda_1 = \frac{1}{4}\|U_\dagger; L^2(\varpi_\flat)\|^{-2}J(h). \tag{3.25}$$

The next assertion will be proven in Sections 4 and 5.

Theorem 3.1. *Assume that the conditions* $\mathbf{1}^\circ$–$\mathbf{3}^\circ$ *hold true and that* $I(h)=0$*, see* (2.28)*. Then, there exist* $\varepsilon_2=\varepsilon_2(\theta,h_\pm)>0$ *and* $c_2>0$ *such that the problem* (1.8)–(1.10) *has for every* $\varepsilon\in(0,\varepsilon_2]$ *a unique eigenvalue* (3.2) *inside the segment* $(0,\Lambda_\dagger]$.

The correction term $\lambda_0 > 0$ *is given by* (3.25), (3.22) *and the asymptotic remainder meets the estimate*

$$|\widetilde{\lambda}^\varepsilon| \le c_2 \varepsilon^{9/2}. \tag{3.26}$$

4. Uniqueness assertions.

4.1. **Additional assertions.** We first prove that the problem (1.25)–(1.28) has no trapped modes at the threshold $\Lambda = \Lambda_\dagger$, cf. Section 2.3 and Theorems 1.1, 3.1. Let $u^0 \in H^1_0(\Omega^0_\flat; \Upsilon^0)$ be a solution of this homogeneous problem. The Green formula gives

$$\int\limits_{\Omega^0_\flat} \Big| \frac{\partial u^0}{\partial x_1}(x) \Big|^2 dx + \int\limits_{\Omega^0_\flat} |\nabla' u^0(x)|^2 dx - \int\limits_{\Gamma^0_\flat} |u^0(x_1, x_2, 0)|^2 dx_1 dx_2. \tag{4.1}$$

The max-min-principle (1.23) implies for all $V \in H^1_0(\varpi_\flat; \upsilon)$

$$\int\limits_{\Omega^0_\flat} |\nabla' V(x')|^2 dx' \ge \Lambda_\dagger \int\limits_{\gamma_\flat} V(x_2, 0) dx_2. \tag{4.2}$$

Setting $V(x') = u^0(x_1, x')$ in (4.2) and integrating the result in $x_1 \in (-\infty, 0) \cup (0, +\infty)$ shows that the difference of the second and third integrals in (4.1) is non-negative. Hence,

$$\int\limits_{\Omega^0_\flat} |\partial_1 u^0(x)|^2 dx \le 0$$

and therefore u^0 does not depend on the longitudinal variable x_1. Owing to the decay of u^0 at infinity, this is possible only, if $u^0 = 0$.

A similar consideration proves the key inequality (3.24) of Section 3.5. Indeed, we have

$$J(h) = \int\limits_{\Omega^0_\flat} \Big| \frac{\partial v_1}{\partial x_1}(x) \Big|^2 dx + \int\limits_{-\infty}^{\infty} j(v_1; x_1) dx_1,$$

where the first integral converges, because the x_1-derivative of the function (2.23) decays exponentially. The integrand (3.23) is non-negative due to the inequality (4.2), and thus $J(h) \ge 0$. The equality $J(h) = 0$ is possible only in the case $\partial_1 v_1 = 0$ in $\Omega^0_\flat$. The asymptotic behaviour (3.8) shows that $v_1(x) = \pm b^0_1 U_\dagger(x')$ for $\pm x_1 > 0$, and the continuity of v_1 requires the equality $b^0_1 = 0$. Of course, $u^0 = 0$ cannot be a solution of the problem (2.13)–(2.16) with inhomogeneous boundary conditions (2.19).

4.2. **Asymptotics of eigenvalues in a bounded domain.** In the next section we shall need some information on the eigenvalues of the problem

$$-\Delta w^\varepsilon(x) = 0, \qquad x \in \Omega^\varepsilon_\flat(R), \tag{4.3}$$

$$\partial_\nu w^\varepsilon(x) = 0, \qquad x \in \Sigma^\varepsilon_\flat(R) \cup \textstyle\bigcup_\pm \big(\theta^\varepsilon_{+,\flat} \cup \varpi_\flat(\pm R)\big), \tag{4.4}$$

$$\partial_z w^\varepsilon(x) = \beta^\varepsilon w^\varepsilon(x), \ x \in \Gamma^\varepsilon_\flat(R), \tag{4.5}$$

$$w^\varepsilon(x) = 0, \qquad x \in \Upsilon^\varepsilon(R) \tag{4.6}$$

in the bounded domain $\Omega^\varepsilon_\flat(R) = \{x \in \Omega^\varepsilon_\flat : |x_1| < R\}$ with some fixed $R > 0$; the sets $\Sigma^\varepsilon_\flat(R)$, $\Gamma^\varepsilon_\flat(R)$, and $\Upsilon^\varepsilon(R)$ are defined similarly. On the truncated cross-sections $\varpi_\flat(\pm R)$, an artificial Neumann condition is imposed in (4.4), and the other conditions are inherited from (1.16)–(1.18).

Putting $\varepsilon = 0$ leads to the limit problem in the bounded cylinder $\Pi_\flat(R) = (-R, R) \times \varpi_\flat$ with the incision $\Theta^0_\flat$. For this problem, we readily find the eigenvalue $\beta^0_1 = \Lambda_\dagger$ and the corresponding eigenfunction $w^0_1(x) = U_\dagger(x')$. Since $U_\dagger > 0$ in $\varpi_\flat$, the strong maximum principle shows that this is the first, simple eigenvalue. We also need the second eigenvalue

$$\beta^0_2 > \beta^0_1 = \lambda_\dagger, \tag{4.7}$$

which of course may be multiple.

In view of the assumption **3**°, Section 1.2 and the definition (1.3) of Θ^ε, there exists a diffeomorphism $\varkappa$ of class $H^{1,\infty}$ which transforms $\Omega^\varepsilon_\flat(R)$ into $\Omega^0_\flat(R)$ and which is "almost identical",

$$|\varkappa^\varepsilon(x) - x| \le c\varepsilon\ , \ \left|\frac{d\varkappa^\varepsilon}{dx} - \mathrm{Id}\right| \le c\varepsilon.$$

According to [10, 7.6.5], see also [26, Ch. 5], this means that $\beta^\varepsilon_p = \beta^\varepsilon_p + O(\varepsilon)$ and in particular

$$\beta^\varepsilon_2 > \Lambda_\dagger \quad \text{for } \varepsilon \in (0, \varepsilon_0] \tag{4.8}$$

by virtue of (4.8). Let us compute the asymptotics of β^ε_1.

In spite of the edge Ψ, the transition from $\Theta^\varepsilon_\flat$ to $\Theta^0_\flat$ can be regarded as a regular perturbation of the boundary, cf. Section 3.3, and we thus choose the standard ansätze

$$\begin{aligned} \beta^\varepsilon_1 &= \Lambda_\dagger - \alpha\varepsilon + \widetilde{\beta}^\varepsilon_1, \\ w^\varepsilon_1(x) &= U_\dagger(x') + \varepsilon W(x) + \widetilde{w}^\varepsilon_1(x). \end{aligned} \tag{4.9}$$

We insert them into the problem (4.3)–(4.6), repeat the arguments of Section 2.2 and thus obtain the following problem for the corrections terms in (4.9):

$$\begin{aligned} -\Delta W x) &= 0, \quad x \in \Omega^0_\flat(R), \\ \partial_\nu W(x) &= 0, \quad x \in \Sigma^0_\flat(R)\ , \quad \pm\partial_1 W(\pm R, x') = 0, \quad x \in \varpi_\flat, \\ \pm\partial_1 W(\pm 0, x') &= -\nabla' h_\pm(x') \cdot \nabla' U_\dagger(x'), \quad x' \in \theta_\flat \\ \partial_z W(x) &= \Lambda_\dagger W(x) - \alpha U_\dagger(x'), x \in \Gamma^0_\flat(R), \\ W(x) &= 0, \quad x \in \Upsilon^0(R). \end{aligned}$$

Moreover, since the eigenvalue $\Lambda_\dagger$ is simple, the only compatibility condition in this problem reads as

$$\begin{aligned} 0 &= \int\limits_{\partial\Omega^0_\flat} \big(U_\dagger(x')\partial_\nu W(x) - W(x)\partial_\nu U_\dagger(x')\big) ds_x \\ &= -\alpha \int\limits_{\Gamma^0_\flat(R)} |U_\dagger(x_2, 0)|^2 dx_1 dx_2 - \sum_\pm \int\limits_{\theta_\flat} U_\dagger(x')\nabla' h_\pm(x') \cdot \nabla' U_\dagger(x') dx'. \end{aligned}$$

Hence, (1.24), (2.26), (2.28) imply

$$\alpha = (2R)^{-1} I(h). \tag{4.10}$$

Finally, again accoding to [10, 7.6.5], the remainder in (4.9) can be bounded by

$$|\widetilde{\beta}_1^\varepsilon| \le c\varepsilon^2. \tag{4.11}$$

Remark 4.1. We emphasize the obvious difference of the asymptotic ansätze (2.1) and (4.9) for the eigenvalues in the infinite waveguide $\Omega_\flat^\varepsilon$ and its truncated part $\Omega_\flat^\varepsilon$. Moreover, the relations (2.31) and (4.10) have been derived with crucially different arguments. These observations are discussed in detail in the paper [20].

4.3. **Max-min-principle.** Following [19, 21] we equip the Sobolev-space $\mathcal{H}^\varepsilon = H_0^1(\Omega_\flat^\varepsilon; \Upsilon^\varepsilon)$ with the scalar product

$$\langle u^\varepsilon, v^\varepsilon \rangle = (\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Omega_\flat^\varepsilon} \tag{4.12}$$

and define the operator $\mathcal{T}^\varepsilon$ by the identity

$$\langle \mathcal{T}^\varepsilon u^\varepsilon, v^\varepsilon \rangle = (u^\varepsilon, v^\varepsilon)_{\Gamma_\flat^\varepsilon}, \tag{4.13}$$

where $(\cdot,\cdot)_\Xi$ stands for the natural scalar product of the Lebesgue space $L^2(\Xi)$. The inequality

$$\|u^\varepsilon; L^2(\Omega_\flat^\varepsilon)\|^2 + \|u^\varepsilon; L^2(\Gamma_\flat^\varepsilon)\|^2 \le c\|\nabla u^\varepsilon; L^2(\Omega_\flat^\varepsilon)\|^2 \tag{4.14}$$

follows from the standard Friedrichs inequality in the truncated channel,

$$\|u^\varepsilon; L^2(\Omega_\flat^\varepsilon(R))\|^2 + \|u^\varepsilon; L^2(\Gamma_\flat^\varepsilon(R))\|^2 \le c\|\nabla u^\varepsilon; L^2(\Omega_\flat^\varepsilon(R))\|^2, \tag{4.15}$$

and the trace inequality in the cross-section ϖ,

$$\|U^\varepsilon; L^2(\varpi_\flat)\|^2 + \|U^\varepsilon; L^2(\gamma_\flat)\|^2 \le c\|\nabla' U^\varepsilon; L^2(\varpi_\flat)\|^2. \tag{4.16}$$

These inequalities are valid owing to the Dirichlet conditions (1.18) and (1.22), respectively. In (4.16) we set $U^\varepsilon(x') = u^\varepsilon(x)$ and in addition integrate over $x_1 \in (-\infty, -R) \cup (R, +\infty)$. The constant c in (4.15) does not depend on ε, since the parts of the surface $\partial\Omega_\flat^\varepsilon$ which are inside $\Pi_\flat$ can be considered as graphs of functions in the variable x', cf. [35].

The inequality (4.14) and the definition of the inner product (4.12) imply that the operator $\mathcal{T}^\varepsilon$ is continuous, positive, and symmetric, hence, self-adjoint. Moreover, by (4.12) and (4.13), the variational formulation of the problem (1.15)–(1.18),

$$(\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Omega_\flat^\varepsilon} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Gamma_\flat^\varepsilon} \quad \forall v^\varepsilon \in H_0^1(\Omega_\flat^\varepsilon; \Upsilon^\varepsilon), \tag{4.17}$$

can be formulated as the abstract equation

$$\mathcal{T}^\varepsilon u^\varepsilon = \tau^\varepsilon u^\varepsilon \ \text{ in } \mathcal{H}^\varepsilon, \tag{4.18}$$

where

$$\tau^\varepsilon = 1/\lambda^\varepsilon. \tag{4.19}$$

This relation implies that the continuous spectrum of $\mathcal{T}^\varepsilon$ is $[0, \lambda_\dagger^{-1}]$. Moreover, the operator $-\mathcal{T}^\varepsilon$ (with the minus sign) is bounded from below and eigenvalues $\tau_1^\varepsilon, \dots, \tau_N^\varepsilon$ in its discrete spectrum can be obtained from the max-min-principle

$$-\tau_n^\varepsilon = \max_{\mathcal{H}_n^\varepsilon} \min_{v^\varepsilon \in \mathcal{H}_n^\varepsilon \setminus \{0\}} \frac{-\langle \mathcal{T}^\varepsilon v^\varepsilon, v^\varepsilon \rangle}{\langle \mathcal{T}^\varepsilon v^\varepsilon, v^\varepsilon \rangle}, \tag{4.20}$$

where $\mathcal{H}_n^\varepsilon$ is any subspace of $\mathcal{H}^\varepsilon$ with codimension $n-1$. More precisely, Theorem 10.2.2. of [2] or Th XXX of [31] state that if the right hand side of (4.20) is less than $\Lambda_\dagger^{-1}$, then the discrete spectrum of $\mathcal{T}^\varepsilon$ as well as the discrete spectrum of the problem (1.15)–(1.18) contains at least n points, which thus are isolated eigenvalues.

Let us assume that $I(h)$, (2.28), is negative. Then, by (4.10) and (4.11), the first eigenvalue (4.9) of the auxiliary problem (4.3)–(4.6) satisfies $\beta_1^\varepsilon \geq \lambda_\dagger$ for small $\varepsilon \in (0, \varepsilon_0]$ and therefore we have in inequality [1]

$$\|\nabla v^\varepsilon; L^2(\Omega_\flat^\varepsilon(R))\|^2 \geq \beta_\flat^\varepsilon \|v^\varepsilon; L^2(\Gamma_\flat^\varepsilon(R))\|^2. \tag{4.21}$$

We take the inequality (4.2) with $V(x') = v^\varepsilon(x_1, x')$, integrate it in $x_1 \in (-\infty, -R) \cup (R, +\infty)$, add it to (4.21) and obtain

$$\|\nabla v^\varepsilon; L^2(\Omega_\flat)\|^2 \geq \Lambda_\dagger \|v^\varepsilon; L^2(\Omega_\flat)\|^2. \tag{4.22}$$

Thus, for all $v^\varepsilon \in \mathcal{H}^\varepsilon$ we have

$$\frac{-\langle \mathcal{T}^\varepsilon v^\varepsilon, v^\varepsilon \rangle}{\langle v^\varepsilon, v^\varepsilon \rangle} = -\frac{\|v^\varepsilon; L^2(\Gamma_\flat^\varepsilon)\|^2}{\|\nabla v^\varepsilon; L^2(\Omega_\flat^\varepsilon)\|^2} \geq -\frac{1}{\Lambda_\dagger}, \tag{4.23}$$

so that the right hand side of (4.20) with $n = 1$ exceeds $-\lambda_\dagger$. By the above mentioned theorems of [2] and [31], the discrete spectrum of $\mathcal{T}^\varepsilon$ is empty. This first assertion of Theorem 1.1 has been verified.

Let $I(h) \geq 0$. We now deal with the second eigenvalue β_2^ε and introduce the subspace of codimension 1,

$$\mathcal{H}_\perp^\varepsilon = \Big\{ v^\varepsilon \in H_0^1(\Omega_\flat^\varepsilon; \Gamma_\flat^\varepsilon) \, : \int\limits_{\Omega_\flat^\varepsilon(R)} v^\varepsilon(x) w_1^\varepsilon(x) dx = 0 \Big\}. \tag{4.24}$$

In (4.24), w_1^ε is the first eigenfunction of the problem (4.3)–(4.6). Owing to the orthogonality condition (4.24), any function $v^\varepsilon \in \mathcal{H}_\perp^\varepsilon$ satisfies the relation (4.21) with $p = 2$, which is an inequality of Poincaré type. We obtain the formula (4.22) by (4.7) and (4.2) and conclude that

$$\inf_{v^\varepsilon \in \mathcal{H}^\varepsilon \setminus \{0\}} \frac{-\langle \mathcal{T}^\varepsilon v^\varepsilon, v^\varepsilon \rangle}{\langle v^\varepsilon, v^\varepsilon \rangle} = -\sup_{v^\varepsilon \in \mathcal{H}^\varepsilon \setminus \{0\}} \frac{\|v^\varepsilon; L^2(\Gamma_\flat^\varepsilon)\|^2}{\|\nabla v^\varepsilon; L^2(\Omega_\flat^\varepsilon)\|^2} \geq -\frac{1}{\Lambda_\dagger} \geq -\frac{1}{\Lambda_\dagger}.$$

Once more, the above mentioned theorems of [2] or [31] implies that the discrete spectrum of the operator $\mathcal{T}^\varepsilon$ cannot contain two eigenvalues.

The uniqueness statements in Theorems 1.1, (2), and 3.1 have been confirmed.

5. Existence of an eigenvalue.

5.1. **Searching for an eigenvalue.** We shall construct a non-trivial function $u_{as}^\varepsilon \in H^1(\Omega_\flat^\varepsilon, v^\varepsilon)$ and a positive number τ_0^ε such that

$$\|\mathcal{T}^\varepsilon u_{as}^\varepsilon - \tau_{as}^\varepsilon u_{as}^\varepsilon; \mathcal{H}^\varepsilon\| = \delta \|u_{as}^\varepsilon; \mathcal{H}^\varepsilon\|, \tag{5.1}$$

$$\tau_{as} - \delta > \Lambda_\dagger^{-1}. \tag{5.2}$$

A classical lemma on "approximate eigenvalues", see e.g. [36], and the formulas (5.1), (5.2) guarantee that the segment $[\tau_{as}^\varepsilon - \delta, \tau_{as}^\varepsilon + \delta]$ does not intersect the continuous spectrum $[0, \lambda_\dagger^{-1}]$ and contains an eigenvalue τ_1^ε of the operator $\mathcal{T}^\varepsilon$. Then, the relation (4.19) of the spectral parameters implies the existence of the eigenvalue $\lambda_1^\varepsilon = 1/\tau_1^\varepsilon \in (0, \Lambda_\dagger)$ of the problem (1.15)–(1.18), as well as the estimate

$$|\lambda_1^\varepsilon - \lambda_{as}^\varepsilon| \leq C_\varepsilon \delta \;\text{ with }\; \lambda_{as}^\varepsilon = \frac{1}{\tau_{as}^\varepsilon}\,,\; C_\varepsilon = \frac{\delta}{\tau_{as}^\varepsilon(\tau_{as} - \delta)}. \tag{5.3}$$

[1]This inequality is quite similar to (4.2) and both of them can be derived by using a reduction to an abstract equation and applying the min-principle.

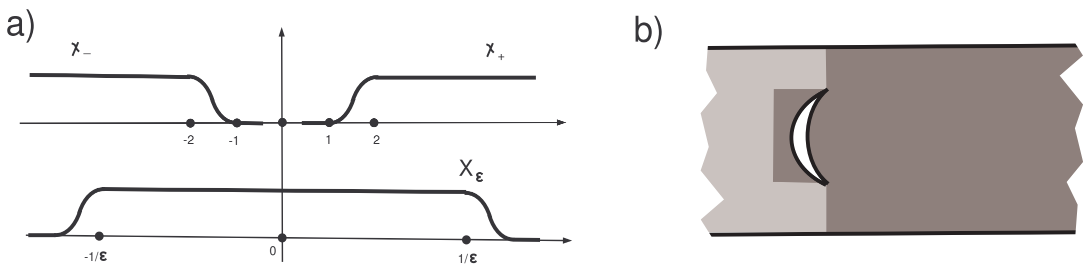

FIGURE 5.1. a) Cut-off-functions. b) Extension through the screen.

In the previous section we have proved that an eigenvalue in $(0, \Lambda_\dagger)$ is unique, if it exists. That is why the above mentioned lemma additionally yields an eigenfunction $u_1^\varepsilon \in H_0^1(\Omega_\flat^\varepsilon; \Upsilon^\varepsilon)$, which corresponds to λ_1^ε, but is not necessarily normed in $\mathcal{H}^\varepsilon$, and satisfies the estimate

$$\|u_1^\varepsilon - u_{\rm as}^\varepsilon; \mathcal{H}^\varepsilon\| \leq \delta \|u_{\rm as}^\varepsilon; \mathcal{H}^\varepsilon\|. \tag{5.4}$$

The simplest way to derive these and similar facts is to apply elementary tools of the theory of the spectral measure; in this way the reduction to the abstract equation becomes very important. In particular, the key estimate

$$|\tau_1^\varepsilon - \tau_{\rm as}^\varepsilon| \leq \delta \tag{5.5}$$

is a consequence of the spectral decomposition of the resolvent, see [2, §6.2], which includes an estimate of the distance of a point to the spectrum in terms of the norm of the resolvent, see (5.1). The estimate (5.4) for the eigenfunction follows by using the spectral projection. A detailed explanation of this technique is given for example in [29].

5.2. **Approximate eigenvalue and eigenfunction.** We assume the condition $I(h) > 0$ and set

$$\lambda_{\rm as}^\varepsilon = \Lambda_\dagger - \varepsilon^2 \lambda_0 \tag{5.6}$$

and correspondingly $\tau_{\rm as}^\varepsilon = (\Lambda_\dagger - \varepsilon^2\lambda_0)^{-1}$; here λ_0 is taken from (2.31). Moreover, by $\{\mu_{\rm as}(\varepsilon), V_{\rm as}(\varepsilon; x')\}$ we understand the solution (2.4) of the model problem (2.3) on $\varpi_\flat$ with the spectral parameter (5.6).

We glue the inner and outer expansions (2.11) and (2.2) of Section 2 by using the smooth cut-off functions (2.21) and the function

$$X_\varepsilon(x_1) = 1 \text{ for } |x_1| < 1/\varepsilon\ ,\ \ X_\varepsilon(x_1) = 0 \text{ for } |x_1| > 1 + 1/\varepsilon\ ,\ \ 0 \leq X_\varepsilon \leq 1. \tag{5.7}$$

Namely, we set

$$\begin{aligned} u_{\rm as}^\varepsilon(x) = {}& X_\varepsilon(x_1)\big(v_0(x) + \varepsilon v_1(x)\big) + \sum_\pm \chi_\pm(x_1)(1 \pm \varepsilon\beta_1^0)e^{\mp\mu(\varepsilon)x_1} V(\varepsilon; x') \\ & - X_\varepsilon(x_1) \sum_\pm \chi_\pm(x_1)\big(1 + \varepsilon(\beta_1^1|x_1| \pm \beta_1^0)\big)U_\dagger(x'). \end{aligned} \tag{5.8}$$

The supports of the cut-off-functions (2.21) and (5.7) overlap like in Fig. 5.1.a. Therefore the terms which have been matched in Section 2.3 are taken into account twice in the first and second terms, but this duplication is compensated by subtracting the third term.

We emphasize that in the case $h_\pm > 0$, when $\overline{\Theta^0} \subset \Theta^\varepsilon$ and $\theta^\varepsilon_\pm$ lays inside $\Pi_\pm = \{x \in \Pi : \pm x_1 > 0\}$, we may just use the function v_1 in (5.8), but in the case when the surfaces $\theta^\varepsilon_\pm$ penetrate in Θ^0, this function must be substituted by its extension $v_1^\pm$ through the screen $\Theta^0_\flat$. For example, if $-h_- < h_+ < 0$ in θ, see Fig. 3.1.b, v_1 must be extended from $\Pi_{+,\flat}$ to the domain

$$\{x \in \Pi_\flat : x_1 > 0 \text{ for } x' \in \varpi_\flat \setminus \overline{\theta}_\flat,\ x_1 > \varepsilon h_+(x') \text{ for } x' \in \theta_\flat\}. \tag{5.9}$$

We take a smooth extension, which has a singularity at the edge (1.30); the larger domain is denoted in Fig. 5.1.b. More precisely, we use the representation (3.17) near the edge and keep the form of $K_0(s)$, $K_1(s) r^{1/2}\cos(\varphi/2)$ unchanged and handle the remainder $\widehat{v}_1$ only. In this way the extensions $\widehat{v}_1^\pm$ will still satisfy the estimates (3.18).

To avoid superfluous technical details we describe the first case and only comment the second one at the end.

Let us derive a lower estimate for the L^2-norm of the function (5.8) written in the form

$$u^\varepsilon_{\rm as}(x) = \Big(1 - \sum_\pm \chi_\pm(x_1)\Big) U_\dagger(x') + X_\varepsilon(x_1)\varepsilon\widetilde{v}_1(x)$$
$$+ \sum_\pm \chi_\pm(x_1)(1 \pm \varepsilon b_1^0) e^{\mp\mu(\varepsilon)} U_\dagger(x'),$$

where the formulas (2.12) for v_0 and (2.23) for v_1 were applied. The first term on the right has compact support and the second one decays exponentially, but, according to (2.4), the decay of the third term is very slow. Thus,

$$\|u^\varepsilon_{\rm as}; L^2(\Omega^\varepsilon_\flat)\|^2 \geq \|u^\varepsilon_{\rm as}; L^2(\Omega^\varepsilon_\flat \setminus \Omega^\varepsilon_\flat(2))\|^2$$
$$\geq C_1 \int_2^\infty e^{-2\varepsilon\mu_0|x_1|} d|x_1| - C_0\varepsilon \geq \frac{C_2}{\varepsilon}\ , \quad C_p > 0. \tag{5.10}$$

5.3. **Further calculations.** Let us compute in the equation (1.15) the discrepancy of the function (5.8), which is written more briefly as follows:

$$u^\varepsilon_{\rm as} = X_\varepsilon u^\varepsilon_{\rm in} + \sum_\pm \chi_\pm u^\varepsilon_{{\rm out},\pm} - X_\varepsilon \sum_\pm \chi_\pm u^\varepsilon_{{\rm mat},\pm}. \tag{5.11}$$

We denote by $[\Delta, X_\varepsilon]$ the commutator of the Laplace operator with the cut-off-function X_ε and observe that

$$[\Delta, X_\varepsilon\chi_\pm] = \chi_\pm[\Delta, X_\varepsilon] + [\Delta, \chi_\pm],$$

hence,

$$\Delta u^\varepsilon_{\rm as} = X_\varepsilon \Delta u^\varepsilon_{\rm in} + \sum_\pm \chi_\pm \big(\Delta u^\varepsilon_{{\rm out},\pm} - X_\varepsilon u^\varepsilon_{{\rm mat},\pm}\big)$$
$$+ \sum_\pm \chi_\pm[\Delta, X_\varepsilon]\big(u^\varepsilon_{{\rm in},\pm} - u^\varepsilon_{{\rm mat},\pm}\big) + \sum_\pm [\Delta, X_\varepsilon]\big(u^\varepsilon_{{\rm out},\pm} - u^\varepsilon_{{\rm mat},\pm}\big). \tag{5.12}$$

Since $u^\varepsilon_{\rm in}$, $u^\varepsilon_{{\rm out},\pm}$, and $u^\varepsilon_{{\rm mat},\pm}$ are harmonic, the first three terms on the right vanish. Coefficients of the differential operator $\chi_\pm[\Delta, X_\varepsilon]$ are supported in $\{x \in \overline{\Pi}_\flat : \pm x_1 - 1/\varepsilon \in [0,1]\}$, where the difference $u^\varepsilon_{\rm in} - u^\varepsilon_{{\rm mat},\pm} = \varepsilon\widetilde{v}_1$ is exponentially small, see (2.12) and (2.23). Recalling the asymptotic formulas (2.4) and (2.8) and the decomposition (2.10) specified in (2.9), (2.12), (2.29), and (2.25), we conclude that the difference

$u^\varepsilon_{\mathrm{out},\pm} - u^\varepsilon_{\mathrm{mat},\pm}$ is of the order ε^2 in the set $\{x \in \overline{\Pi}_\flat : 1 \le \pm x_1 \le 2\}$. In this set the commutator $[\Delta, \chi_\pm]$ is not null, in view of (2.21). Hence,

$$|\Delta u^\varepsilon_{\mathrm{as}}(x)| \le C\varepsilon^2 e^{-\alpha_0 |x_1|} \quad \text{for some } \alpha_0 > 0. \tag{5.13}$$

On the cylindrical surface $\partial\Pi$ the normal derivative annihilates all three cut-off functions depending on the longitudinal variable only. We thus find that the asymptotic solution (5.8) satisfies the Neumann condition

$$\partial_\nu u^\varepsilon_{\mathrm{as}}(x) = 0 \ , \quad x \in \Sigma^\varepsilon_\flat \ , \tag{5.14}$$

and the discrepancy in the Steklov condition looks as follows:

$$\partial_z u^\varepsilon(x) - (\Lambda_\dagger - \varepsilon^2\lambda_0) u^\varepsilon_{\mathrm{as}}(x) = \varepsilon^2 \lambda_0 X_\varepsilon(x_1)\Big(u^\varepsilon_{\mathrm{in}}(x) - \sum_\pm \chi_\pm(x_1) u^\varepsilon_{\mathrm{mat},\pm}(x)\Big)$$
$$= \varepsilon^2\lambda_0\Big(1 - \sum_\pm \chi_\pm(x)\Big) U_\dagger(x') + X_\varepsilon(x_1)\varepsilon^3 \widetilde{v}_1(x) \ , \quad x \in \Gamma^\varepsilon_\flat.$$

Recalling the exponential decay of the remainder in (2.23) we see that

$$|\partial_z u^\varepsilon(x_1,x_2,0) - \lambda^\varepsilon_{\mathrm{as}} u^\varepsilon(x_1,x_2,0)| \le c\varepsilon^2 e^{-\alpha_0|x_1|} \ , \ x \in \Omega^\varepsilon_\flat. \tag{5.15}$$

We are left with examining the boundary conditions (1.16) on the screen surfaces (1.5). Clearly, $u^\varepsilon_{\mathrm{as}}(x) = U_\dagger(x') + \varepsilon v_1(x)$ in a neighbourhood of $\Theta^\varepsilon_\flat$. We use the representation (2.17) for the normal derivative and the relation (2.19). As a result, we obtain

$$\big(1 + \varepsilon^2 |\nabla' h_\pm(x')|^2\big)^{1/2} \partial_{\nu^\varepsilon_\pm}\big(U_\dagger(x') + \varepsilon v_1(x)\big)\Big|_{x_1 = \pm\varepsilon h(x')}$$
$$= \varepsilon(\nabla' h_\pm(x') \cdot \nabla' U_\dagger(x') \mp \partial_1 v_1(\pm\varepsilon h_\pm(x'), x') + \varepsilon \nabla' h_\pm(x') \cdot \nabla' v_1(\pm\varepsilon h_\pm(x'), x')$$
$$= \pm\varepsilon(\partial_1 v_1(\pm 0, x') - \partial_1 v_1(\pm\varepsilon h_\pm(x'), x') + \varepsilon^2 \nabla' h_\pm(x') \cdot \nabla' v_1(\pm\varepsilon h_\pm(x'), x')$$

Applying the Taylor formula and the estimates

$$|\nabla^p v(x)| \le c_p(1 + r^{-p+1/2} + r^{1/2}(L-s)^{1-p}) \ , \ p = 0,1,2, \tag{5.16}$$

which follow for example from the relations (3.17) and (3.18), yield the inequality

$$\big|\partial_{\nu^\varepsilon_\pm}\big(U_\dagger(x') + \varepsilon v_1(\pm\varepsilon h_\pm(x'), x')\big)\big| \le c\varepsilon^2 r^{-1/2} \ , \quad x \in \theta^\varepsilon_{\pm,\flat}. \tag{5.17}$$

5.4. **Final estimate.** By the definition of the Hilbert space norm and the formulas (4.12), (4.13) we have

$$\|\mathcal{T}^\varepsilon u^\varepsilon_{\mathrm{as}} - \tau^\varepsilon_{\mathrm{as}} u^\varepsilon_{\mathrm{as}}; \mathcal{H}^\varepsilon\| = \inf \big|\langle \mathcal{T}^\varepsilon u^\varepsilon_{\mathrm{as}}, w^\varepsilon\rangle - \tau^\varepsilon_{\mathrm{as}}\langle u^\varepsilon_{\mathrm{as}}, w^\varepsilon\rangle\big|$$
$$= \tau^\varepsilon_{\mathrm{as}} \inf \big|\lambda^\varepsilon_{\mathrm{as}}(u^\varepsilon_{\mathrm{as}}, w^\varepsilon)_{\Gamma^\varepsilon_\flat} - (\nabla u^\varepsilon_{\mathrm{as}}, \nabla w^\varepsilon)_{\Omega^\varepsilon_\flat}\big|$$
$$= \tau^\varepsilon_{\mathrm{as}} \inf \Big|(\Delta u^\varepsilon_{\mathrm{as}}, \nabla w^\varepsilon)_{\Omega^\varepsilon_\flat} - (\partial_z u^\varepsilon_{\mathrm{as}} - \lambda^\varepsilon_{\mathrm{as}} u^\varepsilon_{\mathrm{as}}, w^\varepsilon)_{\Gamma^\varepsilon_\flat} - \sum_\pm (\partial_{\nu^\varepsilon_\pm} u^\varepsilon_{\mathrm{as}}, w^\varepsilon)_{\theta^\varepsilon_{\pm,\flat}}\Big|.$$

Here, the infimum is calculated over all functions $v^\varepsilon \in \mathcal{H}^\varepsilon$ such that

$$\|w^\varepsilon; \mathcal{H}^\varepsilon\| = \|\nabla w^\varepsilon; L^2(\Omega^\varepsilon_\flat)\| = 1;$$

according to (4.15) and (3.16), these functions also satisfy

$$\|w^\varepsilon, L^2(\Omega^\varepsilon_\flat)\| + \|w^\varepsilon, L^2(\Gamma^\varepsilon_\flat)\| + \sum_\pm \|r^{-1/2}(1 + |\ln r|)^{-1} w^\varepsilon; L^2(\theta^\varepsilon_{\pm,\flat})\| \le C. \tag{5.18}$$

Now the estimates (5.13), (5.15), and (5.17) imply the inequality

$$\|\mathcal{T}^\varepsilon u^\varepsilon_{\mathrm{as}} - \tau^\varepsilon_{\mathrm{as}} u^\varepsilon_{\mathrm{as}}; \mathcal{H}^\varepsilon\| \le c\varepsilon^2, \tag{5.19}$$

which together with (5.10) show that the factor in (5.1) does not exceed $c\varepsilon^{5/2}$, and therefore (5.2) is true. Hence, the operator $\mathcal{T}^\varepsilon$ has an eigenvalue $\tau_1^\varepsilon \in [\tau_{\rm as}^\varepsilon - \delta, \tau_{\rm as}^\varepsilon + \delta]$. Finally, the calculation (5.3) and the formula (5.6) assure the relations (2.1) and (1.12) for the eigenvalue $\lambda_\bullet^\varepsilon = \lambda_1^\varepsilon = (\tau_1^\varepsilon)^{-1}$ of the problems (1.15)–(1.18) and (1.8)–(1.10). Theorem 1.1 is proved.

Let us comment on the case $h_+ < 0$ depicted in Fig. 5.1.b, and outlined at the end of Section 3.3. The formulas (5.14) and (5.17) remain unchanged. The extension v_1^+ is not harmonic in the thin domain $\Xi_+^\varepsilon = \{x : 0 > x_1 > \varepsilon h_+(x'),\ x' \in \theta_\flat\}$, and therefore

$$\Delta u_{\rm as}^\varepsilon(x) = 0 \text{ in } \Pi_{+,\flat} \quad \text{but} \quad \Delta u_{\rm as}^\varepsilon(x) = \varepsilon \Delta v_1^+(x) \text{ in } \Xi_+^\varepsilon. \tag{5.20}$$

However, according to the relations (3.17), (3.18) and the Taylor formula in the variable x_1, we have

$$|\nabla v_1^+(x)| = |\Delta v_1^+(x) - \Delta v_1^+(+0,x')| \le C|x_1| r^{-3/2}(1 + |\ln \varrho|).$$

Furthermore, a direct consequence of the Newton-Leibnitz formula

$$\int\limits_{\Xi_+^\varepsilon} |w^\varepsilon(x)|^2 dx \le c\varepsilon \int\limits_{\Omega_\flat^\varepsilon} \big(|\nabla w^\varepsilon(x)|^2 + |w^\varepsilon(x)|^2\big) dx \tag{5.21}$$

shows that

$$\begin{aligned} &\varepsilon \Big| \int\limits_{\Xi_+^\varepsilon} w^\varepsilon(x) \Delta v_1^+(x) dx \Big|^2 \\ &\le c\varepsilon\varepsilon^{1/2} \|w^\varepsilon; \Omega_\flat^\varepsilon(R)\| \Big(\int\limits_{\varepsilon h_+(x')}^{0} |x_1|^2 \int\limits_{\theta_\flat} r^{-3}(1 + |\ln \varrho|)^2 dx' dx_1 \Big)^{1/2} \\ &\le c\varepsilon^3 \Big(\int\limits_{\theta_\flat} h_+(x')^3 r^{-3} (1 + |\ln \varrho|)^2 dx' \Big)^{1/2} \le c\varepsilon^3. \end{aligned}$$

Here we used the relation (5.18) for w^ε and observed that the last integral converges because the singular factor r^{-3} is compensated by $h_+(x')^3$, owing to the assumption **3°**. A similar calculation shows that

$$\big|(\partial_z u_{\rm as}^\varepsilon - \lambda_{\rm as}^\varepsilon u_{\rm as}^\varepsilon, w^\varepsilon)_{\Gamma_\flat^\varepsilon}\big| \le c\varepsilon^2, \tag{5.22}$$

and hence our previous conclusion (5.19) as well as Theorem 1.1, (2) are still valid. It should be mentioned that instead of (5.21) the derivation of (5.22) can be based on the estimate

$$\int\limits_0^l \int\limits_{\varepsilon h_+(x_2,0)}^0 |w^\varepsilon(x_1,x_2,0)|^2 dx_1 dx_2 \le c\varepsilon(1 + |\ln \varepsilon|)^2 \|w^\varepsilon; H^1(\Omega_\flat^\varepsilon(R))\|^2$$

which follows from a Hardy-type trace inequality analogous to (3.16).

Theorem 1.1 can be proven in the same way but the extension of the asymptotic ansätze in Section 3.1 requires much more cumbersome but still routine calculations, which we omit here for brevity.

6. Concluding remarks.

6.1. **A criterion.** The following inequality

$$\int_{\Theta_\flat^\varepsilon} |\nabla' U_\dagger(x')|^2 dx - \Lambda_\dagger \int_{\Gamma\cap\Theta_\flat^\varepsilon} |U_\dagger(x')|^2 dx_1 dx_2 \geq 0 \tag{6.1}$$

was shown in [19] to be a sufficient condition for the existence of a trapped mode. The result holds without a restriction on the parameter $\varepsilon > 0$, i.e., for a massive obstacle Θ^ε, and it was derived by imposing the artificial Dirichlet condition (1.18), by assuming $\mathbf{1}^\circ$ and $\mathbf{2}^\circ$ and applying the minimum principle (4.20) with $n = 1$ and $\mathcal{H}_1^\varepsilon = H_0^1(\Omega_\flat^\varepsilon; \Upsilon^\varepsilon)$. By (1.3), an integration with respect to x_1 converts (6.1) into the inequality $\varepsilon I(h) \geq 0$, where $I(h)$ is the expression (2.28). If $\mathbf{3}^\circ$ is in addition assumed, Theorems 1.1 and 3.1 yield a positive number $\varepsilon_0(\theta, h_\pm)$ depending on the screen profiles such that if $0 < \varepsilon < \varepsilon_0(\theta, h_\pm)$, then (6.1) is also a necessary condition for a unique trapped mode. For large ε this necessity and uniqueness may of course be lost.

6.2. **Higher order asymptotic terms.** The asymptotic procedures described in Sections 2 and 3 can be continued to construct infinite asymptotic series for the eigenvalue $\lambda_\bullet^\varepsilon$ and the corresponding eigenfunction $u_\bullet^\varepsilon$. Indeed, the use of the Taylor formula for the asymptotic terms $\varepsilon^m v_m(\pm\varepsilon h_\pm(x'), x')$ of the inner expansion cannot be avoided, and this produces derivatives of v_m of order n, but their singularities at $r \to +0$ are compensated by the related small factors $h_\pm(x')^n = O(r^n)$, see the assumption $\mathbf{3}^\circ$. Hence, this again guarantees the existence of $v_m \in H^1_{\rm loc}(\Omega_\flat^\varepsilon)$.

However, in our scheme it is not possible to control the upper bound $\varepsilon_m(\theta, h_\pm)$ for the small parameter ε, and it may happen that $\varepsilon_m(\theta, h_\pm)$ tends to zero at a very high rate as $m \to \infty$. From this point of view it is doubtful, if the infinite asymptotic series is useful.

6.3. **Particular screens.** As was mentioned in Section 1.2 and follows from the sufficient condition (6.1), any submerged screen $\Theta^\varepsilon \subset \Pi$ traps a surface wave, with the exception of the case of a vertical planar screen. Let us discuss the boundary layer phenomenon for the flattened ellipsoid

$$\Theta^\varepsilon = \{x \,:\, R^{-2}(x_2^2 + (x_3 - z_0)^2) + \varepsilon^{-2} x_1^2 \leq 1\} \tag{6.2}$$

and for the penny-shaped obstacle

$$\Theta^\varepsilon = \{x \,:\, x_2^2 + (x_3 - z_0)^2 \leq R^2,\ |x_1| < \varepsilon\}. \tag{6.3}$$

Both screens (6.2) and (6.3) are submerged and do not touch the wetted surface Σ of the channel Π, see Fig. 6.1.a,c.

The ellipsoid (6.2) is given by the formula (1.3), where θ is a disc of radius R and

$$h_\pm(x') = \sqrt{1 - R^{-2}(x_2^2 + (x_3 - z_0)^2)} = \sqrt{r}(R^{-1} + O(r)). \tag{6.4}$$

Since $h_\pm$ vanish on the circle $\psi = \partial\theta$, all calculations of Section 2 can be repeated word-to-word to derive the asymptotic formula (2.1) for the single eigenvalue $\lambda_\bullet^\varepsilon \in (0, \Lambda_\dagger)$ of (1.8)–(1.10). However, the decay rate $O(r^{1/2})$ in (6.4) is not enough to compensate the growth $O(r^{-3/2})$ of the second order derivatives of v, cf. the right hand side of (3.10). As a result, higher order terms mentioned in Section 6.2 cannot be found using the above presented asymptotic method. Indeed, it was shown in [9], see also [15], that the boundary layer phenomenon occurs in the vicinity of the

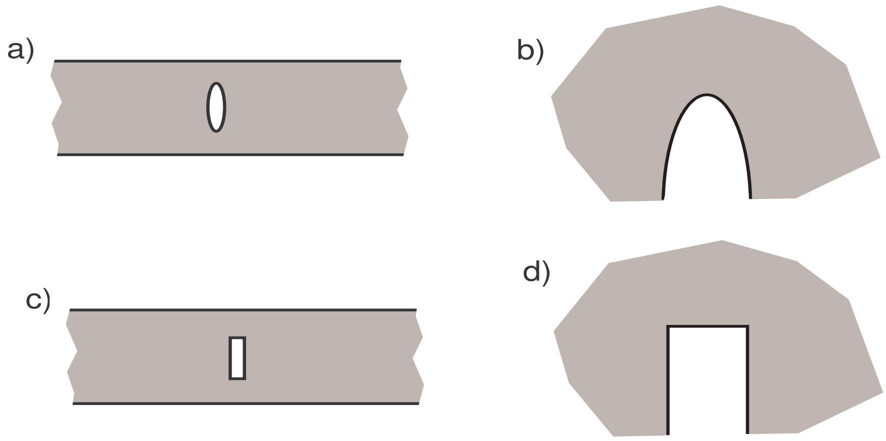

FIGURE 6.1. Particular screens.

edge $\Psi = \{x : x_1 = 0, \ \mathbf{r} := (x_2^2 + (x_2 - z_0)^2)^{1/2} = R\}$. Namely, dilating coordinates as

$$(x_1, \mathbf{r}) \mapsto \xi = (\xi_1, \xi_2) = (\varepsilon^{-2} x_1, \varepsilon^{-2}(\mathbf{r} - R)),$$

using the arc length $s \in [0, 2\pi R)$ on Ψ and setting $\varepsilon = 0$ lead to a Neumann problem for the two-dimensional Laplacian Δ_ξ in the plane $\mathbb{R}^2$ with parabolic notch

$$\mathbb{P} = \{\xi : \xi_2 < 0, \ |\xi_1| \le (2|\xi_2|/R)^{1/2}\},$$

see Fig. 6.1.b. Detailed analysis of the boundary can be found in [9] and [15, Ch. 5].

For the penny-shaped screen (6.2) we have $h_\pm(x') = 1$, and we come across a notable inconsistency in the previous calculations: the right hand side of (2.19) vanishes and the problem (2.13)–(2.16), (2.19) thus turns homogeneous, but according to (2.28), the coefficient b_1^1 in (2.23) takes the form (2.25) with $I(h) = 2\|\nabla' U_\dagger; L^2(\theta_\flat)\|^2$. This contraduction is of course caused by the boundary layer effect. Using the coordinate dilation

$$(x_1, \mathbf{r}) \mapsto \xi = (\varepsilon^{-1} x_1, \varepsilon^{-1}(\mathbf{r} - R)),$$

the effect is described by the solutions of the Neumann problem for Δ_ξ in the plane without the semi-strip

$$\mathbb{S} = \{\xi : \xi_2 \le 0, \ |\xi_1| < 1\},$$

see Fig. 6.1.d. Indeed, the function $v_0(x) = U_\dagger(x')$ has the discrepancy $G(s) = \partial_\mathbf{r} U_\dagger(x')\big|_{\mathbf{r}=R}$ in the Neumann condition on the lateral side of the circular cylinder (6.3). Therefore the main asymptotic term $\varepsilon W(\xi, s)$ of the boundary layer is to be chosen as a solution of the following problem with parameter s:

$$\begin{aligned} -\Delta_\xi W(\xi, s) &= 0, \quad \xi \in \mathbb{S}, \\ \mp \partial_1 W(\pm 1, \xi_2, s) &= 0, \quad \xi_2 < 0, \\ -\partial_2 W(\xi_1, 0, s) &= G(s), \quad \xi_1 \in (-1, 1). \end{aligned} \tag{6.5}$$

Unfortunately, this problem has no solutions which decay at infinity. This is why we employ the traditional method of matched asymptotic expansions, see [34, 9] and change the essence of $\varepsilon W(\xi, s)$: it is regarded as a term in the inner expansion near the edge of the screen, and it is fixed as a solution of the problem (6.5) with logarithmic growth at infinity,

$$W(\xi, s) = -\pi^{-1} G(s) \ln |\xi| + o(1), \quad |\xi| \to +\infty. \tag{6.6}$$

Now, (2.11) is regarded as the outer expansion in a neighbourhood of Θ^ε, and its term $\varepsilon v_1(x)$ must be subject to the asymptotic condition

$$v_1(x) = -\pi^{-1} G(s) \ln r + O(1), \quad r \to +0. \tag{6.7}$$

This term thus becomes a nontrivial singular solution of the homogeneous problem (1.25)–(1.28). This explains why our calculations in Section 2.3 do not work for the penny-shaped screen. In other words, the assumption **3°** simplifies calculations and removing it requires a different asymptotic analysis.

6.4. **Surface-piercing screens.** If the screen thickness function h does not vanish at the endpoints of the line segment $\phi = \partial\theta \cap \gamma$, then, yet another boundary layer must be taken into account, in addition to those discussed in Section 6.3. This amounts to solving a Neumann problem in the lower half-space with the infinite slit of width $h^0 = h^0_+ + h^0_- > 0$,

$$\{\eta = (\eta_1, \eta_2, \eta_3) \,:\, \eta_3 < 0,\ \eta_2 < 0,\ \eta_1 \in [-h^0_-, h^0_+]\}.$$

The authors do not know published results in this direction. Another open question is related to the situation, when h is null on $\psi \setminus \overline{\tau}$ but positive on the arc $\tau = \{s : s \in (-t, t)\}$ of length $2t > 0$.

In addition to the assumption **3°** we have required in Section 1.1 that the angle α between ψ and γ is right. We used this restriction in Section 3.3, since we needed the extension trick to study the singularities of v_2. However, this assumption may be weakened: it was shown in [16] that the exponent β in the "worst" power-law solution (3.13) is a function, which decreases monotonely from 1 to 0, when the variable is the angle α measured from the side of θ. Thus, our calculations remain valid at least for acute angles.

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