SPECTRAL GAPS FOR THE LINEAR SURFACE WAVE MODEL IN PERIODIC CHANNELS

by F.L. Bakharev , K. Ruotsalainen, J. Taskinen

Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178 Russia†

University of Oulu, Department of Electrical and Information Engineering, Mathematics Division, P.O. Box 4500, FI-90401 Oulu, Finland

University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FI-00014 Helsinki, Finland.

Summary

We consider the linear water-wave problem in a periodic channel which consists of infinitely many identical containers connected with apertures of width ϵ. Motivated by applications to surface wave propagation phenomena, we study the band-gap structure of the continuous spectrum. We show that for small apertures there exists a large number of gaps and also find asymptotic formulas for the position of the gaps as $\epsilon \to 0$: the endpoints are determined within corrections of order $\epsilon^{3/2}$. The width of the first bands is shown to be $O(\epsilon)$. Finally, we give a sufficient condition which guarantees that the spectral bands do not degenerate into eigenvalues of infinite multiplicity.

1. Introduction

1.1 *Overview of the results*

Research on wave propagation phenomena in periodic media has been active for many decades. The topics and applications include photonic crystals, meta-materials, Bragg gratings of surface plasmon polariton waveguides, energy harvesting in piezoelectric materials and surface wave propagation in periodic channels, which is the subject of this paper. A standard mathematical approach consists of linearisation and posing a spectral problem for a hopefully self-adjoint elliptic equation or system.

Early on it was noticed that waves propagating in periodic media have spectra with allowed bands separated by forbidden frequency gaps. This phenomenon was first discussed by Lord Rayleigh (**23**). It has also attracted some interest in coastal engineering because it provides a possible means of protection against wave damages (**12**, **15**), for example by varying the bottom topography by periodic arrangements of sandbars. The existence of forbidden frequencies is conventionally related to Bragg reflection of water waves by periodic structures. Here, Bragg reflection is an enhanced reflection which occurs when the wavelength of an incident surface wave is approximately twice the wavelength of the periodic structure. This mechanism works, if the waves are relatively long so that the depth changes can effect them (**15**).

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A similar phenomenon may also happen, when waves are propagating along a channel with periodically varying width. In (**10**), and later (**14**), the authors studied a channel, the wall of which had a periodic stepped structure. Using resonant interaction theory they were able to verify that significant wave reflection could occur. These results are based on the assumption of small wall irregularities.

Gaps in the continuous spectrum for equations or systems in unbounded waveguides have been studied in many papers, and we refer to (**8**) for an introduction to the topic. In (**20**) the authors studied the linear elasticity system and proved the existence of arbitrarily (though still finitely) many gaps, the number of them depending on a small geometric parameter; the approach is similar to Section 3.1, below, and the result is analogous to Corollary 3.2. In the setting of the linear water-wave problem, spectral gaps have been studied in (**4**), (**9**), (**13**), and (**18**).

In this paper we consider surface wave propagation using the linear water wave equation with spectral Steklov boundary condition on the free water surface, see the equations (1.8)–(1.10), which will hereafter be called the "original problem". The water-filled domain Π^ϵ forms an unbounded periodic channel consisting of infinitely many identical bounded containers connected by apertures of width $\epsilon > 0$, see Figure 1.1. The essential spectrum σ of the original problem is expected to be non-empty due to the unboundedness of the domain. The first results, Theorem 3.1 and Corollary 3.2, show that σ has gaps and the number of them can be made arbitrarily large depending on the parameter ϵ. An explanation of this phenomenon can be outlined rather simply using the Floquet-Bloch theory, although a lot of technicalities will eventually be involved. Specifically, if $\epsilon = 0$, the domain becomes a disjoint union of infinitely many bounded containers, and the water-wave problem reduces to a problem on a bounded domain. We shall call this the "limit problem". It has a discrete spectrum consisting of an increasing sequence of eigenvalues $(\Lambda_k^0)_{k=1}^\infty$. On the other hand, for $\epsilon > 0$, one can use the Gelfand transform to render the original problem into another bounded domain problem depending on the additional parameter $\eta \in [0, 2\pi)$. For each fixed η this problem again has a sequence of eigenvalues $(\Lambda_k^\epsilon(\eta))_{k=1}^\infty$. Moreover, by results of (**16**), (**17**), Theorem 2.1, and (**19**), Theorem 3.4.6, the essential spectrum σ of the problem (1.8)–(1.10) equals

$$\sigma = \bigcup_{k=1}^{\infty} \Upsilon_k^\epsilon \ , \quad \Upsilon_k^\epsilon = \{\Lambda_k^\epsilon(\eta) \, : \, \eta \in [0, 2\pi)\}, \tag{1.1}$$

where the sets Υ_k^ϵ are subintervals of the positive real axis, or bands of the spectrum. (For the use of this so called Bloch spectrum in other problems, see for example (**2**) and (**5**).) In general, those bands may overlap, making σ connected, but in Theorem 3.1 we obtain asymptotic estimates for the lower and upper endpoints of Υ_k^ϵ: we show that $\Lambda_k^0 \leq \Lambda_k^\epsilon(\eta) \leq \Lambda_k^0 + C_k\epsilon$ for all k and η and for some constants $C_k > 0$. In view of (1.1) this implies the existence of a spectral gap between Υ_k^ϵ and Υ_{k+1}^ϵ for small ϵ and k whenever $\Lambda_k^0 \neq \Lambda_{k+1}^0$. However, since the estimates depend also on k, we can only open a gap for finitely many k, though the number of gaps tends to infinity as $\epsilon \to 0$.

The asymptotic position (as $\epsilon \to 0$) of the gaps is determined more accurately in Theorems 3.5 and 3.6: those main results state that

$$\Upsilon_k^\epsilon = (\Lambda_k^0 + A_k\epsilon + O(\epsilon^{3/2}), \Lambda_k^0 + B_k\epsilon + O(\epsilon^{3/2}))$$

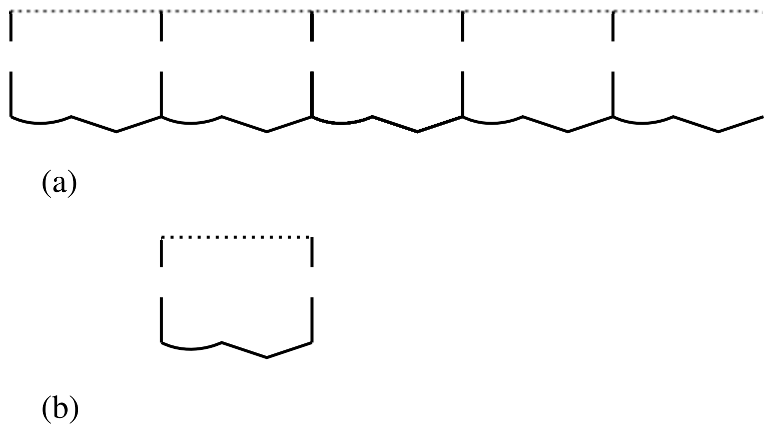

Fig. 1.1 Side view of the waveguide (a) and of the periodicity cell (b)

where the numbers $A_k \leq B_k$ depend linearly on the three dimensional capacity of the set θ describing the apertures. This result also ensures that in the case $A_k \neq B_k$ the bands Υ_k^ϵ do not degenerate into single points, which means that the spectrum of the original problem indeed has a genuine band-gap structure. Facts concerning the numbers A_k, B_k are discussed after Theorem 3.6.

Concerning the structure of this paper, we shall review in Section 1.2 the exact formulation of the linear water-wave problem, its variational formulation as well as the parameter dependent problem arising from the Gelfand transform, and the limit problem. Section 2 contains the formal asymptotic analysis which relates the spectral properties of the original problem to the limit problem and which is rigorously justified in Section 3. The main results, Theorems 3.1, 3.5 and 3.6 as well as Corollary 3.2 are also given in Section 3. The proofs are based on the max-min principle and the construction of suitable test functions adjusted to the geometric characteristics of the domains under study.

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1.2 *Formulation of the problem, operator theoretic tools*

Let us proceed with the exact formulation of the problem. We consider an infinite periodic channel Π^ϵ (see (1.7)), consisting of water containers connected by small apertures of diameter $O(\epsilon)$. The coordinates of the points in the channel are denoted by $x = (x_1, x_2, x_3) = (y_1, y_2, z) = (y, z)$, and $x' = (x_2, x_3)$ stands for the projection of x to the plane $\{x_1 = 0\}$. We choose the coordinate system in such a way that the axis of the channel is in x_1-direction and the free surface is in the plane $\{x_3 = 0\}$.

Definition 1.1. We describe the geometric assumptions on the periodicity cell in detail, as well as some related technical tools including the cut-off functions. Let us denote by $\varpi_\bullet \subset \mathbb{R}^3$ a domain with a Lipschitz boundary and compact closure such that its intersections with $\{x_1 = 0\}$- and $\{x_1 = 1\}$-planes are simply connected planar domains with positive area and contain the points $P^0 = (0, P_2, P_3)$ and $P^1 = (1, P_2, P_3)$ with $P_3 < 0$, respectively; these points are fixed throughout the paper. Then the periodicity cell and its translates are defined by setting (see Figure 1.1)

$$\varpi = \{x \in \varpi_\bullet : x_3 < 0, x_1 \in (0,1)\}, \quad \varpi_j = \{x : (x_1 - j, x_2, x_3) \in \varpi\},\ j \in \mathbb{Z}. \tag{1.2}$$

Furthermore, we assume that the set $\theta \subset \mathbb{R}^2$ is a bounded planar domain containing the origin $(0,0)$ and that the boundary $\partial\theta$ is at least C^2-smooth. We assume that θ is so small that the set $\{0\} \times \overline{\big(2\theta + (P_2, P_3)\big)}$ is contained in $\partial\varpi$ and $\sup\limits_{(x_2,x_3)\in\theta} (x_3 + P_3) =: d_\theta < 0$. We define the apertures between the container walls as the sets

$$\theta_j^\epsilon = \big\{x = (j, x') : \epsilon^{-1}\big(x' - (P_2, P_3)\big) \in \theta\big\},\ j \in \mathbb{Z}. \tag{1.3}$$

It is plain that $x_3 < 0$ for $x \in \theta_j^\epsilon$ for all $0 < \epsilon \le 1$, by the choice of d_θ. We shall need at several places a cut-off function

$$\chi_\theta \in C_0^\infty(\mathbb{R}^3), \tag{1.4}$$

which is equal to one in a neighbourhood of the set $\{0\} \times \overline{\theta}$ and vanishes outside another compact neighbourhood of $\{0\} \times \overline{\theta}$. More precisely, we require that

$$\begin{aligned} &\big(\operatorname{supp}(\chi_\theta) + (0, P_2, P_3)\big) \cap \{x_1 = 0\} \subset \partial\varpi, \\ &\big(\operatorname{supp}(\chi_\theta) + (0, P_2, P_3)\big)\ \cap \{x_1 > 0\} \subset \varpi \end{aligned} \tag{1.5}$$

(this is possible by the specifications made on θ) and $\chi_\theta(x)$ vanishes, if $|x_1| \ge 1/4$ or $x_3 + P_3 \ge d_\theta/2$. We also assume that $\partial_{x_1}\chi_\theta = 0$, when $x_1 = 0$. Furthermore, denoting $\chi_j(x) = \chi_\theta(x - P^j)$, it follows from the above specifications that $\chi_j(x) = 0$, if $x_3 \ge d_\theta/2$; in particular χ_j vanishes on the free water surface γ. Finally, we shall need the scaled cut-off functions

$$X_j^\epsilon = \chi_\theta(\epsilon^{-1}(x - P^j)). \tag{1.6}$$

It is plain that also X_j^ϵ vanishes on γ for $0 < \epsilon \le 1$ and that $X_j^\epsilon(x) = 1$ for $x \in \theta_j^\epsilon$, $j = 0, 1$.

Definition 1.2. The periodic water channel is defined by

$$\Pi^\epsilon = \bigcup_{j\in\mathbb{Z}} (\varpi_j \cup \theta_j^\epsilon), \tag{1.7}$$

and it will be the main object of our investigation. The free surface of the channel is denoted by $\Gamma^\epsilon = \partial\Pi^\epsilon \cap \{x_3 = 0\}$, and the wall and bottom part of the boundary is $\Sigma^\epsilon = \partial\Pi^\epsilon \setminus \overline{\Gamma^\epsilon}$. The boundary of the isolated container ϖ, the periodicity cell, consists of the free surface γ and the wall and bottom σ^ϵ with two apertures θ_0^ϵ and θ_1^ϵ.

Remark 1.3. We shall use the following general notation. Given a domain Ξ, the symbol $(\cdot,\cdot)_\Xi$ stands for the natural scalar product in $L^2(\Xi)$, and $H^k(\Xi)$, $k \in \mathbb{N}$, for the standard Sobolev space of order k on Ξ. The norm of a function f belonging to a Banach function space X is denoted by $\|f; X\|$. For $r > 0$ and $a \in \mathbb{R}^N$, $B_r(a)$ (respectively, $S_r(a)$) stands for the Euclidean ball (resp. ball surface) with centre a and radius r. By C, c (respectively, C_k, c_k, $C(k)$ etc.) we mean positive constants (resp. constants depending on a parameter k) which do not depend on functions or variables appearing in the inequalities, but which may still vary from place to place. The gradient and Laplace operators ∇ and Δ act in variable x, unless otherwise indicated.

In the framework of the linear water-wave theory we consider the spectral Steklov problem in the channel Π^ϵ,

$$-\Delta u^\epsilon(x) = 0 \qquad \text{for all } x \in \Pi^\epsilon, \tag{1.8}$$

$$\partial_n u^\epsilon(x) = 0 \qquad \text{for a.e. } x \in \Sigma^\epsilon, \tag{1.9}$$

$$\partial_z u^\epsilon(x) = \lambda^\epsilon u^\epsilon(x) \qquad \text{for a.e. } x \in \Gamma^\epsilon. \tag{1.10}$$

Here u^ϵ is the velocity potential, $\lambda^\epsilon = g^{-1}\omega^2$ is a spectral parameter related to the frequency of harmonic oscillations $\omega > 0$ and the acceleration of gravity g. By the geometric assumptions made above, the outward normal derivative ∂_n is defined almost everywhere on Σ^ϵ. It coincides with $\partial_z = \partial/\partial_z$ on the free surface Γ^ϵ.

The rest of this section is devoted to presenting the operator theoretic tools which will be needed later to prove our results: the Gelfand transform, the variational formulation of the boundary value problems, and max-min-formulas for eigenvalues. The spectral problem (1.8)–(1.10) can be transformed into a family of spectral problems in the periodicity cell using the Gelfand transform. We briefly recall its definition:

$$v(y,z) \mapsto \; V(y,z,\eta) = \frac{1}{\sqrt{2\pi}} \sum_{j\in\mathbb{Z}} \exp(-i\eta(z+j))v(y,z+j), \tag{1.11}$$

where $(y,z) \in \Pi^\epsilon$ on the left while $\eta \in [0,2\pi)$ and $(y,z) \in \varpi$ on the right. As is well known, the Gelfand transform establishes an isometric isomorphism between the Lebesgue spaces,

$$L^2(\Pi^\epsilon) \simeq L^2(0,2\pi; L^2(\varpi)),$$

where $L^2(0,2\pi;B)$ is the Lebesgue space of functions with values in the Banach space B, endowed with the norm

$$\|V; L^2(0,2\pi;B)\| = \left(\int_0^{2\pi} \|V(\eta);B\|^2 d\eta\right)^{1/2}.$$

The Gelfand transform is also an isomorphism from the Sobolev space $H^l(\Pi^\epsilon)$ onto $L^2(0,2\pi; H^l_{\epsilon,\eta}(\varpi))$ for $l = 1,2$. The space $H^2_{\epsilon,\eta}(\varpi)$ consists of Sobolev functions u which satisfy the quasi-periodicity conditions

$$u(0,x') = e^{-i\eta}u(1,x'), \qquad (0,x') \in \theta_0^\epsilon, \tag{1.12}$$

$$\partial_{x_1} u(0,x') = e^{-i\eta}\partial_{x_1} u(1,x'), \qquad (0,x') \in \theta_0^\epsilon, \tag{1.13}$$

whereas $H^1_{\epsilon,\eta}(\varpi)$ is the Sobolev space with the condition (1.12) only.

Applying the Gelfand transform to the differential equation (1.8) and to the boundary conditions (1.9)–(1.10), we obtain a family of model problems in the periodicity cell ϖ parametrized by the dual variable η,

$$-\Delta U^\epsilon(x;\eta) = 0, \qquad x \in \varpi, \tag{1.14}$$

$$\partial_n U^\epsilon(x;\eta) = 0, \qquad x \in \sigma^\epsilon, \tag{1.15}$$

$$\partial_z U^\epsilon(x;\eta) = \Lambda^\epsilon(\eta)U^\epsilon(x;\eta), \qquad x \in \gamma, \tag{1.16}$$

$$U^\epsilon(0,x';\eta) = e^{-i\eta}U^\epsilon(1,x';\eta), \qquad x \in \theta_0^\epsilon, \tag{1.17}$$
$$\partial_{x_1}U^\epsilon(0,x';\eta) = e^{-i\eta}\partial_{x_1}U^\epsilon(1,x';\eta), \qquad x \in \theta_0^\epsilon. \tag{1.18}$$

Here, $\Lambda^\epsilon = \Lambda^\epsilon(\eta)$ is a new notation for the spectral parameter λ^ϵ. More details on the use of the Gelfand-transform can be found e.g. in (**20**), Section 2.

The apertures disappear at $\epsilon = 0$, hence, in that case the quasi-periodicity conditions cease to exist. We can thus consider the problem (1.14)–(1.18) as a singular perturbation of the limit spectral problem

$$-\Delta U^0(x) = 0, \qquad x \in \varpi, \tag{1.19}$$
$$\partial_n U^0(x) = 0, \qquad x \in \sigma^0, \tag{1.20}$$
$$\partial_z U^0(x) = \Lambda^0 U^0(x), \qquad x \in \gamma \tag{1.21}$$

with Λ^0 as a spectral parameter.

Our approach to the spectral properties of the model and limit problems is similar to (**21**), Sections 1.2, 1.3.. We first write the variational form of the problem (1.14)–(1.18) for the unknown function $U^\epsilon \in H^1_{\epsilon,\eta}(\varpi)$ as

$$(\nabla U^\epsilon, \nabla V)_\varpi = \Lambda^\epsilon (U^\epsilon, V)_\gamma \, , \ V \in H^1_{\epsilon,\eta}(\varpi), \tag{1.22}$$

and the corresponding variational formulation of the limit problem for $U \in H^1(\varpi)$ reads as

$$(\nabla U, \nabla V)_\varpi = \Lambda (U, V)_\gamma \, , \ V \in H^1(\varpi). \tag{1.23}$$

We denote by $\mathcal{H}^\epsilon$ the space $H^1_{\epsilon,\eta}(\varpi)$ endowed with the new scalar product

$$(u,v)_\epsilon = (\nabla u, \nabla v)_\varpi + (u,v)_\gamma, \tag{1.24}$$

and define a self-adjoint, positive and compact operator $\mathcal{B}^\epsilon(\eta) : \mathcal{H}^\epsilon \to \mathcal{H}^\epsilon$ using

$$(\mathcal{B}^\epsilon(\eta)u, v)_\epsilon = (u,v)_\gamma. \tag{1.25}$$

The problem (1.22) is then equivalent to the standard spectral problem

$$\mathcal{B}^\epsilon(\eta)u = M^\epsilon u \tag{1.26}$$

with a spectral parameter

$$M^\epsilon = (1+\Lambda^\epsilon)^{-1}. \tag{1.27}$$

Clearly, the spectrum of $\mathcal{B}^\epsilon(\eta)$ consist of 0 and a decreasing sequence $(M_k^\epsilon(\eta))_{k=1}^\infty$ of eigenvalues, which can be calculated from the usual min-max formula

$$M_k^\epsilon(\eta) = \min_{E_k} \max_{v \in E_k} \frac{(\mathcal{B}^\epsilon(\eta)v, v)_\epsilon}{(v,v)_\epsilon}, \tag{1.28}$$

where the minimum is taken over all subspaces $E_k \subset \mathcal{H}^\epsilon$ of co-dimension $k-1$. Using (1.24) and (1.25), we can write a max-min formula for the eigenvalues of the problem (1.22):

$$\Lambda_k^\epsilon(\eta) = \frac{1}{M_k^\epsilon(\eta)} - 1 = \max_{E_k} \min_{v \in E_k} \frac{(\nabla v, \nabla v)_\varpi + (v,v)_\gamma}{(v,v)_\gamma} - 1$$

$$= \max_{E_k} \min_{v \in E_k} \frac{\|\nabla v; L^2(\varpi)\|^2}{\|v; L^2(\gamma)\|^2}. \tag{1.29}$$

On the other hand, the connection (1.27) and the properties of the sequence $(M_k^\epsilon(\eta))_{k=1}^\infty$ mean that the eigenvalues (1.29) form an unbounded sequence

$$0 \le \Lambda_1^\epsilon(\eta) \le \Lambda_2^\epsilon(\eta) \le \ldots \le \Lambda_k^\epsilon(\eta) \le \ldots \to +\infty. \tag{1.30}$$

The eigenfunctions can be assumed to form an orthonormal basis in the space $L^2(\varpi)$. The functions $\eta \mapsto \Lambda_k^\epsilon(\eta)$ are continuous and 2π-periodic (see for example (**6**), Ch. 9). Hence the sets

$$\Upsilon_k^\epsilon = \{\Lambda_k^\epsilon(\eta) : \eta \in [0, 2\pi)\} \tag{1.31}$$

are closed connected segments, which may degenerate into single points; their relation to the original problem was already mentioned in (1.1).

The spectral concepts of the limit problem (1.19)–(1.21) can be treated in the same way as in (1.24)–(1.30). Since the quasi-periodicity conditions vanish for $\varepsilon = 0$, the space $\mathcal{H}^\epsilon$ is replaced by $H^1(\varpi)$; the norm induced by (1.24) is now equivalent to the original Sobolev norm of $H^1(\varpi)$. We denote by $\mathcal{B} : H^1(\varpi) \to H^1(\varpi)$ the operator, which is defined in the same way as in (1.24)–(1.25). The limit problem has an eigenvalue sequence $(\Lambda_k^0)_{k=1}^\infty$ like (1.30), however, neither the eigenvalues nor the operator $\mathcal{B}$ depend on η (cf. (**20**), Section 3). The first eigenvalue Λ_1^0 equals 0, and the first eigenfunction is the constant function. Analogously to (1.29) we can write

$$\Lambda_k^0 = \max_{F_k} \min_{v \in F_k} \frac{\|\nabla v; L^2(\varpi)\|^2}{\|v; L^2(\gamma)\|^2}, \tag{1.32}$$

where again $F_k \subset H^1(\varpi)$ is running over all subspaces of codimension $k-1$. We denote by

$$(U_k^0)_{k=1}^\infty \tag{1.33}$$

an $L^2(\gamma)$-orthonormal sequence of eigenfunctions corresponding to the eigenvalues (1.32).

Lemma 1.4. *For all k there exists a constant $C_k > 0$ such that*

$$|U_k^0(x)| \le C_k \ , \ |\nabla U_k^0(x)| \le C_k \tag{1.34}$$

for all $x \in \operatorname{supp}(\chi_j) \cap \varpi$, $j = 0, 1$ (and hence for all $x \in \operatorname{supp}(X_j^\epsilon) \cap \varpi$, $0 < \epsilon \le 1$).

Proof. Let for example $j = 0$ (the other case is treated similarly), and define the domains $G_1, G_2 \subset \mathbb{R}^3$ with C^∞ boundary such that $\overline{G_0} := \operatorname{supp}(\chi_j) \subset G_1 \subset \overline{G_1} \subset G_2 \subset \{x_3 < 0\}$ and G_2 still so small that

$$G_2 \cap \{x_1 = 0\} \subset \partial\varpi \quad \text{and} \quad \overline{G_2} \cap \{x_1 > 0\} \subset \varpi. \tag{1.35}$$

As a consequence, these domains are smooth enough so that we can use the local elliptic estimates (**1**), Theorem 15.2, to the solutions U_k^0 of the equation (1.19): this yields for every $l = 1, 2, \ldots$, a constant $C_{l,k} > 0$ such that

$$\|U_k^0; H^{l+1}(G_n \cap \varpi)\| \le C_{l,k}\big(\|U_k^0; H^{l-1}(G_{n+1} \cap \varpi)\| + \|U_k^0; L^2(G_{n+1} \cap \varpi)\|\big)$$

for $n = 0, 1$. Applying this first with $n = 1$ and $l = 1$ we get a bound for $\|U_k^0; H^2(G_1 \cap \varpi)\|$. A second application with $n = 0$ and $l = 2$ yields an estimate for $\|U_k^0; H^3(G_0 \cap \varpi)\|$. The standard embeddings $H^2(G_1 \cap \varpi) \subset C_B(G_0 \cap \varpi)$ and $H^3(G_0 \cap \varpi) \subset C_B^1(G_0 \cap \varpi)$ imply the result. $\square$

2. The formal asymptotic procedure

2.1 *The case of a simple eigenvalue*

To describe the asymptotic behaviour (as $\epsilon \to 0$) of the eigenvalues $\Lambda_k^\epsilon(\eta)$ of the problem (1.14)-(1.18) we consider first the case Λ_k^0 is a simple eigenvalue of the problem (1.19)-(1.21) for some fixed k. Let us make the *ansatz*

$$\Lambda_k^\epsilon(\eta) = \Lambda_k^0 + \epsilon\Lambda_k'(\eta) + \widetilde{\Lambda}_k^\epsilon(\eta), \tag{2.1}$$

where $\Lambda_k'(\eta)$ is a correction term and $\widetilde{\Lambda}_k^\epsilon(\eta)$ a small remainder to be evaluated and estimated. In this section we derive the expression (2.13) for $\Lambda_k'(\eta)$, cf. also (2.18) and (2.19), and the remainder will be treated in Section 3.2

The corresponding asymptotic ansatz for the eigenfunction reads as

$$\begin{aligned} U_k^\epsilon(x;\eta) &= U_k^0(x) \\ &+ \chi_0(x)w_{k0}(\epsilon^{-1}(x-P^0)) + \chi_1(x)w_{k1}(\epsilon^{-1}(x-P^1)) \\ &+ \epsilon U_k'(x;\eta) + \widetilde{U}_k^\epsilon(x;\eta), \end{aligned} \tag{2.2}$$

where $(U_k^0)_{k=1}^\infty$ is as in (1.33). The functions w_{k0} and w_{k1} are of boundary layer type, and χ_j is given above (1.6).

The boundary layers w_{kj} depend on the "fast" variables ("stretched" coordinates)

$$\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j) = \epsilon^{-1}(x - P^j), \quad j = 0, 1.$$

They are needed to compensate the fact that the leading term U_k^0 in the expansion (2.2) does not satisfy the quasi-periodicity conditions (1.17)–(1.18). By Lemma 1.4 and the mean value theorem, the eigenfunction $U_k^0(x)$ has the representation

$$U_k^0(x) = U_k^0(P^j) + O(\epsilon), \quad x \in \theta_j^\epsilon$$

near the points P^j, $j = 0, 1$. We look for w_{k0} and w_{k1} as the solutions of the problems

$$\begin{aligned} \Delta_{\xi^0} w_{k0}(\xi^0) &= 0, && \xi_1^0 > 0, \\ \partial_{\xi_1^0} w_{k0}(\xi^0) &= 0, && \xi^0 \in \{0\} \times (\mathbb{R}^2 \setminus \overline{\theta}), \\ w_{k0}(\xi^0) &= a_{k0}, && \xi^0 \in \{0\} \times \theta, \end{aligned}$$

and

$$\begin{aligned} \Delta_{\xi^1} w_{k1}(\xi^1) &= 0, && \xi_1^1 < 0, \\ \partial_{\xi_1^1} w_{k1}(\xi^1) &= 0, && \xi^1 \in \{0\} \times (\mathbb{R}^2 \setminus \overline{\theta}), \\ w_{k1}(\xi^1) &= a_{k1}, && \xi^1 \in \{0\} \times \theta \end{aligned}$$

in the half spaces $\{\xi_1^0 > 0\}$ and $\{\xi_1^1 < 0\}$, respectively; the meaning of the numbers a_{kj} will be explained below. Both of the functions w_{kj}, $j = 0, 1$, can be extended as even harmonic functions to the exterior of the set $\{0\} \times \theta$:

$$\Delta_{\xi^j} w_{kj}(\xi^j) = 0, \qquad \xi^j \in \mathbb{R}^3 \setminus (\{0\} \times \overline{\theta}), \tag{2.3}$$

$$w_{kj}(\xi^j) = a_{kj}, \qquad \xi^j \in \partial(\{0\} \times \overline{\theta})\,.$$

Furthermore, the problem (2.3) admits a solution (see (**22**))

$$w_{kj}(\xi^j) = a_{kj} \frac{\mathrm{cap}_3\,\theta}{|\xi^j|} + \widetilde{w}_{kj}(\xi^j), \tag{2.4}$$

$$\widetilde{w}_{kj}(\xi^j) = O(|\xi^j|^{-2})\ ,\ \nabla_{\xi^j} \widetilde{w}_{kj}(\xi^j) = O(|\xi^j|^{-3}), \tag{2.5}$$

where $\mathrm{cap}_3(\theta)$ is the 3-dimensional capacity of the set $\{0\} \times \theta$ and (2.5) concerns large ξ^j-behaviour. Moreover, the solution has a finite Dirichlet integral: there holds

$$\int_{\mathbb{R}^3} \left|\nabla_{\xi^j} w_{kj}(\xi^j)\right|^2 d\xi^j \le C \tag{2.6}$$

for some constant $C > 0$.

We aim to choose the coefficients a_{kj} such that U_k^ϵ satisfies the quasi-periodicity conditions (1.17)–(1.18) . Clearly, for each $\epsilon > 0$

$$\begin{aligned} U_k^\epsilon(P^0;\eta) &= e^{-i\eta} U_k^\epsilon(P^1;\eta), \\ \partial_{x_1} U_k^\epsilon(P^0;\eta) &= e^{-i\eta} \partial_{x_1} U_k^\epsilon(P^1;\eta), \end{aligned}$$

which together with the asymptotic expansion (2.2) yield the relations

$$U_k^0(P^0) + a_{k0} = e^{-i\eta}(U_k^0(P^1) + a_{k1}) \text{ and } a_{k0} = -e^{-i\eta} a_{k1}$$

for the coefficients. Hence,

$$a_{k1} = -e^{i\eta} a_{k0}, \quad a_{k0} = \frac{1}{2}\left(e^{-i\eta} U_k^0(P^1) - U_k^0(P^0)\right). \tag{2.7}$$

Now we can write a model problem for the main asymptotic correction term U_k':

$$-\Delta U_k'(x;\eta) = \Delta W_k(x) \qquad x \in \varpi, \tag{2.8}$$

$$(\partial_z - \Lambda_k^0) U_k'(x;\eta) = \Lambda_k'(\eta) U_k^0(x), \qquad x \in \gamma, \tag{2.9}$$

$$\partial_n U_k'(x;\eta) = 0, \qquad x \in \sigma^0\,, \tag{2.10}$$

where we denote

$$W_k(x) = \sum_{j=0}^{1} \chi_j(x) \frac{a_{kj}\,\mathrm{cap}_3(\theta)}{|x - P^j|}, \quad x \in \varpi\,. \tag{2.11}$$

In addition to U_k', the problem (2.8)–(2.10) will also determine the number $\Lambda_k'(\eta)$ in a unique way for every k and η. This will follow by requiring the solvability condition to hold in the Fredholm alternative, see Lemma 2.1 and its proof, below. Indeed, using the Green formula and the normalization in (1.33) we write (ds is the surface measure):

$$\begin{aligned} &\Lambda_k'(\eta) = \Lambda_k'(\eta) \|U_k^0; L^2(\gamma)\|^2 = \\ &= \int_\gamma \left(\partial_z U_k'(x;\eta) - \Lambda_k^0 U_k'(x;\eta)\right) \overline{U_k^0(x)}\, ds(x) = \end{aligned}$$

$$= \int_{\partial\varpi} \Big(\overline{U_k^0(x)} \partial_n U_k'(x;\eta) - U_k'(x;\eta) \overline{\partial_n U_k^0(x)} \Big)\, ds(x) =$$
$$= \int_{\varpi} \overline{U_k^0(x)} \Delta U_k'(x;\eta) = -\int_{\varpi} \overline{U_k^0(x)} \Delta W_k(x)\, dx\,. \tag{2.12}$$

Taking into account that the last integral converges absolutely and using the Green formula again yield

$$\begin{aligned} \Lambda_k'(\eta) &= \lim_{r\to 0} \sum_{j=0}^{1} \int_{S_r(P^j)\cap\varpi} \overline{U_k^0(x)} \partial_n \left(-\frac{a_{kj}\,\mathrm{cap}_3(\theta)}{|x-P^j|} \right) ds(x) \\ &= -2\pi\,\mathrm{cap}_3(\theta) \Big(a_{k0} \overline{U_k^0(P^0)} + a_{k1} \overline{U_k^0(P^1)} \Big)\,; \end{aligned}$$

see (2.11) and Remark 1.3 for notation. According to (2.7), we finally obtain

$$\Lambda_k'(\eta) = \pi\,\mathrm{cap}_3(\theta) |U_k^0(P^0) - e^{-i\eta} U_k^0(P^1)|^2\,. \tag{2.13}$$

Lemma 2.1. *Choosing $\Lambda_k'(\eta)$ as in* (2.13)*, the problem* (2.8)–(2.10) *has a solution $U_k' \in H^1(\varpi)$.*

Proof. The variational formulation of the problem (2.8)–(2.10) reads as

$$(\nabla U_k', \nabla V)_\varpi - \Lambda_k^0 (U_k', V)_\gamma = (\nabla W_k, \nabla V)_\varpi - \Lambda_k' (U_k^0, V)_\gamma. \tag{2.14}$$

We remark that the function $1/|x-P^j|$ is harmonic in ϖ, and since χ_j equals constant one in a neighbourhood of P^j, the function ΔW_k vanishes there, hence, W_k and ∇W_k are smooth as well as uniformly bounded everywhere in ϖ. Moreover, $U_k^0 \in L^2(\gamma)$.

Using the definition of the operator $\mathcal{B} : H^1(\varpi) \to H^1(\varpi)$ (cf. (1.24), (1.25) and the remarks above (1.32)) we can rewrite (2.14) as

$$(U_k', V)_0 - (\Lambda_k^0 + 1)(\mathcal{B} U_k', V)_0 = (W_k, V)_0 - \Lambda_k' (\mathcal{B} U_k^0, V)_0 - (\mathcal{B} W_k, V)_0 \tag{2.15}$$

which means that U_k' must be a solution of the equation

$$(\mathcal{B} - M_k^0) U_k' = -M_k^0 (W_k - \mathcal{B} W_k - \Lambda_k' \mathcal{B} U_k^0). \tag{2.16}$$

Notice that U_k^0 is the solution of the homogeneous problem (2.16), so, by the Fredholm alternative, (2.16) is solvable, if and only if the right hand side of it is orthogonal to the function U_k^0. This condition is satisfied by choosing $\Lambda_k'(\eta)$ as above, since

$$(W_k - \mathcal{B} W_k - \Lambda_k' \mathcal{B} U_k^0, U_k^0)_0 = (\nabla W_k, \nabla U_k^0)_\varpi - \Lambda_k' \| U_k^0; L^2(\gamma) \|^2 = 0\,,$$

by (2.12) and $(\nabla W_k, \nabla U_k^0)_\varpi = -(\Delta W_k, U_k^0)_\varpi$. This last identity follows from the first Green formula, because the normal derivative of W_k vanishes on $\partial\varpi$ due to the properties of the function χ_j, see below (1.5). $\square$

2.2 *The case of a multiple eigenvalue*

In this section we complete the asymptotic analysis by studying the behaviour of eigenvalues $\Lambda_k^\epsilon(\eta)$ in the case some Λ_k^0 has multiplicity m greater than one: we have

$$\Lambda_{k-1}^0 < \Lambda_k^0 = \ldots = \Lambda_{k+m-1}^0 < \Lambda_{k+m}^0.$$

The ansatz (2.1) is used again. Furthermore, as in (1.33), we denote by $(U_{k+j}^0)_{0\le j\le m-1} \subset L^2(\gamma)$ an orthonormal system of eigenfunctions associated with the eigenvalue Λ_k^0. Any eigenfunction U^0 corresponding to Λ_k^0 can be presented as a linear combination

$$U^0(x) = \sum_{j=0}^{m-1} \alpha_j U_{k+j}^0(x).$$

Analogously to (2.2) we introduce the asymptotic ansatz

$$\begin{aligned} U^\epsilon(x;\eta) \,&=\, U^0(x) \\ &+\, \chi_0(x) w_{k0}(\epsilon^{-1}(x-P^0)) + \chi_1(x) w_{k1}(\epsilon^{-1}(x-P^1)) \\ &+\, \epsilon U'(x;\eta) + \widetilde{U}^\epsilon(x;\eta). \end{aligned} \tag{2.17}$$

Using the same argumentation as in the previous section we construct the boundary layers w_{kj}, $j=0,1$, which satisfy the conditions

$$w_{kj}(\xi^j) = a_{kj}\frac{\operatorname{cap}_3\theta}{|\xi^j|} + O(|\xi^j|^{-2});$$

here the coefficients a_{kj} come from the equations (2.7), where U_k^0 is replaced by U^0. The main asymptotic term U' is also treated in the same way as in Section 2.1. To use the Fredholm alternative for finding $\Lambda'_{k+j}(\eta)$, $j=0,\ldots,m-1$, we write

$$\Lambda'_{k+j}(\eta)\alpha_j = \Lambda'_{k+j}(\eta)(U^0, U_{k+j}^0)_\gamma,$$

and making use of the Green formula as above we get

$$\Lambda'_{k+j}(\eta)\alpha_j = \sum_{l=0}^{m-1} \beta_{lj}\alpha_j,$$

where

$$\beta_{lj} = \pi \operatorname{cap}_3(\theta)(U_{k+l}^0(P^0) - e^{-i\eta}U_{k+l}^0(P^1))\overline{(U_{k+j}^0(P^0) - e^{-i\eta}U_{k+j}^0(P^1))}.$$

Hence, $\Lambda'_{k+j}(\eta)$ is an eigenvalue of the matrix $B = (\beta_{lj})_{l,j=0}^{m-1}$. This matrix has rank one, because it can be represented in the form $B = \overline{v}v^\top$, where v is a vector with components $v_{j+1} = U_{k+j}^0(P^0) - e^{-i\eta}U_{k+j}^0(P^1)$, $j=0,\ldots,m-1$. This means that

$$\Lambda'_k(\eta) \,=\, \pi \operatorname{cap}_3(\theta) \sum_{l=0}^{m-1} |U_{k+l}^0(P^0) - e^{-i\eta}U_{k+l}^0(P^1)|^2, \tag{2.18}$$

$$\Lambda'_{k+j}(\eta) \,=\, 0, \quad 1 \le j \le m-1\,. \tag{2.19}$$

3. Existence and position of spectral gaps

3.1 *Existence of gaps*

The first estimate on the eigenvalues of the problem (1.22) can now be stated as follows.

Theorem 3.1. *For any $k \in \mathbb{N}$ there are numbers $\epsilon_k > 0$ and $C_k > 0$ such that for every $\epsilon \in (0, \epsilon_k)$ and any dual variable $\eta \in [0, 2\pi)$, the eigenvalues of the problem* (1.22) *and the eigenvalues of the limit problem* (1.23) *are related as follows:*

$$\Lambda_k^0 \leq \Lambda_k^\epsilon(\eta) \leq \Lambda_k^0 + C_k \epsilon. \tag{3.1}$$

As mentioned in the introduction (see the explanations around (1.1) and (1.31)), this result implies the existence of any prescribed number of gaps in the essential spectrum σ, since (3.1) also establishes an estimate for the endpoints of the intervals Υ_k^ϵ. To prove that result one needs to take enough many distinct eigenvalues Λ_k^0 and a small enough ϵ.

Corollary 3.2. *Given any number $N \in \mathbb{N}$, the essential spectrum σ of the problem* (1.8)–(1.10) *on Π^ϵ has at least N gaps, if ϵ is small enough.*

Proof of Theorem 3.1. We apply the max-min-principle described in Section 1.2 and first prove the estimate

$$\Lambda_k^\epsilon(\eta) \geq \Lambda_k^0. \tag{3.2}$$

Indeed, we recall that in the equations (1.32) and (1.29) both $F_k \subset H^1(\varpi)$ and $E_k \subset \mathcal{H}^\epsilon = H^1_{\epsilon,\eta}(\varpi)$ are arbitrary subspaces of co-dimension $k-1$. Since $H^1_{\epsilon,\eta}(\varpi) \subset H^1(\varpi)$, each E_k is contained in some F_k, and thus the infimum in (1.32) is smaller than that in (1.29).

So we turn to the upper estimate in (3.1) and fix a $k \in \mathbb{N}$. Let the eigenfunctions U_j^0 be as in (1.33) and let $H_k \subset \mathcal{H}^\epsilon(\varpi)$ be a subspace spanned by the functions $Y^\epsilon U_j^0$, where $j = 1, \ldots, k$ and

$$Y^\epsilon = 1 - X_0^\epsilon - X_1^\epsilon \in C^\infty(\varpi) \tag{3.3}$$

and X_j^ϵ are as in (1.6). We remark that the functions $Y^\epsilon U_j^0$ satisfy the quasi-periodicity condition (1.12) in the definition of the space $\mathcal{H}^\epsilon$, since Y^ϵ vanishes in a neighbourhood of the apertures, see the remarks around (1.6). Moreover, the sequence $\big(Y^\epsilon U_1^0,\ Y^\epsilon U_2^0,\ \ldots,\ Y^\epsilon U_k^0\big)$ is still linearly independent, due to the $L^2(\gamma)$-orthogonality in (1.33) and the fact that Y^ϵ equals 1 in the set γ. Hence, the dimension of H_k is k.

If E_k is an arbitrary subspace of $\mathcal{H}^\epsilon$ of co-dimension $k-1$ (cf. (1.29)), the intersection $E_k \cap H_k$ contains a non-trivial linear combination

$$U(x) = Y^\epsilon(x) \sum_{j=1}^k a_j U_j^0(x), \quad \sum_{j=1}^k |a_j|^2 = 1. \tag{3.4}$$

By the remarks just above we have $\|U; L^2(\gamma)\| = 1$. Hence, from (1.29) and (1.33) we infer that

$$\Lambda_k^\epsilon(\eta) \leq \frac{\big\|\nabla U; L^2(\varpi)\big\|^2}{\|U; L^2(\gamma)\|^2} = \|\nabla U; L^2(\varpi)\|^2$$

$$= \Big\| \nabla\Big((Y^\epsilon - 1) \sum_{j=1}^k a_j U_j^0 \Big) + \nabla\Big(\sum_{j=1}^k a_j U_j^0 \Big); L^2(\varpi) \Big\|^2$$

$$= \Big\| \nabla\Big(\sum_{j=1}^k a_j U_j^0 \Big); L^2(\varpi) \Big\|^2 + 2\Big(\nabla\Big((-X_0^\epsilon - X_1^\epsilon) \sum_{j=1}^k a_j U_j^0 \Big), \nabla\Big(\sum_{j=1}^k a_j U_j^0 \Big) \Big)_\varpi$$

$$+ \Big\| \nabla\Big((X_0^\epsilon + X_1^\epsilon) \sum_{j=1}^k a_j U_j^0 \Big); L^2(\varpi) \Big\|^2. \tag{3.5}$$

To evaluate the first term on the right hand side notice that the functions U_j^0 satisfy (1.23) so that the $L^2(\gamma)$-orthogonality of (1.33) implies

$$\Big\| \nabla\Big(\sum_{j=1}^k a_j U_j^0 \Big); L^2(\varpi) \Big\|^2 = \sum_{j,l=1}^k a_j a_l (\nabla U_j^0, \nabla U_l^0)_\varpi$$

$$= \sum_{j,l=1}^k a_j a_l \Lambda_j^0 (U_j^0, U_l^0)_\gamma = \sum_{j=1}^k a_j^2 \Lambda_j^0 \le \Lambda_k^0, \tag{3.6}$$

where the last inequality follows from (3.4) and the fact that the eigenvalues Λ_j^0 are indexed in increasing order. Furthermore, we use Lemma 1.4 as well as the facts that the supports of X_l^ϵ, $l = 0, 1$, have measure of order ϵ^3, $|\nabla X_l^\epsilon|$ are of order ϵ^{-1}, and $|a_j| \le 1$, to estimate

$$\Big| \Big(\nabla\Big((X_0^\epsilon + X_1^\epsilon) \sum_{j=1}^k a_j U_j^0 \Big), \nabla\Big(\sum_{j=1}^k a_j U_j^0 \Big) \Big)_\varpi \Big|$$

$$\le k^2 \Big(\sup_{x\in S} \big(1, |U_j^0(x)|, |\nabla U_j^0(x)|\big) \Big)^2 \sup_{x\in S} \big(1, |\nabla X_l^\epsilon(x)|\big) \int_S dx \le C_k \epsilon^2,$$

$$\Big\| \nabla\Big((X_0^\epsilon + X_1^\epsilon) \sum_{j=1}^k a_j U_j^0 \Big); L^2(\varpi) \Big\|^2$$

$$\le k^2 \Big(\sup_{x\in S} \big(1, |U_j^0(x)|, |\nabla U_j^0(x)|\big) \Big)^2 \Big(\sup_{x\in S} \big(1, |\nabla X_l^\epsilon(x)|\big) \Big)^2 \int_S dx \le C_k \epsilon, \tag{3.7}$$

where $S = \operatorname{supp}(X_0^\epsilon + X_1^\epsilon)$. Combining this with (3.6) and (3.5) yields the result. $\square$

3.2 *Asymptotic position of spectral bands*

In this section we shall prove the validity of the asymptotic ansatz (2.1), see Theorem 3.5. This yields our main result concerning the asymptotic position of the spectral bands, Theorem 3.6.

We start the proof by recalling a classical lemma on near eigenvalues and eigenvectors (see e.g. (**3**), (**24**)).

Lemma 3.3. *Let $\mathcal{T}$ be a self-adjoint, positive, and compact operator in a Hilbert space $\mathcal{H}$. If a number $\mu > 0$ and an element $\mathcal{V} \in \mathcal{H}$ satisfy $\|\mathcal{V}; \mathcal{H}\| = 1$ and $\|\mathcal{T}\mathcal{V} - \mu\mathcal{V}; \mathcal{H}\| = \tau \in (0, \mu)$, then the segment $[\mu - \tau, \mu + \tau]$ contains at least one eigenvalue of $\mathcal{T}$.*

To apply Lemma 3.3 to the operator $\mathcal{B}^\epsilon(\eta)$ of (1.25), we fix an arbitrary k and, keeping in mind the formula (1.27), define the approximate k:th eigenvalue and eigenvector of $\mathcal{B}^\epsilon(\eta)$ by

$$\begin{aligned} \mu_k &= (1+\Lambda_k^0+\epsilon\Lambda_k'(\eta))^{-1}, \\ \mathcal{V}_k(x) &= \|\mathcal{U}_k;\mathcal{H}^\epsilon\|^{-1}\mathcal{U}_k(x), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \mathcal{U}_k(x) &= (1-X_0^\epsilon(x)-X_1^\epsilon(x))U_k^0(x) \\ &+ X_0^\epsilon(x)U_k^0(P^0)+X_1^\epsilon(x)U_k^0(P^1) \\ &+ \chi_0(x)w_{k0}(\epsilon^{-1}(x-P^0))+\chi_1(x)w_{k1}(\epsilon^{-1}(x-P^1)) \\ &+ \epsilon(1-X_0^\epsilon(x)-X_1^\epsilon(x))U_k'(x,\eta), \end{aligned} \tag{3.9}$$

U_k^0 is as in (1.33), $X_j^\epsilon(x)=\chi_\theta(\epsilon^{-1}(x-P^j))$ and χ_j are as in (1.6).

We need a lower bound for the norm of $\mathcal{U}_k$.

Lemma 3.4. *For all k there exists a constant $C_k>0$ such that*

$$\Big|\|\mathcal{U}_k;\mathcal{H}^\epsilon\|^2-1-\Lambda_k^0\Big| \le C_k\epsilon^{1/2}\,. \tag{3.10}$$

Proof. Recall that the expression for $\mathcal{U}_k$, (3.9), contains the term U_k^0; let us denote $\widetilde{\mathcal{U}}_k := \mathcal{U}_k - U_k^0$. By (1.24), (1.32)–(1.33), and (1.23), we have $\|U_k^0;\mathcal{H}^\epsilon\|^2 = \|U_k^0;L^2(\gamma)\|^2 + \|\nabla U_k^0;L^2(\varpi)\|^2 = 1+\Lambda_k^0$. Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned} \Big|\|\mathcal{U}_k;\mathcal{H}^\epsilon\|^2-1-\Lambda_k^0\Big| &= \Big|2(U_k^0,\widetilde{\mathcal{U}}_k)_\epsilon+\|\widetilde{\mathcal{U}}_k;\mathcal{H}^\epsilon\|^2\Big| \\ \le\ 2\sqrt{1+\Lambda_k^0}\|\widetilde{\mathcal{U}}_k;\mathcal{H}^\epsilon\| &+ \|\widetilde{\mathcal{U}}_k;\mathcal{H}^\epsilon\|^2. \end{aligned} \tag{3.11}$$

Taking into account the definition of the norm of $\mathcal{H}^\epsilon$, the formula (3.9) and the fact that the functions χ_j and X_j^ϵ vanish on γ we find that $\|\widetilde{\mathcal{U}}_k;\mathcal{H}^\epsilon\|$ is bounded by the sum of the expressions

$$\big\|\nabla\big(X_j^\epsilon(U_k^0-U_k^0(P^j))\big);L^2(\varpi)\big\|,\ j=0,1, \tag{3.12}$$

$$\big\|\nabla\big(\chi_j w_{kj}(\epsilon^{-1}(x-P^j))\big);L^2(\varpi)\big\|\ \ j=0,1, \tag{3.13}$$

$$\|\epsilon(1-X_0^\epsilon-X_1^\epsilon)U_k';H^1(\varpi)\|. \tag{3.14}$$

First we use the observation that the supports of the functions X_j^ϵ and ∇X_j^ϵ are contained in balls of radius $O(\epsilon)$ and that $|U_k^0(x)|$ and $|\nabla U_k^0(x)|$ are uniformly bounded in these balls (Lemma 1.4), hence $U_k^0(x)-U_k^0(P^j)=O(\epsilon)$ there. So, (3.12) can be bounded by a constant times

$$\begin{aligned} &\int_\varpi |U_k^0-U_k^0(P_j)|^2\,|\nabla X_j^\epsilon|^2dx + \int_\varpi |\nabla U_k^0|^2\,|X_j^\epsilon|^2dx \\ &\le C\Big(\int_{\mathrm{supp}X_j^\epsilon}\epsilon^2\epsilon^{-2}dx+\int_{\mathrm{supp}X_j^\epsilon}dx\Big)^{1/2}\le C\epsilon^{3/2}. \end{aligned} \tag{3.15}$$

We estimate the terms (3.13) using the fact that the support of the function $\nabla\chi_j$ is contained in a set $\{c \le |x-P^j| \le C\} =: \mathcal{S}_j$ for some constants $0<c<C$ (see above (1.6)), hence, by the estimate (2.4)–(2.5),

$$\big|w_{kj}(\epsilon^{-1}(x-P^j))\big| \le C\epsilon|x-P^j|^{-1} \ \text{ for } x\in\mathcal{S}_j. \tag{3.16}$$

Applying (2.6) yields

$$\begin{aligned}
&\int_\varpi \big|\nabla\big(\chi_j(x)w_{kj}(\epsilon^{-1}(x-P^j))\big)\big|^2 dx\\
&\le \int_{\mathcal{S}_j} |w_{kj}(\epsilon^{-1}(x-P^j))|^2dx + \int_\varpi \big|\nabla\big(w_{kj}(\epsilon^{-1}(x-P^j))\big)\big|^2 dx\\
&\le \int_{\mathcal{S}_j} C\epsilon^2|x-P^j|^{-2}dx + \epsilon^3\epsilon^{-2}\int_{\mathbb{R}^3}\big|\nabla_{\xi^j}w_{kj}(\xi^j)\big|^2 d\xi^j\\
&\le C_1\epsilon.
\end{aligned}$$

Finally, by Lemma 2.1, U_k' belongs to the space $H^1(\varpi)$. For the terms (3.14) we thus get the bound

$$\begin{aligned}
&\|\epsilon(1-X_0^\epsilon-X_1^\epsilon)U_k';H^1(\varpi)\|\\
&\le C\epsilon\|U_k';H^1(\varpi)\| + C\epsilon\max_{j=0,1}\|\nabla X_j^\epsilon;L^2(\varpi)\|^{1/2}\|U_k';L^2(\varpi)\|^{1/2} \le C'\epsilon,
\end{aligned}$$

since $\|\nabla X_j^\epsilon;L^2(\varpi)\| \le C\epsilon^{1/2}$, due to the measure of the support of ∇X_j^ϵ. $\square$

As a corollary of this lemma, if $\epsilon\in(0,\epsilon_0]$, then the bounds

$$0<\mu_k\le c_\mu \quad \|\mathcal{U};\mathcal{H}^\epsilon\|\ge c_{\mathcal{U}}>0, \tag{3.17}$$

hold true with some positive constants c_μ and $c_{\mathcal{U}}$ depending on ϖ and θ only.

The next theorem provides quite accurate asymptotic information on the eigenvalues of the model problem and in particular justifies the ansatz (2.1).

Theorem 3.5. *For every $k\ge1$ there exists a constant C_k such that, for each $\eta\in[0,2\pi)$,*

$$|\Lambda_k^\epsilon(\eta)-\Lambda_k^0-\epsilon\Lambda_k'(\eta)| < C_k\epsilon^{3/2}, \tag{3.18}$$

where $\Lambda_k'(\eta)=\pi\,\mathrm{cap}_3(\theta)|U_k^0(P^0)-e^{-i\eta}U_k^0(P^1)|^2$ (cf. (2.13)) in the case the eigenvalue Λ_k^0 is simple and $\Lambda_k'(\eta)$ is given by the formulas (2.18)–(2.19) in the case the eigenvalue Λ_k^0 is multiple.

Proof. We apply Lemma 3.3 to the operator $\mathcal{B}^\epsilon(\eta)$ with $\mu=\mu_k$ and $\mathcal{V}=\mathcal{V}_k$ as in (3.8). Our aim is to show that τ of the lemma can be chosen as small as $C_k\epsilon^{3/2}$. The lemma then gives an eigenvalue $M(\epsilon,\eta)$ of $\mathcal{B}^\epsilon(\eta)$ with the estimate

$$|M(\epsilon,\eta)-\mu_k|\le C_k\epsilon^{3/2}.$$

Using (3.8) and (1.27) this turns into an eigenvalue $\lambda(\epsilon,\eta)$ (of (1.14)–(1.18)) satisfying (3.18) in the place of $\Lambda_k^\epsilon(\eta)$. However, if ϵ is small enough, Theorem 3.1 guarantees that in a neighbourhood of Λ_k^0 there is only one eigenvalue of the model problem, namely $\Lambda_k^\epsilon(\eta)$. So $\lambda(\epsilon, \eta)$ must coincide with it, and the estimate (3.18) follows.

We are thus left with the task of proving

$$\tau = \|\mathcal{B}^\epsilon(\eta)\mathcal{V}_k - \mu_k \mathcal{V}_k; \mathcal{H}^\epsilon\| \le C_k \epsilon^{3/2}.$$

To this end we write, using $\mathcal{V}_k = c_{\mathcal{U}}^{-1}\mathcal{U}_k$, (1.25), (1.24), (3.8), (3.17),

$$\begin{aligned}
\tau &= \sup_Z \left|(\mathcal{B}^\epsilon(\eta)\mathcal{V}_k - \mu_k \mathcal{V}_k, Z)_\epsilon\right| \\
&= c_{\mathcal{U}_k}^{-1} \sup_Z \left|(\mathcal{U}_k, Z)_\gamma - \mu_k(\mathcal{U}_k, Z)_\gamma - \mu_k(\nabla \mathcal{U}_k, \nabla Z)_\varpi\right| \\
&\le c_\mu c_{\mathcal{U}}^{-1} \sup_Z \left|(\Lambda_k^0 + \epsilon \Lambda_k')(\mathcal{U}_k, Z)_\gamma - (\nabla \mathcal{U}_k, \nabla Z)_\varpi\right| =: c_\mu c_{\mathcal{U}}^{-1} \sup_Z |T(Z)|. \qquad (3.19)
\end{aligned}$$

The supremum is calculated here over all functions $Z \in \mathcal{H}^\epsilon$ with unit norm. The expression $T(Z)$ can be represented as a sum of the terms

$$\begin{aligned}
S_1(Z) &= -(\nabla U_k^0, \nabla Z)_\varpi + \Lambda_k^0 (U_k^0, Z)_\gamma, \\
S_{2j}(Z) &= -\big(\nabla(X_j^\epsilon(U_k^0(P^j) - U_k^0)), \nabla Z\big)_\varpi, \\
S_3(Z) &= -\big(\nabla\big(\chi_0 w_{k0}(\epsilon^{-1}(x - P^0)) + \chi_1 w_{k1}(\epsilon^{-1}(x - P^0))\big), \nabla Z\big)_\varpi \\
&\quad + \epsilon(\nabla w_k, \nabla Z)_\varpi, \\
S_4(Z) &= -\epsilon(\nabla U_k', \nabla Z)_\varpi - \epsilon(\nabla w_k, \nabla Z)_\varpi \\
&\quad + \epsilon \Lambda_k'(U_k^0, Z)_\gamma + \epsilon \Lambda_k^0 (U_k', Z)_\gamma, \\
S_5(Z) &= -\epsilon(\nabla((X_0^\epsilon + X_1^\epsilon)U_k'), \nabla Z)_\varpi + \epsilon^2 \Lambda_k'(U_k', Z)_\gamma.
\end{aligned}$$

where w_k is given by

$$w_k(x) = \chi_0(x)\frac{a_{k0}\,\mathrm{cap}_3(\theta)}{|x - P^0|} + \chi_1(x)\frac{a_{k1}\,\mathrm{cap}_3(\theta)}{|x - P^1|}. \qquad (3.20)$$

First we note that $S_1(Z) = S_4(Z) = 0$, because U_k^0 and U_k' are the solutions of the problems (1.19)–(1.21) and (2.8)–(2.10), respectively; see also (2.14).

To estimate $S_{2j}(Z)$ we use the Cauchy-Schwartz inequality:

$$\begin{aligned}
&\left|\big(\nabla(X_j^\epsilon(U_k^0(P^j) - U_k^0)), \nabla Z\big)_\varpi\right| \\
&\le \left\|\nabla(X_j^\epsilon(U_k^0(P^j) - U_k^0)); L^2(\varpi)\right\| \|Z; H^1(\varpi)\| \le C_k \epsilon^{3/2}.
\end{aligned}$$

Here the last inequality follows from an estimate already made for (3.12) (see (3.15)) and the assumption on the norm of Z.

The first term in $S_5(Z)$ can also be treated using the Cauchy-Schwartz inequality, the properties of the cut-off functions X_j^ϵ (once again as in (3.15)) and Lemma 2.1

$$\begin{aligned}
&\epsilon\left|\big(\nabla((X_0^\epsilon + X_1^\epsilon)U_k'), \nabla Z\big)_\varpi\right| \\
\le\ &\epsilon\left\|\nabla((X_0^\epsilon + X_1^\epsilon)U_k'); L^2(\varpi)\right\| \|Z; H^1(\varpi)\| \\
\le\ &\epsilon\left\|\nabla(X_0^\epsilon + X_1^\epsilon); L^2(\varpi)\right\| \left\|U_k'; L^2(\varpi)\right\|
\end{aligned}$$

$$+\ \epsilon\big\|X_0^\epsilon + X_1^\epsilon; L^2(\varpi)\big\|\big\|\nabla U_k'; L^2(\varpi)\big\| \le C_k \epsilon^{3/2}.$$

The surface integral in $S_5(Z)$ can be estimated simply by the trace inequality.

To provide an upper bound for $S_3(Z)$ we notice that by (2.4),

$$w_{kj}(\epsilon^{-1}(x - P^j)) = \epsilon \frac{a_{kj}\,\mathrm{cap}_3(\theta)}{|x - P^j|} + \widetilde{w}_{kj}(\epsilon^{-1}(x - P^j))\,,\ j = 0, 1, \tag{3.21}$$

hence, using (3.20) we can write

$$S_3(Z) = -\sum_{j=0}^{1} \big(\nabla\big(\chi_j \widetilde{w}_{kj}(\epsilon^{-1}(x - P^j))\big), \nabla Z\big)_\varpi.$$

After integrating by parts and taking into account that $\widetilde{w}_{kj}$ are harmonic functions we obtain

$$\begin{aligned} S_3(Z) \,=\, &\sum_{j=0}^{1} \big(\widetilde{w}_{kj}(\epsilon^{-1}(x - P^j))\Delta\chi_j, Z\big)_\varpi \\ &+\ 2\big((\nabla\chi_j)\nabla\big(\widetilde{w}_{kj}(\epsilon^{-1}(x - P^j))\big), Z\big)_\varpi. \end{aligned} \tag{3.22}$$

In the second term, the support of the function $\nabla\chi_j$ is contained in a set $\{c \le |x - P^j| \le C\} =: \mathcal{S}_j$ for some constants $0 < c < C$, hence, by the estimate (2.5),

$$\big|\nabla\big(\widetilde{w}_{kj}(\epsilon^{-1}(x - P^j))\big)\big| \le C\epsilon^{-1}\big(\epsilon^{-1}|x - P^j|\big)^{-3} = C\epsilon^2|x - P^j|^{-3} \ \text{ for } x \in \mathcal{S}_j. \tag{3.23}$$

Hence,

$$\begin{aligned} &\Big|\big((\nabla\chi_j)\nabla\big(\widetilde{w}_{kj}(\epsilon^{-1}(x - P^j))\big), Z\big)_\varpi\Big| \\ \le\ &\Big(\int_{\mathcal{S}_j} \big|\nabla\big(\widetilde{w}_{kj}(\epsilon^{-1}(x - P^j)\big)\big|^2 dx\Big)^{1/2} \|Z; L^2(\varpi)\| \le C'\epsilon^2. \end{aligned} \tag{3.24}$$

The first term in (3.22) is treated with a similar argument, since $\mathcal{S}_j$ still contains the support of $\Delta\chi_j$ and the estimate

$$\big|\widetilde{w}_{kj}(\epsilon^{-1}(x - P^j))\big| \le C\epsilon^2|x - P^j|^{-3} \ \text{ for } x \in \mathcal{S}_j$$

again holds, by (2.5).

We thus get the bound

$$\tau \le C_k(\theta)\epsilon^{3/2}. \ \square$$

As a consequence we can now provide the asymptotic widths and positions of the spectral bands.

Theorem 3.6. *Let the index k be such that the eigenvalue Λ_k^0 is simple. Then, the band Υ_k^ϵ of the continuous spectrum of the problem* (1.8)–(1.10) *has the asymptotic form*

$$\Upsilon_k^\epsilon = [\Lambda_k^0 + A_k\epsilon + O(\epsilon^{3/2}), \Lambda_k^0 + B_k\epsilon + O(\epsilon^{3/2})]$$

where

$$A_k = \pi\,\mathrm{cap}_3(\theta)\min\{|U_k^0(P^0) - U_k^0(P^1)|^2, |U_k^0(P^0) + U_k^0(P^1)|^2\}, \tag{3.25}$$

$$B_k = \pi\,\mathrm{cap}_3(\theta)\max\{|U_k^0(P^0) - U_k^0(P^1)|^2, |U_k^0(P^0) + U_k^0(P^1)|^2\}. \tag{3.26}$$

We recall that U_k^0 is the eigenfunction corresponding to the kth eigenvalue Λ_k^0 of the limit problem in the periodicity cell ϖ, see (1.19)–(1.21), and that the 3-dimensional capacity $\mathrm{cap}_3(\theta)$ of the aperture is defined in (2.4)–(2.5).

The theorem clarifies the relation of the spectral terms of order ε with the geometry of the waveguide, and consequently, the information on the position of the spectral bands and gaps contained in Theorem 3.6 is by far more accurate than in the previous results. This was obtained by a detailed asymptotic ansatz including the boundary layer terms, whereas for example Theorem 3.1 was based on a more straightforward application of the max-min principle.

3.3 *Concluding remarks.*

Let us return to the band-gap structure of the Bloch spectrum (1.1) of periodic waveguides. A spectral band may in general consist of a single eigenvalue of infinite multiplicity, and in this case the band is contained in the essential but not in the continuous spectrum. Such a possibility has not been excluded in the typical existence results of spectral gaps, see for example the discussion after Theorem 2.1 in (**20**). Having accurate enough information on the position of the spectral band of course makes it possible to determine the nature of the band in question. In the situation of Theorem 3.6, if the eigenfunction U_k^0 is such that both $U_k^0(P^0)$ and $U_k^0(P^1)$ are nonzero and ϵ is small enough, then the numbers A_k and B_k are distinct, and in this case the spectral band Υ_k^ϵ is indeed an interval with positive length. This obviously provides a way to construct examples where the spectrum of the linear water-wave problem has a genuine band-gap structure with proper intervals as bands and with at least a given number of spectral gaps, cf. Theorem 3.2.

Mathematical analysis of the band-gap structure of the spectra of periodic waveguides has often been restricted to proving the existence of gaps and giving lower estimates for their number. From the point of view of applications, it would of importance to obtain detailed enough information on the position of the bands using mathematical analysis: the ultimate goal would be the mathematical shape optimization of various materials for desirable wave propagation properties. Our result can be seen as a step to that direction.

Although our considerations are restricted to the linear water wave model, the method is an abstract one and thus capable for generalisations to a priori completely different physical phenomena, provided there still exists the general mathematical framework of a self-adjoint, positive, periodic problem with suitable thin geometric structure admitting a limit process. Indeed, the authors are currently working with applications to spectra in the linear elasticity theory for periodic wavequides, which geometrically resemble the waveguides considered in the present work.

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