UNDERWATER TOPOGRAPHY "INVISIBLE" FOR SURFACE WAVES AT GIVEN FREQUENCIES

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Abstract. We study propagation and scattering of surface waves modelled by the linear water wave equation in an unbounded, two-dimensional water domain of finite depth. We develop a method for constructing perturbations of the bottom shape, which cannot be detected by a distant observer using a given wave frequency. Our approach provides rigorous proofs based on operator theory and a fixed point argument.

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Keywords: linear water-wave problem, asymptotic analysis, two-dimensional problem, invisibility, scattering problem

1. Introduction.

1.1. **Definition of invisible bottom perturbations.** The topic of this work is the propagation and scattering of surface waves over a two-dimensional water layer of finite depth. We aim to find geometric distortions of a straight bottom of the water domain, which are "invisible" on a given wave frequency for an observer located far from the distortion: the bottom perturbations only cause negligible changes, when compared with a planar surface wave propagating over a water domain with constant depth $d > 0$.

We denote by Π the two-dimensional strip $\mathbb{R} \times (-d, 0) \ni (y, z)$, which describes the water domain with constant depth. The main problem is formulated as finding profile functions h of the local water bottom topography (see Figure 1.1) of a water domain

$$\Pi^h = \{(y, z)\,;\, y \in \mathbb{R}, 0 > z > -d - h(y)\} \tag{1.1}$$

such that after passing over the obstacle (or support of h), the surface wave of a given frequency has only an exponentially small perturbation and, therefore, a distant observer cannot recognize the existence of this local warp by measuring the spectral characteristics or amplitude of the wave. The bottom perturbation is assumed to be smooth and situated in the region $\{(y, z)\,;\, |y| < L\}$ for some $L > 0$, so that $h \in C_c^\infty(-L, L)$, which is the space of infinitely smooth functions with compact support in the segment $(-L, L)$. In this paper, such invisible bottom perturbations are found in Section 2.2 by presenting h as a suitable, small linear combination of functions with certain orthogonality conditions. The coefficients of the linear

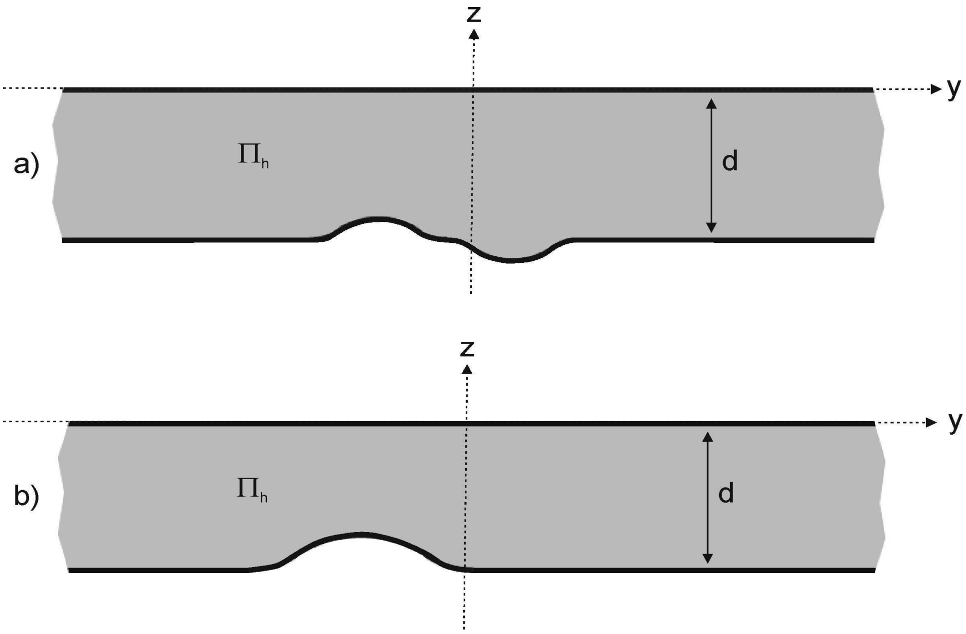

FIGURE 1.1. Bottom topography.

combination are found as a unique solution of a fixed point equation. This approach is still generalized in Section 3 by introducing a method suitable for iteration to build further invisible bottom perturbations. In Section 5.1 we discuss how our results can be interpreted as cloaking of a given bottom distortion.

Let us next present the mathematical model for surface waves studied in this paper. We start with plane waves over the unperturbed three dimensional water layer $\mathbb{R} \times \Pi \ni (x, y, z)$, assuming that $k \geq 0$ is the wave number in the x-direction, which is perpendicular to Π. The propagation of such waves is described by the velocity potential φ satisfying the Helmholtz equation

$$-\partial_y^2 \varphi(y,z) - \partial_z^2 \varphi(y,z) + k^2 \varphi(y,z) = 0 \ , \quad (y,z) \in \Pi, \tag{1.2}$$

$\partial_y := \partial/\partial y$, $\partial_z := \partial/\partial z$, with the kinematic Steklov condition

$$\partial_z \varphi(y,0) = \lambda \varphi(y,0) \ , \quad y \in \mathbb{R} = (-\infty, \infty), \tag{1.3}$$

on the free surface, and the Neumann "no-flow" condition

$$-\partial_z \varphi(y,-d) = 0 \ , \quad y \in \mathbb{R}, \tag{1.4}$$

on the bottom. Here , $\lambda = g^{-1}\omega^2$ is a spectral parameter with the acceleration of gravity $g > 0$ and frequency of time harmonic oscillations $\omega > 0$. The plane wave is thus given by

$$e^{ikx} w(y,z) = e^{ikx} e^{i\ell y} W(z), \tag{1.5}$$

where

$$\begin{aligned} W(z) &= e^{mz} + e^{-m(z+2d)} \ , \ m = \sqrt{k^2 + \ell^2} > 0 \ , \\ \lambda &= \lambda(m) := m \frac{1 - e^{-2md}}{1 + e^{-2md}}. \end{aligned} \tag{1.6}$$

Note that the function $m \mapsto \lambda \in \mathbb{R}_+$ is strictly monotone increasing so that ℓ and m are uniquely determined by k and $\lambda \geq \lambda_\dagger$ with the cut-off value $\lambda_\dagger = \lambda(k)$, see [1].

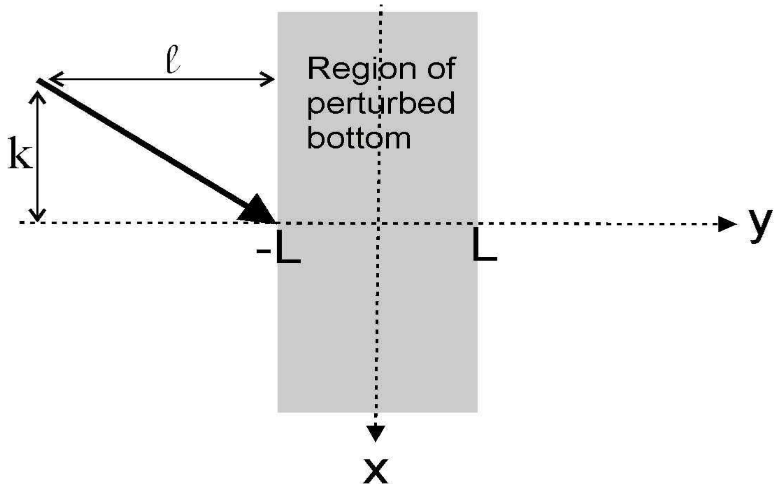

FIGURE 1.2. The direction of the incident wave.

Let us proceed to the case of the distorted bottom of the water domain (1.1). Since the perturbation profile h is assumed constant in the x-direction, the problem remains two-dimensional, cf. Section 5.4 for a more general case. The perturbation of the bottom causes scattering of the plane waves (1.5); the parameter k is related to the angle of incidence of the plane wave at the perturbation, and $k = 0$ corresponds to the case of normal incidence, cf. Figure 1.2. The resulting surface waves are solved from the linear water-wave equations in the domain (1.1),

$$-\partial_y^2 \varphi^h(y,z) - \partial_z^2 \varphi^h(y,z) + k^2 \varphi^h(y,z) = 0 \ , \qquad (y,z) \in \Pi^h, \tag{1.7}$$

$$\partial_z \varphi^h(y,0) = \lambda \varphi^h(y,0) \ , \ y \in \mathbb{R}, \tag{1.8}$$

$$\partial_n \varphi^h(y, -1 + h(y)) = 0 \ , \qquad y \in \mathbb{R}, \tag{1.9}$$

where φ^h again denotes the velocity potential and ∂_n stands for the derivative along the outward normal. For the needs of the forthcoming asymptotic analysis we introduce the normalized waves

$$w_\pm(y,z) = e^{\pm i\ell y} N^{-1/2} W(z), \tag{1.10}$$

where ℓ and W are given in (1.6),

$$N = 2\ell \|W; L^2(-1,0)\|^2 = 2\ell\big((2m)^{-1}(1 - e^{-4md}) + 2de^{-2md}\big) > 0, \tag{1.11}$$

and, as a consequence, $\|w^\pm(y,\cdot); L^2(-1,0)\|^2 = 1/(2\ell)$. Because of the wave number $\pm\ell$ the wave $w_\pm$ travels in the channel Π from $\mp\infty$ to $\pm\infty$. In general, the wave w_+, incoming from the left side of the channel scatters from the bottom perturbation and thus gives rise to a solution of the homogeneous problem (1.7)–(1.9) in the form

$$\varphi^h_\Rightarrow(y,z) = \chi_-(y) w_+(y,z) + \sum_\pm \chi_\pm(y) s^h_\pm w_\pm(y,z) + \widetilde{\varphi}^h_\Rightarrow(y,z), \tag{1.12}$$

where $\chi_\pm$ are smooth cut-off functions,

$$\chi_\pm(y) = 1 \ \text{ for } \pm y > 2L \ \text{ and } \ \chi_\pm(y) = 0 \ \text{ for } \pm y < L, \tag{1.13}$$

s^h_+ and s^h_- are the transmission and reflection coefficients, respectively, and the remainder $\widetilde{\varphi}^h_\Rightarrow(y,z)$ decays exponentially as $|y| \to \infty$. The coefficients are related by

$$|s_+^h|^2 + |s_-^h|^2 = 1. \tag{1.14}$$

However, the *invisibility of the warp* means for us that the velocity potential (1.12) takes the form

$$\varphi_{\Rightarrow}^h(y,z) = w_+(y,z) + \widehat{\varphi}_{\Rightarrow}^h(y,z) \tag{1.15}$$

with the exponentially decaying component $\widehat{\varphi}_{\Rightarrow}^h$. From the point of view of a distant observer, this is the situation described in the beginning of the section. The solution (1.12) reduces to (1.15), if and only if

$$s_-^h = 0\ ,\ s_+^h = 1. \tag{1.16}$$

In what follows we shall find an appropriate profile h just by solving the equations (1.16).

We also mention that since $\overline{w_+} = w_-$, complex conjugation in (1.12) yields

$$\overline{\varphi_{\Rightarrow}^h} =: \varphi_{\Leftarrow}^h = w_- + \widehat{\varphi}_{\Leftarrow}^h. \tag{1.17}$$

In other words, the wave w_- incoming from the right side of the channel also sustains only an exponentially decaying perturbation.

Our technique for the construction of the profiles h is quite similar to that supporting the so-called *enforced stability* of embedded eigenvalues, see [2, 3], and it has been first applied to invisibility of two-dimensional acoustic waveguides in [4]. There are several differences between the results of [4] and the present paper. In particular, unlike the case of acoustic waveguides, we are able to achieve here perfect invisibility, even without any phase shift of the transmitted wave. The generalization presented in Section 3 is also a new idea, which was not developed in the acoustic case (though it could have been).

We continue in Section 5.1 the discussion on the relation of our work with existing literature.

1.2. **Plan of the proof, structure of the paper.** In Section 2 we study low, almost flat profile functions h of the form $h(y) = -\varepsilon H(y)$, where ε is a small parameter and H is a smooth function such that $H(y) = 0$ for $|y| > L$. Our purpose is to find sufficient conditions for the function H in order to make the bottom perturbation invisible; the main result of this section is formulated in the beginning of Section 2.2. As explained above, h should support the relations (1.16) (thus also (1.15) and (1.17)), and to achieve this we employ asymptotic analysis of elliptic problems in regularly perturbed domains and construct two-term asymptotics in ε of the solution $\varphi_{\Rightarrow}^h$ in Π^h as well as of its transmission and reflection coefficients $s_\pm^{\varepsilon H}$. The crucial link between the bottom topography and the coefficients $s_\pm^{\varepsilon H}$ of the solution $\varphi_{\Rightarrow}^h$ is found by calculations in Section 2.1. In particular the main correction terms $s'_\pm$ in the expression (2.11) for $s_\pm^{\varepsilon H}$ are explicit integrals of the function H, which can be interpreted as orthogonality relations, see (2.9). These hint to an ansatz (2.10), $H(y) = H_0(y) + \sum_{j=1}^3 \tau_j H_j(y)$ including three additional free parameters

$$\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3, \tag{1.18}$$

and the functions H_j connected with the mentioned relations (2.9), which lead to orthogonality and normalization conditions (2.15)–(2.18) determining them. The condition (1.16) will be reduced to a nonlinear equation $\tau = T^\varepsilon(\tau)$ in $\mathbb{R}^3$, where T^ε

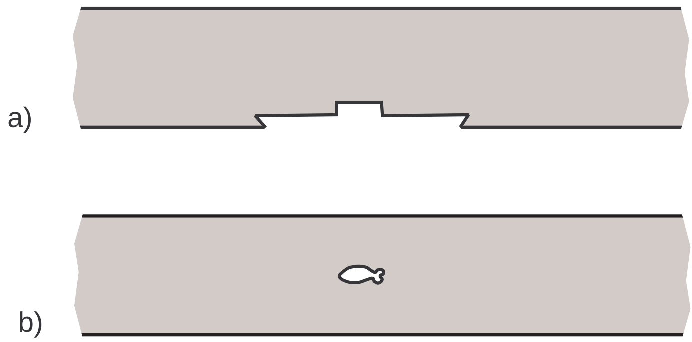

FIGURE 1.3. Geometries with open problems.

is a contraction in a subset of $\mathbb{R}^3$. The contraction mapping principle (Banach fixed point theorem) then gives a small solution τ in the case $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$.

In Section 3 we present a further development of the previous section, the aim of which is to provide an approach for larger invisible warps. The scheme consists of a general perturbation analysis: we make an a priori assumption that for some bottom profile h, (1.1), the scattered surface wave (1.12) takes the form (1.15) and seek for small distortions of the warp still keeping this property. The method will be similar to Section 2, though the result will be less explicit and definite while some additional assumptions will be needed, in particular we have to assume that trapped modes do not exist in the channel (1.1) with the reference profile h. However, all the requirements in Section 3 are quite computable, and the perturbation analysis can help to develop numerical algorithms for producing invisible warps of larger magnitude. Namely, since the choice of the perturbation profiles is quite arbitrary, one may try to apply this result repeatedly: starting with the straight bottom $h(y) = 0$ and almost flat perturbation (2.1), on may try to "cultivate" an invisible warp of big size.

In Section 4 we complete the proofs by justifying the asymptotic expansions and also examine the properties of the operator T^ε. This study relies upon the operator formulation of the water-wave problem (1.7)–(1.9) (with traditional radiation conditions) in suitable function spaces, namely weighted Sobolev spaces describing asymptotic behaviour of the waves at infinity, cf. the monograph [5, Ch. 5] and the review papers [6, 7]. Such a formulation and a standard trick of rectifying the boundary [8, § 7.6] allow us to apply perturbation theory of linear operators in Banach spaces, cf. [8, § XX], which makes all proofs rather simple. For the sake of simplicity we always assume that the problem data is C^∞-smooth, though C^4-smoothness would be enough in view of the proofs in Section 4.

Bottom perturbations with edges like in Figure 1.3. a) cannot be studied with these tools although asymptotic analysis augmented with boundary layers would be very similar to that of Sections 2 and 3 (see e.g. [9, Ch. 5] and [10]). Other open questions and generalizations of our results are given in Section 5. We for example explain how to create a gently sloping warp which cannot be detected by any waves with a prescribed finite set of frequencies $\omega_1, \ldots, \omega_N$ or wave numbers $k_1, \ldots, k_J$.

There remains the open question, what happens in the case $k = \ell$. This corresponds to the situation, when the incident angle is $\pi/4$, and it is excluded in (2.21) of Section 2 for technical reasons.

2. Gently sloping warps.

In this section we consider a water domain Π^h with a gently sloping perturbation of the straight bottom $\{(y,z)\,;\,z=-d\}$ of Π, taking the function

$$h := \varepsilon H \in C_c^\infty(-L,L) \tag{2.1}$$

with small $\varepsilon > 0$ for the bottom topography in (1.1). Starting from an asymptotic ansatz for the solution $\varphi^h_{\Rightarrow}$ (cf. (2.2) below), we shall derive the equations for its main correction term φ'. These, the asymptotic boundary condition on the distorted bottom, and the Sommerfeld radiation condition yield the formulas (2.9) expressing the coefficients $s_\pm^{\varepsilon H}$ with the help of the function H. This relation and the orthogonality conditions (2.15)–(2.18) will motivate the form of the ansatz (2.10) for the function H.

2.1. **Constructing the asymptotics.** We accept the simplest asymptotic ansatz for regularly perturbed domains,

$$\varphi^h_{\Rightarrow}(y,z) = w_+(y,z) + \varepsilon\varphi'(y,z) + \dots, \tag{2.2}$$

cf. [9, Ch. 5]. Notice that both w_+ and φ' are originally defined in the straight strip Π but can be extended smoothly to the lower half-plane; in this way the expansion (2.2) is well-understood in Π^h, too. The dots in (2.2) stand for higher order terms inessential for our asymptotic analysis. Inserting (2.2) into (1.7)–(1.8) we readily conclude that

$$-\partial_y^2\varphi'(y,z) - \partial_y^2\varphi'(y,z) + k^2\varphi'(y,z) = 0 \quad , \quad (y,z)\in\Pi, \tag{2.3}$$

$$\partial_z\varphi'(y,0) = \lambda\varphi'(y,0) \quad , \quad y\in\mathbb{R}. \tag{2.4}$$

To derive a boundary condition on the rectified bottom, we use the Taylor formula together with the Helmholtz equation (1.2) for w_+ and the representation

$$\partial_n = \big(1+\varepsilon^2|\partial_y H(y)|^2\big)^{-1/2}\big(-\partial_z - \varepsilon\partial_y H(y)\partial_y\big) = -\partial_z - \varepsilon\partial_y H(y)\partial_y + \dots$$

for the normal derivative. We then have

$$\begin{aligned}
&\partial_n\varphi^h_{\Rightarrow}(y,-d-\varepsilon H(y))\\
&= -\partial_z w_+(y,-d-\varepsilon H(y)) + \varepsilon\partial_y H(y)\partial_y w_+(y,-d-\varepsilon H(y)) + \dots\\
&\quad - \varepsilon\partial_z\varphi'(y,-d-\varepsilon H(y)) + \dots\\
&= -\partial_z w_+(y,-d) + \varepsilon H(y)\partial_z^2 w_+(y,-d) - \varepsilon\partial_y H(y)\partial_y w_+(y,-d)\\
&\quad - \varepsilon\partial_z\varphi'(y,-d) + \dots\\
&= \varepsilon\Big(-\partial_z\varphi'(y,-d) - H(y)\partial_y^2 w_+(y,-d) + H(y)k^2 w_+(y,-d)\\
&\quad - \partial_y H(y)\partial_y w_+(y,-d)\Big) + \dots,
\end{aligned} \tag{2.5}$$

and hence

$$-\partial_z\varphi'(y,-d) = \partial_y\big(H(y)\partial_y w_+(y,-d)\big) - H(y)k^2 w_+(y,-d) \quad , \quad y\in\mathbb{R}. \tag{2.6}$$

It is known (see, e.g., [11]) that the problem (2.3), (2.4), (2.6) has a unique solution subject to the Sommerfeld radiation condition

$$\varphi'(y,z) = \sum_\pm \chi_\pm(y)s'_\pm w_\pm(y,z) + \widetilde{\varphi}'(y,z), \tag{2.7}$$

where the notation is quite similar to (1.12), in particular $s'_\pm$ are some coefficients and the remainder decays exponentially as $|y| \to \infty$. To compute the coefficients we insert the function φ' and the waves w_α, $\alpha = \pm$, into the Green formula on the long ($R \to +\infty$) rectangle $(-R, R) \times (-d, 0)$ and obtain

$$\int_{-R}^{R} \overline{w_\alpha(y,-d)} \partial_z \varphi'(y,-d) dy$$

$$= \sum_\pm \pm \int_{-1}^{0} \Big(\overline{w_\alpha(\pm R, z)} \partial_y \varphi'(\pm R, z) - \varphi'(\pm R, z) \overline{\partial_y w_\alpha(\pm R, z)} \Big) dz = i s'_\alpha. \tag{2.8}$$

The last equality is due to the normalization factor $N^{-1/2}$ in (1.10). Integrating by parts in the segment $(-L, L)$ we observe using (2.6) that the left-hand side of (2.8) converts into

$$-\int_{-L}^{L} H(y) \Big(\partial_y w_+(y,-d) \overline{\partial_y w_\alpha(y,-d)} + k^2 w_+(y,-d) \overline{w_\alpha(y,-d)} \Big) dy.$$

Thus we finally obtain the expressions

$$s'_+ = 4i \frac{k^2 + \ell^2}{N} e^{-2md} \int_{-L}^{L} H(y) dy,$$

$$s'_- = 4i \frac{k^2 - \ell^2}{N} e^{-2md} \int_{-L}^{L} e^{2i\ell y} H(y) dy, \tag{2.9}$$

where N is taken from (1.11).

2.2. **Main result for gently sloping warps.** We show in this section (completing the proof in Section 4) that the bottom perturbation $h = -\varepsilon H$, (2.1), is invisible for small enough ε, if H is of the form

$$H(y) = H_0(y) + \sum_{j=1}^{3} \tau_j H_j(y), \tag{2.10}$$

where the functions $H_j \in C_c^\infty(-L, L)$, $j = 0, 1, 2, 3$, will be subject to the conditions (2.15)–(2.18) and $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ is a new small vector parameter to be solved from the equations (2.20), or equivalently (2.22).

Comparing (1.12) and (2.2), (2.7) we derive the representations

$$s_+^{\varepsilon H}(\tau) = 1 + \varepsilon s'_+(\tau) + \varepsilon^2 \tilde{s}_+^{\varepsilon H}(\tau) \ , \ s_-^{\varepsilon H}(\tau) = 0 + \varepsilon s'_-(\tau) + \varepsilon^2 \tilde{s}_-^{\varepsilon H}(\tau), \tag{2.11}$$

where the dependence on τ is displayed explicitly. In Section 4 we shall demonstrate that the functions

$$(\varepsilon, \tau) \mapsto s_\pm^{\varepsilon H}(\tau) \tag{2.12}$$

are analytic in the cylinder

$$\mathcal{Q} = \{(\varepsilon, \tau) \in \mathbb{R}^4 ;\, |\varepsilon| \le \varepsilon_0 \ , \ |\tau| \le \tau_0\} \tag{2.13}$$

for suitable positive ε_0 and τ_0. Moreover, the estimates

$$|\tilde{s}^{\varepsilon H}_{\pm}(\tau)| \leq C_0, \tag{2.14}$$

which justify the asymptotic analysis, are also verified there, see (4.12).

Taking into account the formulas (2.9) we subject the terms of (2.10) to the orthogonality conditions

$$\int_{-L}^{L} H_0(y)dy = 0, \tag{2.15}$$

$$\int_{-L}^{L} \cos(2\ell y)H_0(y)dy = 0 \ , \ \int_{-L}^{L} \sin(2\ell y)H_0(y)dy = 0, \tag{2.16}$$

$$\int_{-L}^{L} H_j(y)dy = \delta_{j,1}, \tag{2.17}$$

$$\int_{-L}^{L} \cos(2\ell y)H_j(y)dy = \delta_{j,2} \ , \ \int_{-L}^{L} \sin(2\ell y)H_j(y)dy = \delta_{j,3}, , \tag{2.18}$$

where $\delta_{j,p}$ is the Kronecker symbol. As a result, the relations

$$\operatorname{Im} s^{\varepsilon H}_{+}(\tau) = 0 \ , \ \operatorname{Im} s^{\varepsilon H}_{-}(\tau) = 0 \ , \ \operatorname{Re} s^{\varepsilon H}_{-}(\tau) = 0 \tag{2.19}$$

are converted into the system of transcendental equations

$$\begin{aligned} 4(k^2+\ell^2)e^{-2md}\tau_1 &= -\varepsilon N \operatorname{Im} \tilde{s}^{\varepsilon H}_{+}(\tau) \\ 4(k^2-\ell^2)e^{-2md}\tau_2 &= -\varepsilon N \operatorname{Im} \tilde{s}^{\varepsilon H}_{-}(\tau) \\ 4(k^2-\ell^2)e^{-2md}\tau_3 &= \varepsilon N \operatorname{Re} \tilde{s}^{\varepsilon H}_{-}(\tau). \end{aligned} \tag{2.20}$$

We assume that

$$k \neq \ell \tag{2.21}$$

and dividing by the numbers on the left hand sides, rewrite the system (2.20) in the vector form

$$\tau = T^{\varepsilon}(\tau). \tag{2.22}$$

Due to the estimates (2.14) and the analyticity of the functions (2.12) the operator $T^{\varepsilon} : \mathbb{R}^3 \to \mathbb{R}^3$ becomes a contraction in the ball

$$\mathcal{B} = \{\tau \in \mathbb{R}^3 \, ; \, |\tau - T^{\varepsilon}(0)| \leq \varrho_0\}, \tag{2.23}$$

if ε and ϱ_0 are small enough. Hence, the contraction mapping principle implies the existence of a unique solution $\tau = \tau(\varepsilon) \in \mathcal{B}$ of (2.22) such that the estimate

$$|\tau(\varepsilon)| \leq C_0\varepsilon \tag{2.24}$$

holds in addition.

The desired profile function (2.10) in (1.1) has been found. Indeed, the relations (2.19) are fulfilled, hence, $s^{\varepsilon H}_{-}(\tau(\varepsilon)) = 0$, and so the identity (1.14) implies $|s^{\varepsilon H}_{+}(\tau(\varepsilon))| = 1$. Moreover,

$$s^{\varepsilon H}_{+}(\tau(\varepsilon)) = 1 + O(\varepsilon) \tag{2.25}$$

by (2.11) and (2.12) so that the first relation (2.19) yields the last inequality (1.16) for small ε. Thus, both relations (1.16) are verified and the solution (1.12) of the problem (1.7)–(1.9) takes the form (1.15) with an exponentially decaying remainder $\widehat{\varphi}^h_{\Rightarrow}$.

2.3. **Discussing the result.** Clearly, all the requirements (2.15)–(2.18) can easily be satisfied. The inequality (2.24) shows that

$$H(y) = H_0(y) + O(\varepsilon), \tag{2.26}$$

and therefore the profile $h(y) = \varepsilon H(y)$ is mainly defined by H_0 while the functions $\tau_j(\varepsilon)H_j$ play the role of small correction terms which besides depend on the parameter ε.

The condition (2.15) implies that the volume increment of the bottom perturbation becomes $O(\varepsilon^2)$, which may be disappointing for some types of applications. However, putting $\tau_1 = 0$ in (2.10) and skipping the restrictions (2.15) and (2.17) do not prevent us from finding a parameter vector $\tau(\varepsilon) = (\tau_1(\varepsilon), \tau_2(\varepsilon))$ such that $s_-^{\varepsilon H}(\tau(\varepsilon)) = 0$ and thus $|s_-^{\varepsilon H}(\tau(\varepsilon))| = 1$, i.e.,

$$s_+^{\varepsilon H}(\tau(\varepsilon)) = e^{i\theta_\varepsilon} \tag{2.27}$$

with some exponent $\theta_\varepsilon \in [-\pi, \pi)$. Furthermore,

$$\theta_\varepsilon = O(\varepsilon) \tag{2.28}$$

in view of the asymptotics (2.25).

Formulas (2.27), (2.28) mean that the wave (1.5) incoming from the left side of the channel Π^h gains nothing but a small phase shift after passing the bottom perturbation. In other words, a distant observer who is only able to measure the wave amplitude, cannot recognize the warp of this particular shape.

The orthogonality conditions (2.16) may be satisfied by a non-positive function with

$$V_0 = -\int_{-L}^{L} H_0(y)dy > 0. \tag{2.29}$$

By (2.26), the total volume increment of the bottom perturbation becomes $\varepsilon V_0 + O(\varepsilon^2)$ and thus stays positive for small ε, cf. Figures 1.1.b) and 1.1.a).

The condition (2.21) was introduced for a technical reason: if $k = \ell$, the left hand sides of the last two equations in (2.20) vanish, and this makes our previous conclusion on the existence of the vector $\tau(\varepsilon)$ impossible. The authors do not know a physical reason for this restriction, and it is very probable that (2.21) can be removed by processing the higher order terms. However, we do not extend the asymptotic analysis of this paper that far.

3. Perturbation analysis of "invisible" warps.

This section contains a generalization of the approach and methods of Section 2. As a starting point we assume to be given a suitable "invisible" profile h, and we aim to present a method which allows to make a new perturbation of h while still preserving the invisibility properties. As a consequence, this procedure can be iterated to yield more general invisible bottom profiles, hopefully with larger perturbations.

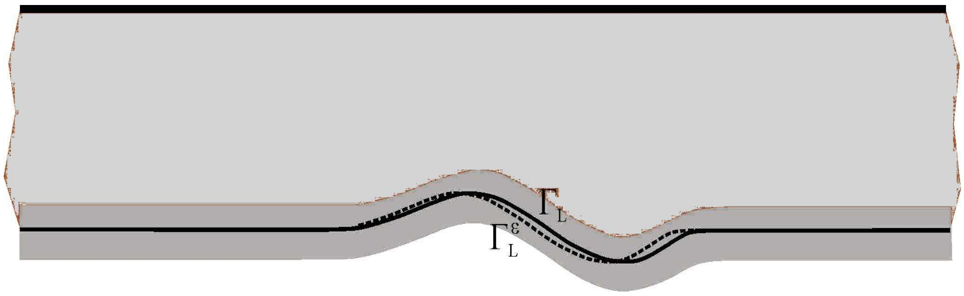

FIGURE 3.1. Tubular neighbourhood $\mathcal{V}$.

3.1. **Assumptions on the reference profile.** We assume to be given a reference profile h as in (1.1), such that the solution $\varphi^h_{\Rightarrow}$ in (1.12) of the problem (1.7)–(1.9) gets the form (1.15), i.e., the propagating wave (1.5) is not affected at infinity by this particular warp shape. We also agree that there is no trapped mode of the frequency λ for this particular underwater topography. This means that any solution of the homogeneous problem (1.7)–(1.9) which belongs to $L^2(\Pi^h)$, or equivalently decays at infinity, is nothing but null. All these assumptions are met for example by the gently sloping bottom studied in Section 2, but h does not necessarily need to be of that form.

The bottom perturbation is described as follows. Let $\mathcal{V}$, Figure 3.1, be a tubular neighbourhood of the curve $\Gamma_L = \{(y,z) \,:\, |y| < L \,,\ z = -d + h(y)\}$ endowed with the natural curvilinear coordinate system (s,n), where n is the oriented distance to Γ_L, $n \geq 0$ outside Π^h and $s \in (0, s_L)$ is the arc length measured along Γ_L from the point $(-L,-d)$. We define the perturbed curve

$$\Gamma^{\varepsilon}_L = \{(y,z) \in \mathcal{V} \,:\, s \in (0,s_L) \ ,\ n = \varepsilon K(s)\}, \tag{3.1}$$

where ε is again a small positive parameter and $K \in C^{\infty}_c(0,s_L)$. The water domain $\Pi^{h,\varepsilon K}$ is defined to lie between the free surface $\mathbb{R} \times \{0\}$ and the perturbed bottom Γ^{ε} consisting of the arc (3.1) and the two semi-axis $(-\infty,-L]$ and $[L,+\infty)$. Our assumptions ensure that K vanishes near the points $(\pm L, -d)$ so that Γ^{ε} is still a smooth curve. We shall next derive sufficient conditions for K, which ensure that the invisibility properties are preserved. The procedure for the construction of the asymptotics remains similar to that in Section 2, but concomitant calculations become a bit more complicated.

3.2. **Asymptotic analysis.** Given h and $\varphi^h_{\Rightarrow}$ of the form (1.15) as explained in Section 3.1, we introduce the ansätze generalizing (2.2) and (2.11),

$$\varphi^{h,\varepsilon K}_{\Rightarrow}(y,z) = \varphi^h_{\Rightarrow}(y,z) + \varepsilon\varphi'(y,z) + \varepsilon^2\widetilde{\varphi}^{h,\varepsilon K}_{\Rightarrow}(y,z), \tag{3.2}$$

$$s^{h,\varepsilon K}_{\pm} = s^h_{\pm} + \varepsilon s'_{\pm} + \varepsilon^2 \widetilde{s}^{h,\varepsilon K}_{\pm}. \tag{3.3}$$

So, in (3.2), $\varphi^{h,\varepsilon K}_{\Rightarrow}$ and $\varphi^h_{\Rightarrow}$ stand for velocity potentials in $\Pi^{h,\varepsilon K}$ and Π^h, respectively, generated by the same incoming wave $w_+(y,z)$ on the left sides of these channels. The functions $\varphi^h_{\Rightarrow}$ and φ' are defined in Π^h, but if necessary, they can be extended smoothly to the neighbourhood of Γ_L covering the set $\Pi^{h,\varepsilon K} \setminus \Pi^h$.

Clearly, φ' satisfies the Helmholtz equation (1.7) and the Steklov condition (1.8). Let us derive the boundary condition on Γ for it; we keep the notation $\varphi'(s,n)$ also in the curvilinear coordinates. Since

$$\nabla = \big(\partial_n, (1+\kappa(s)n)^{-1}\partial_s\big),$$

where κ is the curvature of Γ, the normal derivative ∂_{n^ε} on Γ^ε looks as follows:

$$\frac{\partial}{\partial n^\varepsilon} = \Big(1+\Big|\frac{\varepsilon\partial_s K(s)}{1+\varepsilon\kappa(s)K(s)}\Big|\Big)^{-1/2}\Big(\frac{\partial}{\partial n} - \frac{\varepsilon\partial_s K(s)}{\big(1+\varepsilon\kappa(s)K(s)\big)^2}\frac{\partial}{\partial s}\Big)$$

$$= \partial_n - \varepsilon\partial_s K(s)\partial_s + \dots. \tag{3.4}$$

Hence, similarly to (2.9) we obtain

$$\begin{aligned}&\partial_{n^\varepsilon}\varphi_{\Rightarrow}^{h,\varepsilon K}(s,\varepsilon K(s))\\ &= \partial_n\varphi_{\Rightarrow}^{h}(s,\varepsilon K(s)) - \varepsilon\partial_s K(s)\partial_s\varphi_{\Rightarrow}^{h}(s,\varepsilon K(s)) + \varepsilon\partial_n\varphi'(s,\varepsilon K(s)) + \dots\end{aligned}$$

$$= \partial_n\varphi_{\Rightarrow}^{h}(s,0) + \varepsilon K(s)\partial_n^2\varphi_{\Rightarrow}^{h}(s,0) - \varepsilon\partial_s K(s)\partial_s\varphi_{\Rightarrow}^{h}(s,0) + \varepsilon\partial_n\varphi'(s,0) + \dots. \tag{3.5}$$

We now recall the Neumann boundary condition $\partial_n\varphi_{\Rightarrow}^{h}(s,0)=0$ and the formula

$$(1+\kappa(s)n)^{-1}\partial_n\big((1+\kappa(s)n)\partial_n\big) + (1+\kappa(s)n)^{-1}\partial_s\big((1+\kappa(s)n)^{-1}\partial_s\big)$$

for the Laplacian in the curvilinear coordinates. Thus, $\partial_n^2\varphi_{\Rightarrow}^{h}(s,0) = -\partial_s^2\varphi_{\Rightarrow}^{h}(s,0) + k^2\varphi_{\Rightarrow}^{h}(s,0)$, and the boundary condition on Γ reads as

$$\partial_n\varphi'(s,0) = \partial_s\big(K(s)\partial_s\varphi_{\Rightarrow}^{h}(s,0)\big) - k^2K(s)\varphi_{\Rightarrow}^{h}(s,0). \tag{3.6}$$

As a consequence of the assumed non-existence of trapped modes, the problem (1.7), (1.8), (3.6) admits a unique solution φ' subject to the radiation conditions (2.7). To compute the arising coefficient s'_+ we insert φ' and $\varphi_{\Rightarrow}^{h}$ into the Green formula on the rectangle $(-R,R)\times(-d,0)$. Making use of (3.6) and integrating by parts along Γ we deduce analogously to (2.8) that

$$\begin{aligned}&\int_\Gamma K(s)\Big(\big|\partial_s\varphi_{\Rightarrow}^{h}(s,0)\big|^2 + k^2\big|\varphi_{\Rightarrow}^{h}(s,0)\big|^2\Big)ds\\ &= -\int_\Gamma \overline{\varphi_{\Rightarrow}^{h}(s,0)}\partial_n\varphi'(s,0)ds\\ &= \lim_{R\to\infty}\sum_{\pm}\pm\int_{-1}^{0}\Big(\overline{\varphi_{\Rightarrow}^{h}(\pm R,z)}\partial_y\varphi'(\pm R,z) - \varphi'(\pm R,z)\overline{\partial_y\varphi_{\Rightarrow}^{h}(\pm R,z)}\Big)dz\end{aligned}$$

$$= is'_+. \tag{3.7}$$

The last equality follows from the representations (1.15), (2.7) and the normalization factor (1.11) in the definition (1.10). In the same way we deal with $\varphi_{\Leftarrow}^{h} = \overline{\varphi_{\Rightarrow}^{h}}$, cf. (1.17), and get

$$\int_\Gamma K(s)\Big(\big(\partial_s\varphi_{\Rightarrow}^{h}(s,0)\big)^2 + k^2\big(\varphi_{\Rightarrow}^{h}(s,0)\big)^2\Big)ds = is'_-. \tag{3.8}$$

We emphasize that the integrand on the left in (3.7) involves moduli of the functions $\partial_s\varphi_{\Rightarrow}^{h}$ and $\varphi_{\Rightarrow}^{h}$, but in (3.8) the functions themselves. In particular the coefficient s'_+ is purely imaginary.

Estimates for the asymptotic remainders in (3.2) and (3.3) will be derived in Section 4. Namely, it will be shown that the asymptotic formulas (2.11), (2.14) remain valid in the channel $\Pi^{h,\varepsilon K}$ after an evident modification. The functions (2.12) become only smooth in the cylinder (2.13), but not in general analytic. However, this smoothness will still be sufficient for argumentation similar to that in Section 2.

3.3. **Proper perturbations of the profile.** The goal is to prove the relations

$$\operatorname{Im} s_+^{h,\varepsilon K} = 0 \ , \ \operatorname{Im} s_-^{h,\varepsilon K} = 0 \ , \ \operatorname{Re} s_-^{h,\varepsilon K} = 0; \tag{3.9}$$

these and (1.16) lead to the equalities

$$s_-^{h,\varepsilon K} = 0 \ , \ s_+^{h,\varepsilon K} = 1, \tag{3.10}$$

as shown for (2.19).

As in Section 2 we work with the representation

$$K(s) = K_0(s) + \sum_{j=1}^{3} \tau_j K_j(s) \tag{3.11}$$

involving the vector (1.18) of small parameters and the functions $K_q \in C_c^\infty(0, s_L)$. In view of (3.7) and (3.8) we impose three orthogonality conditions

$$\int_0^{s_L} K(s) F_p(s) ds = 0 \ , \ p = 1, 2, 3, \tag{3.12}$$

where

$$\begin{aligned} F_1(s) &= \left|\partial_s \varphi^h_{\Rightarrow}(s,0)\right|^2 + k^2 \left|\varphi^h_{\Rightarrow}(s,0)\right|^2, \\ F_2(s) &= \left|\operatorname{Re} \partial_s \varphi^h_{\Rightarrow}(s,0)\right|^2 - \left|\operatorname{Im} \partial_s \varphi^h_{\Rightarrow}(s,0)\right|^2 \\ &\quad + k^2 \Big(\left|\operatorname{Re} \varphi^h_{\Rightarrow}(s,0)\right|^2 - \left|\operatorname{Im} \varphi^h_{\Rightarrow}(s,0)\right|^2 \Big), \\ F_3(s) &= 2 \operatorname{Re} \partial_s \varphi^h_{\Rightarrow}(s,0) \operatorname{Im} \partial_s \varphi^h_{\Rightarrow}(s,0) + 2k^2 \operatorname{Re} \varphi^h_{\Rightarrow}(s,0) \operatorname{Im} \varphi^h_{\Rightarrow}(s,0). \end{aligned} \tag{3.13}$$

We assume that the functions F_j are linearly independent (this will be discussed in the next section) and that also the relations

$$\int_0^{s_L} K_q(s) F_p(s) ds = \delta_{p,q} \ , \ p, q = 1, 2, 3, \tag{3.14}$$

hold with functions taken from (3.11) and (3.13). Owing to (3.7), (3.8) and (3.12), (3.14) we obtain

$$\operatorname{Im} s'_+(\tau) = -\tau_1 \ , \ \operatorname{Im} s'_-(\tau) = -\tau_2 \ , \ \operatorname{Re} s'_-(\tau) = \tau_3,$$

and it is straightforward to convert the relations (3.9) into the abstract equation (2.22). The operator T^ε in (2.22) remains a contraction in the ball (2.23) with some radius $\varrho_0 > 0$, because the functions (2.12) will be proven to be smooth in Section 4. Thus the contraction mapping principle again yields a unique solution $\tau = \tau(\varepsilon) \in \mathcal{B}$ and the estimate (2.23) for some constant C_0.

The desired perturbation profile (3.11) is now constructed by assuming (3.14) and (3.12).

3.4. **Discussing the orthogonality and normalization conditions.** Since $\varphi^h_{\Rightarrow}(s,0)$ in (3.13) is not known explicitly, it is difficult to make conclusions about the linear independence of the functions F_1, F_2, F_3, (3.13). Moreover, we know that they can be dependent, because the last expression in (2.9), which now takes the form

$$-i\int_0^{s_L}\big(F_2(s)+iF_3(s)\big)ds$$

in the new notation, vanishes at $\ell = k$; we have put the restriction (2.21) just to avoid null coefficients of τ_2 and τ_3 in (2.20) and to reduce the relations (2.19) to the solvable equation (2.22).

Evidently, neither $\operatorname{Re}\varphi^h_{\Rightarrow}(s,0)$ nor $\operatorname{Im}\varphi^h_{\Rightarrow}(s,0)$ vanish on any arc $\gamma\subset\Gamma_L$ of positive length (otherwise they would satisfy the Helmholz equation (1.7) with both Dirichlet and Neumann conditions and thus become null everywhere in Π^h). The same is true for the derivative $\partial_s\varphi^h_{\Rightarrow}(s,0)$, cf. [12]. Hence, it is very easy to satisfy the first normalization condition (3.14) with $p=q=1$. The other two conditions $(p=q=2,3)$ need a different argument. However, there is no computational obstruction to verify, if (3.14) can be met or not. Finally, it remains as an open problem to find a non-trivial profile h such that the functions (3.13), restricted to the arc Γ_L, stay linearly dependent.

As in Section 2.3, annulling the function K_1 and skipping the conditions (2.15), (2.17) yield a perturbation of the bottom Γ which does not cause any reflection of the wave w_+ and leads only to a phase shift, like in (2.27) and (2.28).

4. Justification of asymptotics.

To make the analysis in Sections 2 and 3 rigorous, it is necessary to treat the remainder terms in (2.2), (2.11), (3.2) and (3.3). However, we only consider the last two of these in detail, since the analysis performed in Section 2 can be regarded as a special case of Section 3.

4.1. **Operator formulation.** We next present the suitable function spaces, as well as equations and radiation conditions satisfied by the remainder $\widetilde{\varphi}^{h,\varepsilon K}_{\Rightarrow}$. We solve the equations in Sections 4.2 and 4.3, and also show there that the solutions have the desired properties already used in Section 3.

The Kondratiev space (weighted Sobolev space) $W^q_\beta(\Pi^h)$ is the completion of the linear set $C^\infty_c(\Pi^h)$ of compactly supported infinitely smooth functions with respect to the norm

$$\|\psi;W^q_\beta(\Pi^h)\| = \sum_{p=0}^{q}\|e^{\beta|y|}\nabla^p\psi;L^2(\Pi^h)\|, \tag{4.1}$$

where $q=0,1,\dots$ and $\beta\in\mathbb{R}$ are the smoothness and weight exponents and $\nabla^p\psi$ is the collection of all pth order partial derivatives of ψ. In the case $\beta>0$ these functions decay exponentially at infinity. By $W^{1/2}_\beta(\partial\Pi^h)$ we understand the space of traces of functions in $W^1_\beta(\Pi^h)$ with the intrinsic norm

$$\|\Psi;W^{1/2}_\beta(\partial\Pi^h)\| = \inf\big\{\|\psi;W^1_\beta(\Pi^h)\|\,;\,W^1_\beta(\Pi^h)\ni\psi=\Psi\ \text{ on }\partial\Pi^h\big\}. \tag{4.2}$$

We associate to the inhomogeneous problem (1.7)–(1.9),

$$-\partial_y^2\psi(y,z)-\partial_z^2\psi(y,z)+k^2\psi(y,z)=f(y,z)\ ,\quad (y,z)\in\Pi^h,$$

$$\partial_z\psi(y,0) - \lambda\psi(y,0) = f_0(y)\ , \quad y\in\mathbb{R}, \tag{4.3}$$
$$-\partial_n\psi(y,-1-h(y)) = f_1(y)\ , \quad y\in\mathbb{R}$$

the mapping

$$A^h_\beta : W^2_\beta(\Pi^h)\to W^0_\beta(\Pi^h)\times W^{1/2}_\beta(\partial\Pi^h)\times W^{1/2}_\beta(\partial\Pi^h)\ ,\ \psi\mapsto(f,f_0,f_1) \tag{4.4}$$

which is continuous for any $\beta\in\mathbb{R}$ but has "good" properties only under appropriate restrictions on the weight index (see [13] and e.g. [5, Ch. 3 and 5]).

To fix β, we note that the remainder $\widetilde{\varphi}^h_{\Rightarrow}$ in (1.12) satisfies the problem (4.3) with the right hand sides $f_0=f_1=0$ and $f\in C^\infty_c(\Pi^h)$,

$$f(y,z)=\sum_\pm\big(\partial_y^2\chi_\pm(y)+2\partial_y\chi_\pm(y)\partial_y\big)\big(s^h_\pm w_\pm(y,z)+\delta_{\pm,-}w_+(y,z)\big), \tag{4.5}$$

where we have the Kronecker delta of signs at the end. Applying the Fourier decomposition in the straight subdomains $\Pi_\pm := \{(y,z)\in\Pi^h\,;\ \pm y>2\ell\}$, one shows that $\widetilde{\varphi}^h_{\Rightarrow}$ gains the decay rate $O(e^{-\beta_0|y|})$, where $\beta_0=\sqrt{k^2+\mu_1^2}$ and $\mu_1\in(\pi/2,\pi)$ is the first positive root of the equation $\lambda=-\mu\tan\mu$. We now fix β just by requiring

$$\beta\in(0,\beta_0). \tag{4.6}$$

Notice that the spectral parameter $\lambda>0$ has been fixed from the very beginning so that we do not display the dependence on it in (4.6), (4.4) or in what follows.

According to general results in [13] (see also [5, Ch. 2 and 5]), our assumption on the absence of trapped modes in (1.7)–(1.9) means that the operator (4.4) with the exponent (4.6) is a Fredholm monomorphism. We still use the weighted space with *attached asymptotics*, cf. [5, Ch. 6] and [7], namely the space $\mathcal{W}^2_\beta(\Pi^h)$ which is composed of functions

$$\psi(y,z)=\sum_\pm\chi_\pm(y)a_\pm w_\pm(y,z)+\widetilde{\psi}(y,z), \tag{4.7}$$

where $a_\pm\in\mathbb{C}$ and $\widetilde{\psi}\in W^2_\beta(\Pi^h)$, and it is supplied with the norm

$$\|\psi;\mathcal{W}^2_\beta(\Pi^h)\|=\big(|a_+|^2+|a_-|^2+\|\widetilde{\psi};W^2_\beta(\Pi^h)\|^2\big)^{1/2}. \tag{4.8}$$

Note that, similarly to (2.7), the expansion (4.7) is to be regarded as radiation conditions for the problem (4.3). Furthermore, being a proper skew-symmetric extension of the operator (4.4), the operator

$$\mathcal{A}^h_\beta : \mathcal{W}^2_\beta(\Pi^h)\to W^0_\beta(\Pi^h)\times W^{1/2}_\beta(\partial\Pi^h)\times W^{1/2}_\beta(\partial\Pi^h) \tag{4.9}$$

becomes an isomorphism, a useful fact in the later application of the perturbation theory of Banach space operators.

4.2. **Error estimates.** We extend $\varphi^h_{\Rightarrow}$ and φ' smoothly to $\Pi^h\cup\mathcal{V}\supset\Pi^{h,\varepsilon K}$ (the set $\mathcal{V}$ was defined in Section 3.1) and observe that the difference (cf. (3.2))

$$\varepsilon^2\widetilde{\varphi}^{h,\varepsilon K}_{\Rightarrow}=\varphi^{h,\varepsilon K}_{\Rightarrow}-\varphi^h_{\Rightarrow}-\varepsilon\varphi' \tag{4.10}$$

satisfies the problem (4.3) with $\Pi^{h,\varepsilon K}$ instead of Π^h, with the radiation conditions (4.7), and with the right hand sides written in the form

$$f(y,z)=\varepsilon^2\widetilde{f}^\varepsilon(y,z)\ ,\ f_0(y)=0\ ,\ f_1(y)=\varepsilon^2\widetilde{f}^\varepsilon_1(y), \tag{4.11}$$

Here $\widetilde{f}^\varepsilon$ vanishes for $|y|>2\ell$, since $\varphi^h_{\Rightarrow}$ and φ' both solve (1.7) in Π^h and $\varphi^{h,\varepsilon K}_{\Rightarrow}$ in $\Pi^{h,\varepsilon K}$ (see (1.12) and the remarks after (3.3)); also $\widetilde{f}^\varepsilon_1$ is null for $|y|>2\ell$ due to the boundary conditions satisfied by the terms in (4.10). Hence, $\widetilde{f}^\varepsilon \in W^0_\beta(\Pi^{h,\varepsilon K})$ and $\widetilde{f}^\varepsilon_1 \in W^{1/2}_\beta(\partial\Pi^{h,\varepsilon K})$, respectively, for any β, in particular for (4.6). Moreover, the norms of these functions are small: since $\varphi^h_\Rightarrow$ and φ' both obey the Helmholtz equation (1.7), their C^{2+J}-smooth extensions provide the relation

$$\|\widetilde{f}^\varepsilon; W^0_\beta(\Pi^{h,\varepsilon K})\| \leq c\varepsilon^{-2}\varepsilon^J\varepsilon^{1/2}, \tag{4.12}$$

where ε^{-2} comes from the left hand side of (4.10) and $\varepsilon^{1/2} = O\big(\mathrm{mes}_2(\Pi^{h,\varepsilon K}\setminus\Pi^h)^{1/2}\big)$. Also, making the formal calculations in (3.4) and (3.6) rigorous, namely, restoring the higher order terms denoted by dots and taking the condition (3.6) into account yield the inequalities $|\widetilde{f}^\varepsilon_1(y)| \leq c$ and $|\partial_y \widetilde{f}^\varepsilon_1(y)| \leq c$, which imply

$$\|\widetilde{f}^\varepsilon_1; W^{1/2}_\beta(\partial\Pi^{h,\varepsilon K})\| \leq c, \tag{4.13}$$

in view of (4.2).

An argument in the next section shows that the operator $\mathcal{A}^{h,\varepsilon K}_\beta$ remains an isomorphism for small ε and furthermore the norm of its inverse stays uniformly bounded in $\varepsilon \in [0, \varepsilon_0]$. The estimates (4.12) and (4.13) with $J \geq 2$ imply the inequality

$$\|\widetilde{\varphi}^{h,\varepsilon K}_\Rightarrow; \mathcal{W}^2_\beta(\Pi^{h,\varepsilon K})\| \leq c. \tag{4.14}$$

Since the norm (4.8) involves the coefficients $a_\pm$ of the expansion (4.7), we easily deduce from (4.14) and (1.12), (2.7) that

$$|\tilde{s}^{h,\varepsilon K}_\pm| \leq c. \tag{4.15}$$

The estimates (4.14) and (4.15) for the remainders in (3.2) and (3.3) make the formal asymptotic analysis in Sections 2 and 3 rigorous.

4.3. **Smoothness of the scattering coefficients.** In this section we complete the proof by performing a coordinate change which turns $\Pi^{h,\varepsilon K}$ into the reference channel Π^h and by treating the operators of the preceding section in Π^h with a perturbation argument. First, we make the change of curvilinear coordinates (K is as in (3.11))

$$(s,n) \mapsto \big(s(\varepsilon,\tau), n(\varepsilon,\tau)\big) = \big(s, n - \varepsilon K(s)\big)$$

in the tubular neighbourhood $\mathcal{V}$ of Γ_L, and this transforms Γ^ε_L into Γ_L. In Cartesian coordinates we write the change as $(y,z) \mapsto (y(\varepsilon,\tau), z(\varepsilon,\tau))$. The domain $\Pi^{h,\varepsilon K}$ is then mapped onto the reference channel Π^h by the global coordinate change

$$\begin{aligned}(y,z) &\mapsto \big(Y(\varepsilon,\tau), Z(\varepsilon,\tau)\big)\\ &= \mathcal{X}(y,z)\big(y(\varepsilon,\tau), z(\varepsilon,\tau)\big) + \big(1-\mathcal{X}(y,z)\big)(y,z),\end{aligned} \tag{4.16}$$

where $\mathcal{X} \in C^\infty_c(\mathcal{V})$ is a cut-off function such that $\mathrm{supp}\, K \subset \{(y,z) \in \Gamma_L\,;\, \mathcal{X}(y,z) = 1\}$. The transform (4.16) is nonsingular and moreover "almost identical", if ε and τ are small, or, belong to the cylinder (2.13). This means that

$$\big|\nabla^p(y - Y(\varepsilon,\tau))\big| + \big|\nabla^p(z - Z(\varepsilon,\tau))\big| \leq c_p(|\varepsilon| + |\tau|)\ ,\ p = 0,1,\dots, \tag{4.17}$$

and moreover that $Y(\varepsilon,\tau)$ and $Z(\varepsilon,\tau)$ depend smoothly on the parameters.

Near the free surface the Cartesian coordinates are not perturbed by the change (4.16). In the new coordinates the Helmholtz operator and the normal derivative differ from $-\partial^2_Y - \partial^2_Z + k^2$ and $\partial_{n(Y,Z)}$ only by small terms, see (4.16) again. The key point of our argument is that these terms have compact supports in $\mathcal{V} \cap \Pi^h$ and depend smoothly on $(\varepsilon,\tau) \in \mathcal{Q}$; in the framework of Section 2 the dependence is even analytic, since we had $s = y + L$ and $n = -z - d$ there. Due to compact supports, the coordinate change turns the operator $\mathcal{A}_\beta^{h,\varepsilon K}$ into $\mathcal{A}_\beta^h + \mathcal{T}_\beta^{h,\varepsilon K}(\tau)$, where only the small continuous addendum $\mathcal{T}_\beta^{h,\varepsilon K}$ depends on (ε, τ) .

We are now in a position to apply a classical result of operator theory. Keeping in mind that $\mathcal{A}_\beta^h$ is invertible, see (4.9), the inverse of the operator $\mathcal{A}_\beta^h + \mathcal{T}_\beta^{h,\varepsilon K}$ can be presented as

$$\big(\mathcal{A}_\beta^h + \mathcal{T}_\beta^{h,\varepsilon K}\big)^{-1} = \big(I + (\mathcal{A}_\beta^h)^{-1}\mathcal{T}_\beta^{h,\varepsilon K}\big)^{-1}(\mathcal{A}_\beta^h)^{-1}, \tag{4.18}$$

where we write the Neumann series

$$(I + (\mathcal{A}_\beta^h)^{-1}\mathcal{T}_\beta^{h,\varepsilon K})^{-1} = \sum_{n=0}^{\infty}\Big((\mathcal{A}_\beta^h)^{-1}\mathcal{T}_\beta^{h,\varepsilon K}\Big)^n \tag{4.19}$$

and I denotes the identity operator $\mathcal{W}_\beta^2(\Pi^h) \to \mathcal{W}_\beta^2(\Pi^h)$. The series (4.19) converges in the operator norm, since the operator norm of $(\mathcal{A}_\beta^h)^{-1}\mathcal{T}_\beta^{h,\varepsilon K} : \mathcal{W}_\beta^2(\Pi^h) \to \mathcal{W}_\beta^2(\Pi^h)$ is small. In addition to the fact that $\mathcal{A}_\beta^{h,\varepsilon K}$ becomes an isomorphism, we now see from (4.18)–(4.19) that the norm of its inverse is bounded uniformly in $(\varepsilon, \tau) \in \mathcal{Q}$. Moreover, the problem (4.3), (4.7) for the difference

$$\widehat{\varphi}_{\Rightarrow}^{h,\varepsilon K} = \varphi_{\Rightarrow}^{h,\varepsilon K} - \chi_- w_+ \in \mathcal{W}_\beta^2(\Pi^{h,\mathcal{H}}) \tag{4.20}$$

can be interpreted as the abstract equation

$$\begin{aligned}\big(\mathcal{A}_\beta^h + \mathcal{T}_\beta^{h,\varepsilon K}\big)\widehat{\varphi}_{\Rightarrow}^{h,\varepsilon K} &= (f, f_0, f_1) \\ &\in W_\beta^0(\Pi^h) \times W_\beta^{1/2}(\partial\Pi^h) \times W_\beta^{1/2}(\partial\Pi^h),\end{aligned} \tag{4.21}$$

where $f_0 = f_1 = 0$ and

$$f(y, z) = \big(\partial_y^2\chi_-(y) + 2\partial_y\chi_-(y)\partial_y\big)w_+(y, z), \tag{4.22}$$

cf. (4.5). Now (4.22) is independent of (ε, τ) and has support in the set $\{(y, z)\,;\, -2L \le y \le -L\,,\, 0 \ge z \ge -d\}$ with $(y, z) = (Y; Z)$. We can thus deduce that the solution of (4.21) in $\mathcal{W}_\beta^2(\Pi^h)$ depends smoothly on $(\varepsilon, \tau) \in \mathcal{Q}$ and even analytically in the case of the perturbation (2.1) of Section 2, because the dependence of the solution operator on (ε, τ) is smooth or analytic, due to the representation (4.18)–(4.19).

This solution is nothing but the function (4.20) rewritten in the coordinates $(Y(\varepsilon, \tau), Z(\varepsilon, \tau))$. Since the change (4.16) is the identity outside a compact set, the coefficients $s_\pm^{h,\varepsilon K}$ in the norm $\|\widehat{\varphi}_{\Rightarrow}^{h,\varepsilon K}; \mathcal{W}_\beta^2(\Pi^h)\|$, see (4.8), are also smooth and analytic as required in Sections 3 and 2, respectively. This completes the proofs of the existence of "invisible" warps.

5. Concluding remarks.

5.1. **Discussion on cloaking in the linear water wave model.** The results in Sections 2 and 3 can be seen as cloaking of a bottom perturbation, when observations are made on a fixed surface wave frequency. (For extensions of the result to the case of a finite number of frequencies, see the next section.) For example in Section 2.2, one can think the function H_0 as a given bottom perturbation, which only needs to satisfy the mild conditions (2.15), (2.16). Then, choosing the functions H_j, $j = 1, 2, 3$, properly as described in Section 2.2, one can build an invisible obstacle determined by the sum function (2.10). Notice that we do not pose any condition

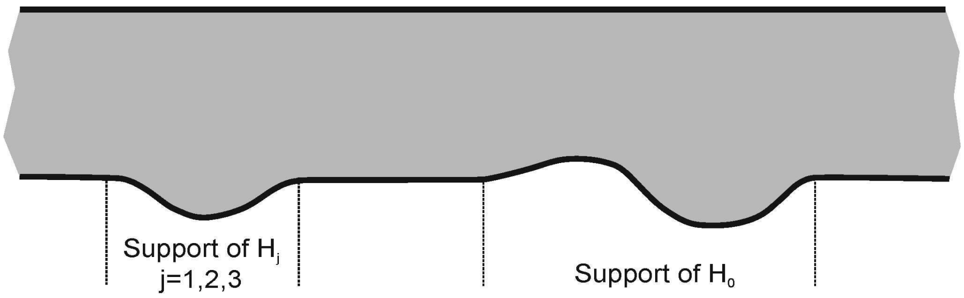

FIGURE 5.1. Cloaking the obstacle H_0.

on the supports of H_j, $j = 1, 2, 3$, so they can be situated at some distance of the support of H_0 as in Figure 5.1

We mention two related recent works, [14] and [15], which also deal with cloaking of an object in the linear water wave model with a single wave frequency. To compare the results we remind that the cloaking obtained in the present work is perfect, the assumptions on the geometric form of the distortion are very mild, and, finally, the results are rigorously proven. However, the price to pay is the smallness assumption of the bottom distortion, and the setting is two dimensional. In [15] the object to be cloaked is a vertical cylinder. The framework is three dimensional, the liquid is assumed homogeneous, and the cylinder does not need to be a small one. Yet, the methods are numerical algorithms, and the obtained cloaking is nearly, but not absolutely, perfect. The paper [14] concerns the cloaking of a floating body. The setting is two dimensional, stratification of the liquid is assumed, the methods consist of physical arguments and the obtained cloaking is not quite perfect.

As a conclusion, the overlapping of the results and especially methods of the three works is negligible.

5.2. **Set of prescribed frequencies.** Let us fix the values

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_N \tag{5.1}$$

for the spectral parameter. Searching for the profile function in the form (2.1) with

$$K(y) = K_0(y) + \tau_{m,1} K_{m,1}(y) + \sum_{q=1}^{N} \sum_{j=1}^{2} \tau_{q,j} K_{q,j}(y), \tag{5.2}$$

where $m \in \{1, \ldots, N\}$ is fixed, we impose $2N + 1$ orthogonality conditions on K_0, namely (2.15) and

$$\int_{-L}^{L} \cos(2\ell_p y) K_0(y) dy = \int_{-L}^{L} \sin(2\ell_p y) K_0(y) dy = 0, \ p = 1, \ldots, N, \tag{5.3}$$

where ℓ_p is determined by λ_p and k through (1.6). Moreover, since the trigonometric functions in (5.3) and (2.15) are linearly independent, we can also require the conditions

$$\int_{-L}^{L} K_{m,1}(y) dy = 1, \int_{-L}^{L} \cos(2\lambda_p y) K_{m,1}(y) dy = \int_{-L}^{L} \sin(2\lambda_p y) K_{m,1}(y) dy = 0,$$

$$\int_{-L}^{L} K_{q,j}(y)dy = 0, \int_{-L}^{L} \cos(2\lambda_p y) K_{q,j}(y)dy = \delta_{p,q}\delta_{2,j}, \tag{5.4}$$

$$\int_{-L}^{L} \sin(2\lambda_p y) K_{q,j}(y)dy = \delta_{p,q}\delta_{3,j} \ , \ p,q = 1,\dots,N \ , \ j = 2,3. \tag{5.5}$$

For each $p = 1,\dots,N$ and the corresponding frequency $\omega_p = \sqrt{g\lambda_p}$ we consider the corrections terms $\varepsilon s'_+(\lambda_p;\tau)$ in the representations (2.11). Repeating the asymptotic analysis of Section 2.1 we derive for them the formulas (2.9) with $\ell, m \mapsto \ell_p, m_p = \sqrt{k^2+\ell^2}$. We also assume that $k \neq \ell_p$ for all p. There is no problem to convert the relations

$$\operatorname{Im} s_+^{h,\varepsilon K}(\lambda_m;\tau) = 0 \ , \ \operatorname{Im} s_-^{h,\varepsilon K}(\lambda_p;\tau) = 0 \ , \ \operatorname{Re} s_-^{h,\varepsilon K}(\lambda_p;\tau) = 0 \ , \tag{5.6}$$

$p = 1,\dots,N$, into the equation (2.22) such that the operator T^ε is still a contraction in a small ball. Thus the solution $\tau = (\tau_{m,1}, \tau_{1,2}, \tau_{1,3}, \dots, \tau_{N,2}, \tau_{N,2}) \in \mathbb{R}^{1+2N}$ exists and we have detected the bottom profile which causes no reflection for the waves $w_+(\lambda_1;x,y),\dots,w_+(\lambda_N;x,y)$ of (1.10) with λ as in (5.1). We emphasize that the condition $s_+^{h,\varepsilon K}(\lambda_m;\tau) = 1$ of the intact passing wave can be achieved for one predetermined frequency ω_m, and for $\omega_p \neq \omega_m$ a phase shift occurs.

Using the same approach one may consider waves with the spectral characteristics k_p, λ_p, where $k_1,\dots,k_N$ can be chosen arbitrarily. However, in this situation the assumption $\ell_p \neq k_p$ must be supplemented with $\ell_p \neq \ell_q$; here $p = 1,\dots,N$ and $q = 1,\dots,p-1$.

5.3. **Submerged fixed and freely floating bodies.** The general method [9, Ch. 2,5,9] provides rather explicit asymptotic formulas for the velocity potential $\varphi^\varepsilon_\Rightarrow(y,z)$, when one or several small diameter bodies are immersed into the straight channel Π, cf. Figure 1.2.b). Introducing free parameters, which for example are related to the disposition of the bodies, may lead to the same formal inferences as in Section 2. However, similarly to the case of piecewise smooth bottom profiles in Figure 1.2.a) and Section 1.3, a gap appears in the justification scheme because the rectification trick of [8, §7.6] does not work any more. This is why the existence of submerged fixed or freely floating bodies, which cannot be observed by surface waves at given frequencies, remains an open problem.

5.4. **A warp localized in all directions.** The reduction of the originally three dimensional water-wave problem to the two dimensional problem (1.7)–(1.9) becomes possible only when the warp has a strictly cylindrical shape. It seems that the perturbation technique developed here allows to deal with the full problem in the domain

$$\Xi^{\varepsilon H} = \{(x,y,z) \,:\, (x,y) \in \mathbb{R}^2 \ , \ 0 > z > -d - \varepsilon H(x,y)\},$$

where H is a compactly supported smooth function and ε is a small positive parameter. Namely, fixing a finite number of frequencies and several emitters and receivers, to find the profile H such that the corresponding bottom perturbation cannot be observed by these particular surface waves and from the fixed directions.

5.5. **Higher order asymptotic terms.** The most challenging open question is, if it is possible to construct an invisible warp in the case $k = \ell$, that is, when the condition (2.21) is violated. It is quite probable that this can be done with help of higher order asymptotic terms, which however were not considered in this paper.

Acknowledgements. The second named author was partially supported by RFFI, grant 12-01-00348 and by the Academy of Finland grant no. 139545. The third named author was partially supported by the Academy of Finland project "Functional analysis and applications".

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