# LOCALIZATION EFFECT FOR DIRICHLET EIGENFUNCTIONS IN THIN NON-SMOOTH DOMAINS

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Abstract. We study the localization effect for the eigenfunctions of the Laplace-Dirichlet problem in a thin three-dimensional plate with curved non-smooth bases. We show that the eigenfunctions are localized at the thickest region, or the longest traverse axis, of the plate and that the magnitude of the eigenfunctions decays exponentially as a function of the distance to this axis. We consider some extensions like mixed boundary value problems in thin domains. The obtained asymptotic formulas for eigenfunctions prove the existence of gaps in the essential spectrum of the Dirichlet Laplacian in an unbounded double-periodic curved piecewise smooth thin layer.

## 1. Introduction

**1.1. Formulation of the problem.** Our aim is to study the asymptotic behavior of the Laplace-Dirichlet eigenvalues and eigenfunctions in thin three dimensional plates with non-smooth bases, when the thickness of the plate is proportional to a small parameter $\varepsilon \to 0^+$. We will study various aspects of the phenomenon of localization of the eigenfunctions. Our results (cf. Theorems 3.1, 5.2, 5.3) state that the eigenfunctions are *localized at the thickest region*, or the longest traverse axis, of the plate and that the magnitude of the eigenfunctions decays exponentially as a function of the distance to this axis. This behavior differs completely from the evenly distributed eigenfunctions of the Dirichlet-Laplacian, when the domain is a cylindrical plate with constant thickness. Our approach consists, among other things, of asymptotic analysis and a study of the eigenvalues and eigenfunctions of a spectral limit problem (Section 2). At the end of the paper we give generalizations and applications for example by proving the existence of a large number of spectral gaps for the Laplace-Dirichlet problem in an unbounded double-periodic thin domain (Section 6.4).

The main results are presented in detail in Sections 1.2 and 1.3, but we start by describing the geometric setting of the problem and some elementary facts on its spectrum and eigenfunctions. Let $\omega$ be a domain in the plane $\mathbb{R}^2$ bounded by a simple closed Lipschitz contour $\partial\omega$ and let $H_\pm$ be smooth profile functions in $\overline{\omega} = \omega \cup \partial\omega$ such that

$$H(y) := H_+(y) + H_-(y) > 0, \quad y = (y_1, y_2) \in \overline{\omega}. \tag{1.1}$$

We assume that the origin $y = 0$ is contained in $\omega$ and that it is the unique global strict maximum point of the function $H$, moreover,

$$\begin{aligned} & H(y) < H(0) =: h \in \mathbb{R}_+ = (0, +\infty) \quad \text{for} \quad y \in \overline{\omega} \setminus \{0\}, \\ & H_\pm(y) = l_\pm - r^2 \mathcal{H}_\pm(\varphi) + O(r^3), \\ & \big|\nabla_y \big(H_\pm(y) + r^2 \mathcal{H}_\pm(\varphi)\big)\big| \le c r^2 \quad \text{for a.e.} \quad y \in \omega. \end{aligned} \tag{1.2}$$

*Key words and phrases.* Dirichlet problem, asymptotics of eigenfunctions and eigenvalues, localization effect, spectral gaps.

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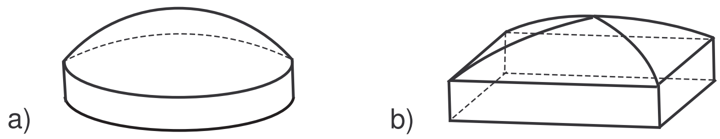


FIGURE 1.1. Thin plates with smooth and nonsmooth bases

Here $l_\pm$ are constants, $l_- + l_+ = H(0)$, $(r, \varphi) \in \mathbb{R}_+ \times \mathbb{S}^1$ are polar coordinates in the plane $\mathbb{R}^2 \ni y$, $\mathbb{R}_+ = (0, +\infty)$, and $\mathbb{S}^1$ is the unit circle,

$$\mathcal{H}_\pm \in H^{1,\infty}(\mathbb{S}^1), \quad \mathcal{H}(\varphi) = \mathcal{H}_+(\varphi) + \mathcal{H}_-(\varphi) \geq \mathcal{H}_0 > 0 \quad \text{for} \quad \varphi \in \mathbb{S}^1. \tag{1.3}$$

All these assumption are readily satisfied, if for example

$$\begin{gathered} H(y) = h - a_{11}y_1^2 - 2a_{12}y_1y_2 - a_{22}y_2^2 + O(r^3) \text{ , where} \\ a_{11} > 0, \ a_{22} > 0, \ a_{11}a_{22} > a_{12}^2. \end{gathered} \tag{1.4}$$

Given a small parameter $\varepsilon > 0$, we introduce the thin plate (see Fig. 1.1. a,b )

$$\Omega^\varepsilon = \{x = (y, z) \in \mathbb{R}^2 \times \mathbb{R} : \, y \in \omega, -\varepsilon H_-(y) < z < \varepsilon H_+(y)\} \tag{1.5}$$

with gently sloping bases

$$\Sigma_\pm^\varepsilon = \{x : \, y \in \omega, z = \pm\varepsilon H_\pm(y)\}, \tag{1.6}$$

which may in general be non-smooth. Let us consider the spectral Dirichlet problem for the Laplace operator:

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Omega^\varepsilon, \tag{1.7}$$

$$u^\varepsilon(x) = 0, \quad x \in \partial\Omega^\varepsilon. \tag{1.8}$$

In the case $H_\pm \in C^2(\overline{\omega})$ the thin plate (1.5) lays between smooth surfaces, cf. Fig. 1.1. a, but we are mostly interested in the non-smooth bases (1.6) as depicted in Fig. 1.1. b and described above in (1.2), (1.3).

The variational formulation of the problem (1.7), (1.8) reads as the integral identity (see [23])

$$(\nabla_x u^\varepsilon, \nabla_x v^\varepsilon)_{\Omega^\varepsilon} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Omega^\varepsilon} \qquad \forall \ v^\varepsilon \in H_0^1(\Omega^\varepsilon), \tag{1.9}$$

where $( \, , )_{\Omega^\varepsilon}$ stands for the natural scalar product in the Lebesgue space $L^2(\Omega^\varepsilon)$ and $H_0^1(\Omega^\varepsilon)$ is the Sobolev space of functions satisfying the Dirichlet condition (1.8). Furthermore, $\nabla_x$ is the gradient operator and $\Delta_x = \nabla_x \cdot \nabla_x$ the Laplacian in the variables $x = (x_1, x_2, x_3) = (y, z)$. The spectrum of the problem (1.7), (1.8) consists of the eigenvalue sequence

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \lambda_3^\varepsilon \leq \ldots \leq \lambda_k^\varepsilon \leq \ldots \to +\infty \tag{1.10}$$

with the standard convention on repeated multiple eigenvalues. As known, the first eigenvalue $\lambda_1^\varepsilon$ is simple and the corresponding eigenfunction $u_1^\varepsilon$ can be taken positive in $\Omega^\varepsilon$. The eigenfunctions $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, \ldots, u_k^\varepsilon, \ldots$ are subject to the orthogonality and normalization conditions

$$(u_j^\varepsilon, u_k^\varepsilon)_{\Omega^\varepsilon} = \delta_{j,k}, \quad j, k \in \mathbb{N} := \{1, 2, 3, \dots\}, \tag{1.11}$$

where $\delta_{j,k}$ is the Kronecker symbol. The eigenfunctions are infinitely differentiable inside the plate $\Omega^\varepsilon$, but in general they do not belong to the Sobolev space $H^2(\Omega^\varepsilon)$ due to possible singularities on the non-smooth bases (1.6) and the lateral side

$$\Sigma_0^\varepsilon = \{x \in \mathbb{R}^3 : y \in \partial\omega, -\varepsilon H_-(y) < z < \varepsilon H_+(y)\}. \tag{1.12}$$

**1.2. Localization effect.** We now describe the general aim of the paper concerning the localization, i.e., pointwise magnitude estimates, of the eigenfunctions $u^\varepsilon$. It turns out that the behavior of $u^\varepsilon$ is in this respect totally different from the case of a thin plate with constant thickness. Indeed, if the functions $H_\pm$ are constant, e.g., equal to 1/2, and $\varepsilon$ is just the thickness of the cylindrical plate (1.5) with the straight bases (1.6), then $\Omega^\varepsilon = \omega \times (-\varepsilon/2, \varepsilon/2)$ and it is possible to solve the spectral problem (1.7), (1.8) "almost explicitly" using separation of variables:

$$\lambda_{pq}^\varepsilon = \frac{p^2\pi^2}{\varepsilon^2} + \beta_q,\ u_{pq}^\varepsilon(x) = \sqrt{\frac{2}{\varepsilon}} \sin\left(\pi p\Big(\frac{z}{\varepsilon} + \frac{1}{2}\Big)\right) \varphi_q(y). \tag{1.13}$$

Notice that in (1.13) it is convenient to re-enumerate the eigenpairs $\{\lambda_k^\varepsilon, u_k^\varepsilon\}$ with the double index $(p,q) \in \mathbb{N}^2$. The numbers $\beta_q$ belong to the spectrum of the Dirichlet problem in the longitudinal cross-section $\omega$ of the plate $\Omega^\varepsilon$,

$$-\Delta_y\varphi(y) = \beta\varphi(y), \quad y \in \omega, \quad \varphi(y) = 0, \quad x \in \partial\omega,$$

and the eigenfunctions $\varphi_q$ are subject to the orthogonality and normalization conditions

$$(\varphi_j, \varphi_k)_\omega = \delta_{j,k}, \quad j,k \in \mathbb{N}. \tag{1.14}$$

The variational formulation of this problem consists of the integral identity

$$(\nabla_y\varphi, \nabla_y\psi)_\omega = \beta(\varphi,\psi)_\omega \qquad \forall\ \psi \in H_0^1(\omega)$$

It is plain, cf. (1.13), that the eigenfunctions of the problem (1.9) in $\Omega^\varepsilon = \omega\times(-\varepsilon/2, \varepsilon/2)$ are characterized by practically uniform distribution in the plate and do not become very small inside any subdomain of a fixed positive measure.

The main goal of the paper is to show that, for small $\varepsilon$, the eigenfunctions $u_k^\varepsilon$ behave in a different way, if the geometric condition (1.2) is assumed. Namely, we show that for some $\varepsilon_k > 0$ and all $\varepsilon \in (0, \varepsilon_k)$, the eigenfunction $u_k^\varepsilon$ is localized in a $c\sqrt{\varepsilon}$-neighborhood of the origin $x = 0$, i.e., in the vicinity of the longest interval in $\Omega^\varepsilon$ parallel to the $z$-axis. At the same time, $u_k^\varepsilon(x)$ is of the exponentially small order $O(\exp(-\varepsilon^{-1}b_k))$, $b_k > 0$, outside any fixed neighborhood of the origin. More precisely, we derive the asymptotic formula

$$u_k^\varepsilon(x) \sim \varepsilon^{-1}\alpha_k(\varepsilon)\sin\left(\pi\frac{z + \varepsilon H_-(y)}{\varepsilon H(y)}\right) w_k(\eta),$$

where $\alpha_k(\varepsilon)$ is normalization factor, $\eta$ denotes the stretched variable

$$\eta = \varepsilon^{-1/2}y. \tag{1.15}$$

Moreover, $w_k$ is an eigenfunction associated with the $k$th eigenvalue of the limit spectral differential equation

$$-\Delta_\eta w(\eta) + A(\eta)w(\eta) = \mu w(\eta), \quad \eta \in \mathbb{R}^2, \tag{1.16}$$

where

$$A(\eta) = 2\frac{\pi^2\rho^2}{h^3}\mathcal{H}(\varphi) =: \rho^2\mathcal{A}(\varphi) \tag{1.17}$$

and $(\rho,\varphi)$ are the polar coordinates in the plane $\mathbb{R}^2 \ni \eta$ with $\rho = |\eta|$ and $\rho = \varepsilon^{-1/2}r$ (see (1.15)).

The eigenvalues $\mu_k$ of the problem (1.16) appear in the asymptotic formula

$$\lambda_k^\varepsilon(x) \sim \varepsilon^{-2}\frac{\pi^2}{h^2} + \varepsilon^{-1}\mu_k. \tag{1.18}$$

Note that in the case $h = 1$ the right-hand side of (1.18) is nothing but a relatively small perturbation of the eigenvalue $\lambda_{1q}^\varepsilon$ in (1.13).

**1.3. Main results and structure of the paper.** We now describe the results and the contents of the paper in detail. Section 2 contains the analysis of the limit spectral problem (1.16). We prove there that its spectrum is discrete and forms an unbounded positive sequence $\{\mu_k\}$, while the eigenfunctions belong to $H^2_{\rm loc}(\mathbb{R}^2)$ and decay as follows:

$$|w_k(\eta)| \le c_k \exp(-b_k|y|^2), \quad b_k > 0. \tag{1.19}$$

This estimate will be derived in Proposition 2.4, and it is preceded by a corresponding integral estimate in Proposition 2.3.

Section 3 contains a proof of the following convergence result (Theorem 3.5) for the eigenvalues:

$$\varepsilon(\lambda_l^\varepsilon(x) - \pi^2\varepsilon^{-2}h^2) \to \mu_{J(l)} \quad \text{as} \quad \varepsilon \to 0^+. \tag{1.20}$$

On the left of (1.20) we have the rescaled eigenvalue of the problem (1.9) and on the right an eigenvalue of the limit equation (1.16). We emphasize that at this stage we cannot yet establish the equality $J(l) = l$ (contrary to the conclusions of Theorem 4.2). The way to verify (1.20) is quite standard: using $u_k^\varepsilon$ we construct an appropriate function $\eta \mapsto W_k^\varepsilon(\eta)$ (cf. (3.16), (3.24), (3.43)) and then pass to the limit $\varepsilon \to 0^+$ in the integral identity (1.9) with a properly chosen test function. However, due to the unboundedness of the domain, the embedding $H^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ is not compact, hence, this limit procedure needs an additional supporting argument, and we thus prove in Theorem 3.1 integral estimates for $\nabla_x u^\varepsilon$ and $u^\varepsilon$ with the exponentially large weight function (3.3). On one hand, this weighted estimate helps to overcome the difficulty of the above mentioned absence of the compact embedding, and, on the other hand, it contains our first result on the localization of eigenfunctions. However, this result will be improved in Section 5.3 by proving pointwise estimates with exponential weights.

In Section 4 we establish the convergence rate $O(\varepsilon^{1/2})$ in formula (1.20) (see Theorem 4.2). This conclusion is based on Lemma 4.1 about "near eigenvalues and eigenvectors" and the weighted estimates in Section 3. In parallel we will show that, for some $C_p > 0$, the interval

$$\Big(\pi^2\varepsilon^{-2}h^{-2} + \varepsilon^{-1}\mu_p - C_p\varepsilon^{-1/2}, \pi^2\varepsilon^{-2}h^{-2} + \varepsilon^{-1}\mu_p + C_p\varepsilon^{-1/2}\Big)$$

includes at least one entry of the eigenvalue sequence (1.10). The statement of Theorem 4.2 includes the fact that $J(l) = l$ in (1.20), but as regards the proof, the case of possible multiple eigenvalues $\lambda_l^\varepsilon$ and $\mu_p$ is only treated in Section 5.1 (see Remark 5.1).

In Theorem 5.3 we state one of the main results of the paper, the localization estimate for the eigenfunctions $u_k^\varepsilon$, which implies a pointwise bound for the function

$$\exp(\varepsilon^{-1}b|y|^2)|u_k^\varepsilon(x)| \le C\varepsilon^{-3/2} \tag{1.21}$$

for some constant $b > 0$. The estimate (1.21) is valid, roughly, in the interior of $\Omega^\varepsilon$ and means that the eigenfunctions decay exponentially, when the distance (in $y$-variable) to the maximum thickness point $0 \in \mathbb{R}^2$ of the plate increases.

In Section 6 we discuss various related results, in particular we comment on stable asymptotic forms of eigenvalues in the high-frequency range of the spectrum. This observation allows to discover similar localization effects in the spectral Neumann problem under the symmetry assumption $H_- = H_+$.

Moreover, with the help of the obtained asymptotic formulas for eigenvalues we detect gaps in the spectrum of the Dirichlet Laplacian in the double-periodic layer

$$\Pi^\varepsilon = \big\{(y,z) : \, y \in \mathbb{R}^2, -\varepsilon H_-(y) < z < \varepsilon H_+(y)\big\},$$

of variable thickness, where the profile functions $H_\pm$ are $l_i$-periodic in $y_i$, $l_i > 0$, $i = 1,2$, and their restrictions on the rectangle $\overline{\omega} = \{y = (y_1,y_2) : \, y_i \in [0,l_i], i = 1,2\}$ possess all properties mentioned above.

**1.4. Literature review.** Using an analysis of the resolvent, localization of the eigenfunctions of the Dirichlet Laplacian was proved in [16] (see also [5]) in the case of a thin curved two-dimensional trapezoid

$$\Omega^\varepsilon = \{(y,z) \in \mathbb{R}^2 : |y| < 1, 0 < z < \varepsilon H(y)\}, \tag{1.22}$$

where the smooth profile function $H$ has just one strict global maximum at $y = 0$. Although we deal here with the three-dimensional thin domain (1.6), these phenomena are of a similar nature.

Other types of localized eigenfunctions are found in [7]. Namely, assume that $\mathcal{Q}$ is a semi-infinite cylinder

$$\mathcal{Q} = \{\xi = (\eta,\zeta) \in \mathbb{R}^{n-1} \times \mathbb{R} : \eta \in \omega, \zeta > \mathcal{F}(\eta)\}\ , \quad n \geq 2,$$

where the cross-section $\omega$ is a domain in $\mathbb{R}^{n-1}$ with a smooth boundary, and $\mathcal{F} \in C^2(\overline{\omega})$; also denote $\mathcal{T} = \{\xi : \eta \in \omega, \zeta = \mathcal{F}(\eta)\}$. Posing Dirichlet conditions on the lateral side and Neumann conditions on the end part leads to the problem

$$\begin{aligned} -\Delta_\xi \mathcal{W}(\xi) &= \Lambda \mathcal{W}(\xi),\ \xi \in \mathcal{Q}, \\ \partial_\nu \mathcal{W}(\xi) &= 0,\ \xi \in \mathcal{T}, \\ \mathcal{W}(\xi) &= 0,\ \xi \in \partial\mathcal{Q} \setminus \overline{\mathcal{T}}. \end{aligned} \tag{1.23}$$

Here, $\partial_\nu$ stands for the normal derivative in $\mathcal{T}$. The results of [7] are based on the approach of [11], and they contain the description of the boundary layer phenomenon of the problem (1.23). Also, [7] provides a simple condition for the function $\mathcal{F}$, which ensures that the operator of the problem (1.23) has discrete spectrum situated below the continuous spectrum $[\beta^\dagger, +\infty)$; here $\beta_1 = \beta^\dagger$ is the first eigenvalue of the Dirichlet Laplacian in $\omega$, cf. (1.14). For example in the case $\Delta_\eta \mathcal{F}(\eta) > 0$, $\eta \in \omega$ there appears a so called trapped mode (cf. [25]), which is an eigenvalue $\Lambda_1 \in (0, \Lambda^\dagger)$ of (1.23) (here $\Lambda^\dagger > 0$ comes from $\beta^\dagger$) with the corresponding eigenfunction $\mathcal{W}_1 \in H^1(\mathcal{Q})$. However, the primary object of investigation in [7] is the Laplace equation with mixed boundary conditions in a thin, bounded, straight cylinder $\mathcal{Q}^\varepsilon$ (denoted by $\Omega^h$ in the reference) with two distorted ends. As shown in [7], each of the eigenvalues $\Lambda_k \in (0, \Lambda^\dagger)$ gives rise to an eigenvalue

$$\lambda_k^\varepsilon \sim \varepsilon^{-2}\Lambda_k + O(1)$$

of the spectral problem in the thin cylinder $\mathcal{Q}^\varepsilon$, and the corresponding eigenfunction $u_k^\varepsilon$ concentrates in the vicinity of the ends of $\mathcal{Q}^\varepsilon$. Each of the ends of the thin cylinder generates a problem of type (1.23) and, if at least one of them has non-empty discrete spectrum, the first eigenfunction $u_1^\varepsilon$ is localized, possibly at both ends simultaneously.

Both types of localization effects were investigated for the first time in [10] (see also [33]). The object was a spectral mixed boundary value problem in a thin cylindrical plate in $\mathbb{R}^3$ with a distorted lateral side $\Gamma^\varepsilon$, and it was shown that under a simple geometric condition the first eigenfunction is concentrated in the vicinity of either the whole lateral side, or a single point $x^0 \in \Gamma^\varepsilon$. Moreover, at some distance from $x^0$, the eigenfunction becomes of order $\exp(-\delta_\Gamma \varepsilon^{-1/2})$ near $\Gamma^\varepsilon$ and of order $\exp(-\delta_\Omega \varepsilon^{-1})$ inside the plate; here $\delta_\Gamma, \delta_\Omega > 0$. Other localization effects are discussed in [10] as well.

Results of [16] for the two-dimensional trapezoid (1.22) have been generalized in [6] to the case $\mathbb{R}^d$, $d \geq 3$, for thin domains such that the profile functions $H_\pm$ are $C^\infty$-smooth and

$$H(y) = h - P_{2m}(y) + O(|y|^{2m+1}), \tag{1.24}$$

where $P_{2m}$ is a homogeneous positive polynomial of degree $2m \geq 2$ in the variable $y \in \mathbb{R}^{d-1} \setminus \{0\}$. A rather elaborate and complete formal asymptotic analysis of eigenvalues and eigenfunctions of the problem (1.7), (1.8) is performed in [6], but the justification of the asymptotic formulas for the eigenvalues and eigenfunctions remains incomplete: in [6]

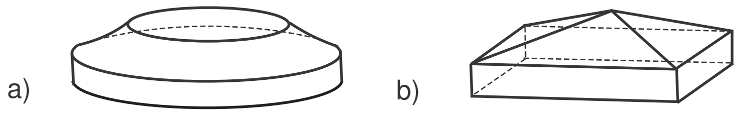


FIGURE 1.2. Other types of non-smooth thin plates

the authors refer to the papers [16] and [5], where the limit problem in the case $m = 1$ is the ordinary differential equation of the harmonic oscillator,

$$-\frac{d^2 w}{d\eta^2}(\eta) + A\eta^2 w(\eta) = \mu w(\eta), \quad \eta \in \mathbb{R},\ A > 0 \text{ constant.} \tag{1.25}$$

The eigenvalues (and the corresponding eigenfunctions) are known exactly, and they are simple, cf. [24]. Thus, the convergence theorem in [16] is useful to justify the asymptotics of *simple* eigenvalues, but in the higher dimensional case $d \geq 3$ the limit problem becomes (1.16) with many multiple eigenvalues. This case is not treated in [16], and it requires additional arguments and computations. Known results on the eigenfunctions of the spectral problem (1.16) in $\mathbb{R}^{d-1}$, are contained in [3, Sect. 3]. These include a proof for the exponential decay under strong enough smoothness hypothesis of the potential. Obviously, our results in Propositions 2.2 and 2.4 and Remark 2.5 are more general.

In our paper the main attention is paid to the justification scheme. We modify the approach from the papers [10, 7] and also [36, 35], where similar localization effects were found for other types of thin domains (see Fig. 1.2, a and b). The present approach is based on weighted a priori estimates, which immediately reveal the localization effect (in contrast to [16, 15, 5, 6]) and require only mild assumptions on the smoothness properties of the profile functions $H_\pm$. Furthermore, the method provides pointwise estimates in the case of smooth data.

The approach also works for other than quadratic decay rates in (1.2), cf. also (6.28). However, the technical details would be quite different for the profile function with $H(y) - H(0) = O(|y|^\kappa)$, $\kappa \in (0, 1)$ or $\kappa > 1$. (The case $\kappa = 1$ has already been considered in [35].) We choose to treat here only the exponent $\kappa = 2$, since this case is still general enough and it on the other hand avoids inessential technical complications.

A similar specific behavior of eigenfunctions was found and studied in [1, 8] for some problems in homogenization theory and in [21] in domains with thin bands.

The papers [16] and [5] contain studies on a planar domain $\Omega_\varepsilon$ with the Dirichlet condition on the entire boundary. As for the corresponding Neumann problem on $\Omega_\varepsilon$, we mention that localized eigenfunctions associated with the middle frequencies can be obtained by computations, which are simplified from those in [21]. In the same way, by methods of [21] one can show that the low frequencies are of the order $O(1)$ and they give rise to longitudinal vibrations. The limit problem is a Neumann problem in dimension one and it contains coefficients with information on the shape of the domain $\Omega_\varepsilon$. For the eigenvalues of order $O(\varepsilon^{-2})$ it is possible to construct the so-called *quasimodes* or *almost eigenfunctions*, which are approximations of certain linear combinations of eigenfunctions associated with eigenvalues in small intervals. The length of these intervals, and estimates for the difference between quasimodes and eigenfunctions, provide useful information for describing the behavior of standing waves, which are solutions of the corresponding time-dependent problems; their long-time asymptotic limits can be constructed explicitly from the quasimodes. From the localization point of view, the supports of the standing waves are asymptotically concentrated near the point of global maxima, and they obey the same spatial decay as described in this paper. We refer to [39] and [40] for the time dependent problems.

The paper [21] contains a study of the asymptotics of eigenfunctions of reinforcement problems. The problem concerns a second order differential operators with piecewise constants coefficients in a domain $\Omega_\varepsilon = \Omega \cup \partial\Omega \cup \omega_\varepsilon \subset \mathbb{R}^2$. The subdomain $\omega_\varepsilon$ is a thin heavy stiff band of a variable width $O(\varepsilon)$ surrounding the fixed domain $\Omega$. The paper highlights the localization phenomena for the eigenfunctions associated with the middle frequencies, and we refer to [21] for precise rates of convergence, and for the asymptotic behavior of the whole spectrum and the associated eigenfunctions. We mention that several different scales are involved in the problem: the size $O(1)$ of $\Omega$, the width $O(\varepsilon)$ of $\omega_\varepsilon$, the density and stiffness $O(1)$ of $\Omega$, the density $O(\varepsilon^{-1})$ and stiffness $O(\varepsilon^{-m})$ (with $m > 2$) of $\omega_\varepsilon$, and finally the intermediate scale $O(\sqrt{\varepsilon})$ which allows to describe the localized eigenfunctions in neighborhoods of points where the "height function" defining $\partial\Omega_\varepsilon$ has the local maxima (see comments above (1.15)). As the references [21], [20], [19] and [22] show, the relation between density and stiffness in reinforcement problems and asymptotics for eigenpairs may vary very much depending on the situation.

Heterogeneities of masses, in particular the so-called *concentrated masses* may cause other kinds of localization phenomena for the supports of eigenfunctions. If the density is of a very big order $O(\varepsilon^{-m})$, $m > 2$, in a small region of diameter $O(\varepsilon)$, the eigenfunctions associated with the low frequency eigenvalues (order $O(\varepsilon^{m-2})$) give rise to vibrations with supports localized at the points of the concentrated masses (cf. [39]). Also the Dirichlet condition plays an important role in these localization phenomena: we refer to [28] for a general bibliography on this subject. We emphasize that the decay rate of eigenfunctions in the case of concentrated masses is polynomial (or logarithmic, depending on the dimension of the space) in all directions. This is different from the the exponential decay in the direction perpendicular to the boundary, which is the behavior described in this paper and in the other above mentioned results for thin domains.

## 2. Formal asymptotics and spectrum of the limit problem

**2.1. Preliminary asymptotic analysis.** We introduce the standard asymptotic ansätze

$$\lambda^\varepsilon = \varepsilon^{-2}\frac{\pi^2}{h^2} + \varepsilon^{-1}\mu + \ldots, \tag{2.1}$$

$$u^\varepsilon(x) = \sin\left(\pi\,\frac{z + \varepsilon H_-(y)}{\varepsilon H(y)}\right) w(\eta) + \ldots, \tag{2.2}$$

where the number $\mu$ and the function $w$ are to be determined and $\eta$ is the rapid variable (1.15). We insert the ansätze into the equation (1.7), perform formal differentiation (recall that $H_\pm$ belong to $H^{1,\infty}_{loc}(\mathbb{R}^2)$ only) and obtain

$$\begin{aligned}
&\Delta_x u^\varepsilon(x) + \lambda^\varepsilon u^\varepsilon(x)\\
=\ &\sin\left(\pi\,\frac{z + \varepsilon H_-(y)}{\varepsilon H(y)}\right)\Big(\varepsilon^{-1}\Delta_\eta w(\eta) - \frac{\pi^2}{\varepsilon^2 H(y)^2} w(\eta) + \ldots\\
+\ &\frac{\pi^2}{\varepsilon^2 h^2} w(\eta) + \varepsilon^{-1}\mu w(\eta) + \ldots\Big)\\
=\ &\frac{1}{\varepsilon}\sin\left(\pi\,\frac{z + \varepsilon H_-(y)}{\varepsilon H(y)}\right)\Big(\Delta_\eta w(\eta) + \mu w(\eta)\\
+\ &\frac{\pi^2}{\varepsilon h^2}\Big(1 - \frac{1}{(1 - \varepsilon h^{-1}|\eta|^2\mathcal{H}(\varphi) + \ldots)^2}\Big) w(\eta) + \ldots\Big) + \ldots
\end{aligned}$$

$$= \frac{1}{\varepsilon}\sin\left(\pi\,\frac{z+\varepsilon H_-(y)}{\varepsilon H(y)}\right)\{\Delta_\eta w(\eta)+\mu w(\eta)-A(\eta)w(\eta)\}+\ldots, \tag{2.3}$$

where the coefficient $A(\eta)$ is given in (1.17) and dots stand for higher-order terms, which are inessential for the following analysis; we have used that $z+\varepsilon H_-(y)<\varepsilon H(y)$. Observing that the Dirichlet conditions (1.8) on the plate bases (1.6) are fulfilled because of the factor sin(...) in (2.2), we see that the original problem (1.7), (1.8) is satisfied asymptotically, if and only if the expression in the curly brackets vanishes. In this way we come across the *limit differential* equation (1.16) for the first time in this paper. We emphasize, that since the solutions (eigenfunctions) of this equation decay exponentially (see Proposition 2.3 below), the right-hand side of (2.2) satisfies approximately the Dirichlet condition on the lateral side $\Sigma_0^\varepsilon$ of $\Omega^\varepsilon$, too.

**2.2. Studying the limit equation.** If $A(\eta)$ is a quadratic polynomial in $\eta$, cf. (1.4), the solutions of the equation (1.16) can be found almost explicitly (see [16, 5] and [6] as well as references therein). However, a general $A(\eta)$, (1.17), requires a more careful analysis of the spectrum of the limit equation.

Let $\mathbf{H}$ denote the Hilbert space obtained as the completion of the linear set $C_c^\infty(\mathbb{R}^2)$ (infinitely differentiable functions with compact supports) with respect to the weighted norm

$$\|w;\mathbf{H}\| = \left(\|\nabla_\eta w; L^2(\mathbb{R}^2)\|^2 + \|(1+\rho)w; L^2(\mathbb{R}^2)\|^2\right)^{1/2}. \tag{2.4}$$

Then, the variational formulation of the problem (1.16) amounts to finding a number $\mu$ and a nontrivial function $w\in\mathbf{H}$ such that

$$(\nabla_\eta w,\nabla_\eta v)_{\mathbb{R}^2} + (Aw,v)_{\mathbb{R}^2} = \mu(w,v)_{\mathbb{R}^2} \qquad \forall \quad v\in\mathbf{H}. \tag{2.5}$$

**Proposition 2.1.** *The spectrum of the problem* (2.5) *is discrete and forms the eigenvalue sequence*

$$0<\mu_1<\mu_2\le\mu_3\le\ldots\le\mu_k\le\ldots\le\ldots\to+\infty, \tag{2.6}$$

*The corresponding eigenfunctions $w_1,w_2,w_3,\ldots,w_k,\ldots$ in $\mathbf{H}$ can be subject to the normalization and orthogonality conditions*

$$(w_j,w_k)_{\mathbb{R}^2}=\delta_{j,k}, \quad j,k\in\mathbb{N}. \tag{2.7}$$

*The first eigenvalue $\mu_1$ is simple and the eigenfunction $w_1$ can be chosen positive.*

**Proof.** We use the following variant of Poincare's inequality,

$$\|v;L^2(\mathbb{B}_R)\|^2\le c_R(\|\nabla_\eta v;L^2(\mathbb{B}_{2R})\|^2+\|v;L^2(\mathbb{B}_{2R}\setminus\mathbb{B}_R)\|^2)\le C_R\|v;\mathbf{H}\|^2,$$

where $\mathbb{B}_R=\{\eta:\ \rho=|\eta|<R\}$ is a disk of radius $R>0$. This and (1.17) imply that the left-hand side of (2.5) is a scalar product in $\mathbf{H}$, denoted by $\langle w,v\rangle$ in the sequel. Let $\mathcal{K}:\mathbf{H}\to\mathbf{H}$ be the operator defined by the equation

$$\langle\mathcal{K}w,v\rangle=(w,v)_{\mathbb{R}^2} \qquad \forall \quad w,v\in\mathbf{H}. \tag{2.8}$$

Clearly, $\mathcal{K}$ is positive, symmetric, and continuous, therefore, self-adjoint. Moreover, it is compact since the embedding $\mathbf{H}\subset L^2(\mathbb{R}^2)$ is compact due to the following observation: the embedding operator is the sum of a small operator with norm of magnitude $O(R^{-1})$ (outside the disk $\mathbb{B}_R$) and a compact operator (inside the disk $\mathbb{B}_R$). Thus, by [4, Thm. 10.1.5, 10.2.2], the discrete spectrum of $\mathcal{K}$ consists of the positive, monotone decreasing sequence

$$\kappa_1\ge\kappa_2\ge\kappa_3\ge\ldots\ge\kappa_k\ge\ldots\to0^+ \tag{2.9}$$

together with the point $\kappa=0$, which is the only element of the essential spectrum. In view of (2.8), the abstract equation

$$\mathcal{K}w=\kappa w \quad \text{in} \quad \mathbf{H}$$

with the new spectral parameter $\kappa = \mu^{-1}$ is equivalent to the variational problem (2.5), and hence the eigenvalue sequence (2.9) turns into (2.6) by inversion. The normalization and orthogonality conditions (2.7) are standard and straightforward to prove. The strong maximum principle yields the simpleness of $\mu_1$ and the positivity of $w_1$. ⊠

In the following $B$ denotes any function such that

$$B(\eta) = \rho^2 \mathcal{B}(\varphi), \tag{2.10}$$

for some function $\mathcal{B} \in H^{1,\infty}(\mathbb{S}^1)$ having the properties

$$\mathcal{B}(\varphi) \geq \mathcal{B}_0 \quad \text{and} \quad |\nabla_\eta \mathcal{B}(\eta)|^2 \leq (1-\beta) A(\eta) \quad \forall\, \eta \in \mathbb{R}^2 \tag{2.11}$$

with constants $\mathcal{B}_0 > 0$, $0 < \beta < 1$. We also define the weight function

$$e_B(\eta) = \exp(B(\eta)). \tag{2.12}$$

**Remark 2.2.** One can take for example

$$B(\eta) = b_1 \eta_1^2 + b_2 \eta_2^2 \tag{2.13}$$

so that $|\nabla_\eta B(\eta)|^2 = 4\eta_1^2 b_1^2 + 4\eta_2^2 b_2^2$. We have, by (1.2), $A(\eta) \geq a_1 \eta_1^2 + a_2 \eta_2^2$ for some $a_m > 0$, hence, to satisfy the second relation in (2.11) it suffices to verify $4b_m^2 \in (0, a_m(1-\beta)]$, $m = 1, 2$. If (1.4) holds, we may take any $b_m$ with $b_m < \frac{1}{2}\sqrt{a_m}$ and then a suitable $\beta > 0$.

The next assertion proves the exponential decay of eigenfunctions of the problem (2.5).

**Proposition 2.3.** *Let $j \in \mathbb{N}$ and $\{\mu_j, w_j\} \in \mathbb{R}_+ \times \mathbf{H}$ be an eigenpair of the problem (2.5), as in Proposition 2.1. The inclusion $e_B w_j \in \mathbf{H}$ holds true for all $B$ satisfying (2.10)–(2.11), and there exists a constant $C_{jB}$ such that*

$$\|e_B \nabla_\eta w_j; L^2(\mathbb{R}^2)\| + \|e_B (1+\rho) w_j; L^2(\mathbb{R}^2)\| \leq C_{jB}. \tag{2.14}$$

**Proof.** Let $\mathbb{E}_R = \mathbb{E}_R(B) := \{\eta : B(\eta) < R^2\}$ be domains exhausting the plane $\mathbb{R}^2$ when $R \to +\infty$. We introduce the weight function

$$\mathcal{R}(\eta) = \begin{cases} e_B(\eta), & \eta \in \mathbb{E}_R, \\ \exp(R^2), & \eta \in \mathbb{R}^2 \setminus \mathbb{E}_R, \end{cases} \tag{2.15}$$

which is positive, belongs to $H^{1,\infty}(\mathbb{R}^2)$ and satisfies

$$|\mathcal{R}(\eta)^{-1} \nabla_\eta \mathcal{R}(\eta)|^2 \leq \begin{cases} (1-\beta) A(\eta), & \eta \in \mathbb{E}_R, \\ 0, & \eta \in \mathbb{R}^2 \setminus \mathbb{E}_R, \end{cases}$$

see (2.10). Since (2.15) equals a constant near infinity, we can consider the integral identity (2.5) with the test function $v = \mathcal{R} W_j = \mathcal{R}^2 w_j \in \mathbf{H}$ and the eigenpair $\{\mu_j, w_j\}$. A simple calculation shows that

$$\begin{aligned}
& \mu_j \|W_j; L^2(\mathbb{R}^2)\|^2 \\
= \; & (\nabla_\eta w_j, \mathcal{R} \nabla_\eta W_j)_{\mathbb{R}^2} + (\nabla_\eta w_j, W_j \nabla_\eta \mathcal{R})_{\mathbb{R}^2} + (A W_j, W_j)_{\mathbb{R}^2} \\
= \; & \|\nabla_\eta W_j; L^2(\mathbb{R}^2)\|^2 - (w_j \nabla_\eta \mathcal{R}, \nabla_\eta W_j)_{\mathbb{R}^2} + (\nabla_\eta W_j, w_j \nabla_\eta \mathcal{R})_{\mathbb{R}^2} \\
- \; & \|W_j \mathcal{R}^{-1} \nabla_\eta \mathcal{R}; L^2(\mathbb{R}^2)\|^2 + (A W_j, W_j)_{\mathbb{R}^2} \\
= \; & \|\nabla_\eta W_j; L^2(\mathbb{R}^2)\|^2 + (A W_j, W_j)_{\mathbb{R}^2} - \|W_j \mathcal{R}^{-1} \nabla_\eta \mathcal{R}; L^2(\mathbb{R}^2)\|^2 \\
\geq \; & \|\nabla_\eta W_j; L^2(\mathbb{R}^2)\|^2 + \beta (A W_j, W_j)_{\mathbb{R}^2}.
\end{aligned} \tag{2.16}$$

Furthermore, there exists $r_j(\beta) > 0$ such that

$$\mu_j < \frac{\beta}{2} A(\eta) \quad \text{for} \quad \eta \in \mathbb{R}^2 \setminus \mathbb{E}_{r_j(\beta)}.$$

We clearly also have the estimate

$$\mu_j \|W_j; L^2(\mathbb{E}_{r_j(\beta)})\|^2 \leq \mu_j \exp(2 r_j(\beta)) \|w_j; L^2(\mathbb{E}_{r_j(\beta)})\|^2 \leq$$

$$\leq \mu_j \exp(2r_j(\beta))\|w_j; L^2(\mathbb{R}^2)\|^2 = \mu_j \exp(2r_j(\beta)).$$

Hence, the following bound holds uniformly for $R \geq 1$:

$$\begin{aligned}
&\|\nabla_\eta W_j; L^2(\mathbb{R}^2)\|^2 + \frac{\beta}{2}(AW_j, W_j)_{\mathbb{R}^2} \\
&\leq \|\nabla_\eta W_j; L^2(\mathbb{R}^2)\|^2 + \beta(AW_j, W_j)_{\mathbb{R}^2} - \mu_j\|W_j; L^2(\mathbb{E}_{r_j(\beta)})\|^2 \\
&\leq \mu_j \exp(2r_j(\beta)). \qquad (2.17)
\end{aligned}$$

Since the weight function (2.15) is monotone, passing to the limit $R \to +\infty$ proves that

$$\|\nabla_\eta(e_B w_j); L^2(\mathbb{R}^2)\|^2 + \frac{\beta}{2}\|A^{1/2}e_B w_j; L^2(\mathbb{R}^2)\|^2 \leq \mu_j \exp(2r_j(\beta)).$$

So, (1.17), (1.3), and (2.12) yield the inequality (2.14). ⊠

**2.4. Pointwise estimates of the decay rate.** In the following our aim is to describe the behavior of $w_k$ at infinity by estimating its weighted norms with the weight $e_B$ given by (2.12) and (2.13); see (1.19). More precisely, we derive weighted Sobolev and Hölder (pointwise) estimates for the functions $e_B w_k$ – Sobolev estimates will be provided in the case of a general function $\mathcal{H}$, see (1.3), and Hölder estimates in the case of a smooth $H$ in (1.2). The definitions of these norms are standard,

$$\begin{aligned}
\|v; H^l(\mathbb{R}^2)\| &= \left(\sum_{j=0}^{l} \|\nabla_\eta^j v; L^2(\mathbb{R}^2)\|^2\right)^{1/2}, \\
\|v; C^{l,\alpha}(\mathbb{R}^2)\| &= \sum_{j=0}^{l} \sup_{\eta\in\mathbb{R}^2} |\nabla_\eta^j v(\eta)| \\
&\quad + \sup_{\eta\in\mathbb{R}^2} \sup_{\zeta\in\mathbb{R}^2:\,|\eta-\zeta|\leq 1} \left(|\eta-\zeta|^{-\alpha}|\nabla_\eta^j v(\eta) - \nabla_\eta^j v(\zeta)|\right), \qquad (2.18)
\end{aligned}$$

where $l \in \{0, 1, 2, \dots\}$, $\alpha \in (0,1)$ and $\nabla_\eta^j v$ is the collection of all order $j$ derivatives of the function $v$.

**Proposition 2.4.** *Let the weight $e_B$ be as in* (2.12) *with $B$ given by* (2.13)*.*
*1°. If $\mathcal{H} \in H^{2,\infty}(\mathbb{S}^1)$, then $e_B w_k \in H^4(\mathbb{R}^2)$.*
*2°. If $\mathcal{H} \in C^{1,\alpha}(\mathbb{S}^1)$, $\alpha \in (0,1)$, then $e_B w_k \in C^{3,\alpha}(\mathbb{R}^2)$.*

**Proof.** The functions $w_k$ may lack smoothness in any disk $\mathbb{B}_R^2 = \{\eta : |\eta| < R\}$, a fact which can be caused for example by the singularities $O(|\eta|^{-1})$ of the third-order derivatives of $A$ at $\eta = 0$. Indeed, according to classical results in [12] and [30], see also [38, Ch.3], the eigenfunctions have the representation

$$w_k(\eta) = p_k(\eta) + \rho^2\psi_k(\varphi, \ln\rho) + \widetilde{w}_k(\eta), \quad \eta \in \mathbb{B}_R^2, \qquad (2.19)$$

where $p_k$ is a polynomial of degree 3, $\psi_k$ is a linear function in $\ln\rho$ with coefficients in $H^{4,\infty}(\mathbb{S}^1)$ or $C^{3,\alpha}(\mathbb{S}^1)$, and the fast decaying remainder $\widetilde{w}_k(\eta)$ is of order $O(\rho^{2-\delta})$ for $\rho \to 0$. It is easy to see that the second term on the right in (2.19) does not belong to $H^5(\mathbb{S}^1)$ or $C^{4,\alpha}(\mathbb{S}^1)$, $\alpha \in (0,1)$. In what follows we verify the desired inclusions 1° and 2° outside the above-mentioned disk and consider the functions $\mathcal{H} \in H^{2,\infty}(\mathbb{S}^1)$ and $\mathcal{H} \in C^{1,\alpha}(\mathbb{S}^1)$.

Let us define the squares

$$\begin{aligned}
Q_{pq}^m &= \{\eta : |\eta_1 - p| < (m+1)/2, |\eta_2 - q| < (m+1)/2\}, \\
&\quad m = 0, 1,\ p, q \in \mathbb{Z} := \{0, \pm1, \pm2, \dots\}. \qquad (2.20)
\end{aligned}$$

To treat the case 1°, we write the equation (1.16) in the form

$$-\Delta_\eta w_k(\eta) = f_k(\eta) := \mu_k w_k(\eta) - A(\eta)w_k(\eta), \quad \eta \in Q_{pq}^1, \qquad (2.21)$$

and recall the standard local elliptic estimates for solutions of elliptic equations (cf. [2]),

$$\|w_k; H^{s+2}(Q^0_{pq})\|^2 \le c(\|f_k; H^s(Q^1_{pq})\|^2 + \|w_k; L^2(Q^1_{pq})\|^2), \tag{2.22}$$

which holds true for $s = 0, 1, 2$ due to the assumption of $1^\circ$ on $A$ and $\mathcal{H}$. Also note that $\overline{Q^0_{pq}} \subset Q^1_{pq}$ and that the squares $\overline{Q^0_{pq}}$ fill the plane. The same constant $c$ can be chosen in (2.22) for all $s = 0, 1, 2$ and all $p, q \in \mathbb{Z}$, because the measure of $Q^m_{pq}$ does not depend on $p$ and $q$.

Since $|A(\eta)| \le c(p^2 + q^2)$ in $Q^s_{pq}$, we get directly from (2.21)

$$\|f_k; L^2(Q^1_{pq})\|^2 \le c_k(1 + (p^2 + q^2)^2)\|w_k; L^2(Q^1_{pq})\|^2. \tag{2.23}$$

By (2.13), the weight function (2.12) satisfies the relations

$$\begin{aligned} c_B^- \exp(-b_B^-(p^2 + q^2)^{-1/2})e_B(p, q) &\le e_B(\eta) \\ &\le c_B^+ \exp(b_B^+(p^2 + q^2)^{-1/2})e_B(p, q), \quad \eta \in Q^1_{pq}, \end{aligned} \tag{2.24}$$

with some positive constants $b^\pm_B$ and $c^\pm_B$ independent of $p$ and $q$. We now take $s = 0$ in (2.22), multiply both (2.22) and (2.23) with $e_B(p, q)$ and use (2.24) to bring weights inside the norms, and estimate the first term on the right hand side (2.22) by (2.23). This yields

$$\begin{aligned} &\sum_{j=0}^{2} \|(1 + \rho^2)^{-1/2} \exp\Big(-\frac{1}{2}(b_B^+ + b_B^-)\rho\Big) e_B \nabla^j_\eta w_k; L^2(Q^0_{pq})\|^2 \\ &\le c_k\|(1 + \rho)e_B w_k; L^2(Q^1_{pq})\|^2. \end{aligned} \tag{2.25}$$

The weight on the left satisfies for any $\delta > 0$ the bound

$$e_B(\eta)(1 + \rho^2)^{-1/2} \exp\Big(-\frac{1}{2}(b_B^+ + b_B^-)\rho\Big) \ge C_\delta e_{B-\delta}(\eta)$$

where $e_{B-\delta}$ is as in (2.12)–(2.13) with $b_m$ replaced by $b_m - \delta$, $m = 1, 2$.

We now sum up the inequalities (2.25) with respect to $p, q \in \mathbb{Z}$. Since the square $Q^1_{pq}$ intersects only 8 of the neighboring squares, we obtain

$$\sum_{j=0}^{2} \|e_{B-\delta}\nabla^j_\eta w_k; L^2(\mathbb{R}^2)\|^2 \le 9c_k C_\delta^{-1}\|(1 + \rho)e_B w_k; L^2(\mathbb{R}^2)\|^2. \tag{2.26}$$

Note that the right-hand side of (2.26) is finite due to Proposition 2.3. Furthermore, to estimate the Sobolev-norm of $e_{B-\delta}w_k$ one has to commute $e_{B-\delta}$ and $\nabla^j_\eta$ on the left-hand side of (2.26). This produces additional powers of $|\eta| = \rho$, but these can be compensated by replacing the weight $e_{B-\delta}$ by $e_{B-2\delta}$. As a result we find that

$$\|e_{B-2\delta}w_k; H^2(\mathbb{R}^2)\|^2 \le C_k\|e_B(1 + \rho)w_k; L^2(\mathbb{R}^2)\|^2. \tag{2.27}$$

We repeat this argument, replacing (2.23) by

$$\|f_k; H^2(Q^1_{pq})\|^2 \le c_k(1 + (p^2 + q^2)^2)\|w_k; H^2(Q^1_{pq})\|^2 \tag{2.28}$$

(which also follows from (2.21)) and taking $s = 2$ in (2.22), and thus obtain

$$\|e_{B-4\delta}w_k; H^4(\mathbb{R}^2)\|^2 \le C_k\|e_B(1 + \rho)w_k; L^2(\mathbb{R}^2)\|^2. \tag{2.29}$$

The proof of the statement $1^\circ$ is completed by Proposition 2.3 and the remark that $\delta > 0$ and $B$ in Remark 2.2 are arbitrary and, thus, we could have considered from the very beginning the function $e_{B+2\delta}$ with a small $\delta > 0$ instead of $e_B$. This yields the above estimates for $e_B w_k$ in place of $e_{B-2\delta}w_k$.

In the case $2^\circ$ we first observe that $\mathcal{H} \in H^{1,\infty}(\mathbb{S}^1)$ and, using the same argument as above, derive the estimate

$$\|e_{B-4\delta}w_k; H^3(\mathbb{R}^2)\|^2 \le C_k\|e_B(1 + \rho)w_k; L^2(\mathbb{R}^2)\|^2.$$

By the Sobolev embedding $H^3(\mathbb{R}^2) \subset C^{1,\alpha}(\mathbb{R}^2)$, we then have the inclusion $e_{B-4\delta} w_k \in C^{1,\alpha}(\mathbb{R}^2)$. The proof is completed by employing the argument of the case 1° once more, but instead of (2.23) and (2.22) we now use

$$\|f_k; C^{1,\alpha}(Q^1_{pq})\| \leq c_k(1 + p^2 + q^2)\|w_k; C^{1,\alpha}(Q^1_{pq})\|$$

and the local estimate (cf. [2])

$$\|w_k; C^{3,\alpha}(Q^0_{pq})\| \leq c(\|f_k; C^{1,\alpha}(Q^1_{pq})\| + \|w_k; L^2(Q^1_{pq})\|),$$

where the right hand side does not exceed $C_k(1 + p^2 + q^2)\|w_k; C^{1,\alpha}(Q^1_{pq})\|$. ⊠

We again emphasize that, of course, increasing the smoothness of the coefficient $A$ improves the smoothness of the eigenfunctions $w_k$, but increasing the smoothness of the angular part $\mathcal{A}$ in (1.17) is not in general helpful in this respect: the derivative $\nabla^3_\eta w_k$ may remain logarithmically singular at $\eta = (0, 0)$. However, $w_k \in C^\infty(\mathbb{R}^2)$ holds true for a polynomial $A(\eta)$, cf. (1.4).

**Remark 2.5.** Assuming only that $\mathcal{H}$ satisfies (1.3), the above presented argument proves the inclusion $e_B w_k \in H^3(\mathbb{R}^2)$. Since $H^{1,\infty}(\mathbb{S}^1) \subset C^{0,\alpha}(\mathbb{S}^1)$ for all $\alpha \in (0, 1)$, we get in this case $e_B w_k \in C^{2,\alpha}(\mathbb{R}^2)$.

## 3. Convergence theorem

**3.1. Weighted estimates for eigenfunctions.** Our purpose is to prove in this section the convergence result (1.20) for the eigenvalues of the problem (1.9). To this end we need the following theorem, which also yields the localization effect for the eigenfunctions of the problem (1.7), (1.8). The proof of the theorem is similar to that of Proposition 2.3.

**Theorem 3.1.** *Let the function $B$ be as in* (2.10), (2.11), *let $k \in \mathbb{N}$, and assume that the eigenvalue $\lambda^\varepsilon_k$ of the problem* (1.9) *satisfies the bound*

$$\lambda^\varepsilon_k \leq \varepsilon^{-2}\frac{\pi^2}{h^2} + \Lambda\varepsilon^{-1}, \tag{3.1}$$

*for small $\varepsilon > 0$ and a constant $\Lambda > 0$. Then there exist $\varepsilon_k = \varepsilon_k(B, \Lambda) > 0$ and $c_k = c_k(B, \Lambda) > 0$ such that the corresponding eigenfunction $u^\varepsilon_k$ satisfies for $\varepsilon \in (0, \varepsilon_k]$ the estimate*

$$\int\limits_{\Omega^\varepsilon} \mathcal{E}_B(y)^2 \left(\varepsilon|\nabla_y u^\varepsilon_k(x)|^2 + \varepsilon^2|\partial_z u^\varepsilon_k(x)|^2 + (1 + \varepsilon^{-1}|y|^2)|u^\varepsilon_k(x)|^2\right) dx \leq c_k \tag{3.2}$$

*where the normalization* (1.11) *holds,*

$$\mathcal{E}_B(y) = \begin{cases} e_B(\varepsilon^{-1/2}y), & y \in \mathbb{E}_R, \\ \exp\left(\frac{1}{2}\varepsilon^{-1}R^2\right), & y \in \omega \setminus \mathbb{E}_R, \end{cases} \tag{3.3}$$

*$e_B$ is the exponential weight function* (2.12), *the set $\mathbb{E}_R$ is defined above formula* (2.15), *and $R$ is some positive number.*

**Proof.** Since the weight function (3.3) is continuous, we have $v^\varepsilon = \mathcal{E}_B U^\varepsilon_k = \mathcal{E}^2_B u^\varepsilon_k \in H^1_0(\Omega^\varepsilon)$, and we insert $v^\varepsilon$ as a test function into the integral equation (1.9) for the eigenpair $\{\lambda^\varepsilon_k, u^\varepsilon_k\}$. Repeating the calculation (2.16) with small modifications and commuting $\nabla_y$ and $\mathcal{E}_B$ several times, we obtain

$$\begin{aligned} &\lambda^\varepsilon_k \|U^\varepsilon_k; L^2(\Omega^\varepsilon)\|^2 \\ &= \|\partial_z U^\varepsilon_k; L^2(\Omega^\varepsilon)\|^2 + (\nabla_y u^\varepsilon_k, \mathcal{E}_B \nabla_y U^\varepsilon_k)_{\Omega^\varepsilon} + (\nabla_y u^\varepsilon_k, U^\varepsilon_k \nabla_y \mathcal{E}_B)_{\Omega^\varepsilon} \\ &= \|\partial_z U^\varepsilon_k; L^2(\Omega^\varepsilon)\|^2 + \|\nabla_y U^\varepsilon_k; L^2(\Omega^\varepsilon)\|^2 - \|U^\varepsilon_k \mathcal{E}^{-1}_B \nabla_y \mathcal{E}_B; L^2(\Omega^\varepsilon)\|^2; \end{aligned} \tag{3.4}$$

here, $\partial_z = \partial/\partial z$. From (1.17), (3.3), (2.12), and (2.10) we obtain that $A(\varepsilon^{-1/2}y) = \varepsilon^{-1}A(y)$ and

$$|\nabla_y \mathcal{E}_B(y)|^2 = \begin{cases} \varepsilon^{-2}(1-\beta)A(y)|\mathcal{E}_B(y)|^2, & y \in \mathbb{E}_R, \\ 0, & y \in \omega \setminus \mathbb{E}_R. \end{cases} \tag{3.5}$$

We now fix $R > 0$ such that the relation

$$\frac{\pi^2}{H(y)^2} - \frac{\pi^2}{h^2} \geq \begin{cases} (1-\beta)A(y) + t|y|^2, & y \in \mathbb{E}_R, \\ T, & y \in \omega \setminus \mathbb{E}_R, \end{cases} \tag{3.6}$$

is valid for some positive constants $t$ and $T$. This is possible by the following two facts, which are based on the original assumptions in Section 1.1. First, the function $y \mapsto H(y)^{-2}$ has the global strict minimum $h^{-2}$ at the point $y = 0$. Second, owing to the formulas (1.2), (1.3) and (1.17), we have

$$\begin{aligned} \frac{\pi^2}{H(y)^2} - \frac{\pi^2}{h^2} &= \pi^2 \frac{h^2 - (h - r^2\mathcal{H}(\varphi) + O(r^3))^2}{h^2 H(y)^2} \\ &= \pi^2 \frac{2hr^2\mathcal{H}(\varphi) + O(r^3)}{h^2(h + O(r^2))^2} = A(y) + O(r^3). \end{aligned} \tag{3.7}$$

Integrating the Friedrichs inequality

$$\int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} |\partial_z U_k^\varepsilon(y,z)|^2\, dz \geq \frac{\pi^2}{\varepsilon^2 H(y)^2} \int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} |U_k^\varepsilon(y,z)|^2\, dz \tag{3.8}$$

with respect to $\omega \ni y$ yields

$$\|\partial_z U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \geq \pi^2 \varepsilon^{-2} \|H^{-1}U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2. \tag{3.9}$$

Taking into account (3.5), (3.6), and (3.8), we deduce from (3.4)

$$\begin{aligned} &\frac{\Lambda}{\varepsilon} \|U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \\ \geq\ & \|\partial_z U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \|\nabla_y U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 - \|U_k^\varepsilon \mathcal{E}_B^{-1}\nabla_y \mathcal{E}_B; L^2(\Omega^\varepsilon)\|^2 \\ -\ & \varepsilon^{-2}\frac{\pi^2}{h^2} \|U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \\ \geq\ & \|\nabla_y U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \frac{1}{\varepsilon^2} \int_{\Omega_R^\varepsilon} \left( \frac{\pi^2}{H(y)^2} - \frac{\pi^2}{h^2} - (1-\beta)A(y) \right) |U_k^\varepsilon(y,z)|^2 dx \\ +\ & \frac{\pi^2}{\varepsilon^2} \int_{\Omega^\varepsilon \setminus \Omega_R^\varepsilon} \left( \frac{1}{H(y)^2} - \frac{1}{h^2} \right) |U_k^\varepsilon(y,z)|^2 dx \\ \geq\ & \|\nabla_y U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \frac{t}{\varepsilon^2} \int_{\Omega_R^\varepsilon} r^2 |U_k^\varepsilon(y,z)|^2 dx + \frac{T}{\varepsilon^2} \int_{\Omega^\varepsilon \setminus \Omega_R^\varepsilon} |U_k^\varepsilon(y,z)|^2 dx \end{aligned} \tag{3.10}$$

where $\Omega_R^\varepsilon = \{(y,z) \in \Omega^\varepsilon : y \in \mathbb{E}_R\}$ and $r = |y|$. The choice of $\varepsilon_k > 0$ is done at this point as follows. We write $\varrho = \sqrt{2t^{-1}\Lambda}$ and then fix $\varepsilon_k$ and a constant $C_B > 0$ such that in the case $\varepsilon \in (0, \varepsilon_k]$ the following inequalities hold:

$$\begin{aligned} &\frac{2}{\varepsilon}\Lambda \leq \frac{1}{\varepsilon^2}T, \quad \frac{2}{\varepsilon}\Lambda \leq \frac{t}{\varepsilon^2} r^2 \text{ for } r \geq \sqrt{\varepsilon}\varrho, \\ &e_B(\varepsilon^{-1/2}y)^2 = \exp(B(\varepsilon^{-1/2}y)) \leq C_B \text{ for } r \geq \sqrt{\varepsilon}\varrho. \end{aligned} \tag{3.11}$$

The normalization condition $\|u_k^\varepsilon; L^2(\Omega^\varepsilon)\| = 1$ and the latter estimate (3.11) yield

$$\|U_k^\varepsilon; L^2(\widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 \leq C_B^2 \|u_k^\varepsilon; L^2(\widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 \leq C_B^2 \tag{3.12}$$

where $\widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}} = \{x \in \Omega^\varepsilon : |y| < \sqrt{\varepsilon}\varrho\}$. The inequalities (3.10), (3.11), and (3.12) give

$$\begin{aligned} C_B^2 \frac{\Lambda}{\varepsilon} &\geq \frac{\Lambda}{\varepsilon} \| U_k^\varepsilon; L^2(\widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 \\ &= \frac{\Lambda}{\varepsilon} \Big( \| U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 - \| U_k^\varepsilon; L^2(\Omega^\varepsilon \setminus \widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 \Big) \\ &\geq \|\nabla_y U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \frac{t}{\varepsilon^2}\| r U_k^\varepsilon; L^2(\Omega_R^\varepsilon)\|^2 - \frac{\Lambda}{\varepsilon}\| U_k^\varepsilon; L^2(\Omega^\varepsilon \setminus \widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 \\ &+ \frac{T}{\varepsilon^2}\| U_k^\varepsilon; L^2(\Omega^\varepsilon \setminus \Omega_R^\varepsilon)\|^2 - \frac{\Lambda}{\varepsilon}\| U_k^\varepsilon; L^2(\Omega^\varepsilon \setminus \Omega_R^\varepsilon)\|^2 \\ &\geq \|\nabla_y U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \frac{t}{2\varepsilon^2}\| r U_k^\varepsilon; L^2(\Omega_R^\varepsilon)\|^2 + \frac{T}{2\varepsilon}\| U_k^\varepsilon; L^2(\Omega^\varepsilon \setminus \Omega_R^\varepsilon)\|^2. \end{aligned} \tag{3.13}$$

Note that the sum of the last two terms is bigger than $\tau\varepsilon^{-2}\| r U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2$ for some number $\tau > 0$ independent of $\varepsilon$ and $k$, hence,

$$\begin{aligned} &\|\mathcal{E}_B u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 = \|\mathcal{E}_B u_k^\varepsilon; L^2(\widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 + \|\mathcal{E}_B u_k^\varepsilon; L^2(\Omega^\varepsilon \setminus \widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 \\ &\leq C_B^2 \| u_k^\varepsilon; L^2(\widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 + \varepsilon^{-1}\varrho^{-2}\| r U_k^\varepsilon; L^2(\Omega^\varepsilon \setminus \widetilde{\Omega}_\varrho^{\sqrt{\varepsilon}})\|^2 \leq C_B^2(1 + \varrho^{-2}\tau^{-1}\Lambda). \end{aligned} \tag{3.14}$$

We get the bound (3.2) – without the term $\varepsilon^2\mathcal{E}_B^2|\partial_z u_k^\varepsilon|^2$ – by observing that

$$\|\nabla_y U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \geq \frac{1}{2}\|\mathcal{E}_B \nabla_y u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 - \frac{c}{\varepsilon^2}\| r U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2$$

(see (3.5) and (1.17)) and by estimating the last term using (3.13) and (3.14).

The missing term is treated as follows. Since the norm $\|\mathcal{E}_B(1 + \varepsilon^{-1/2}|y|) u_k^\varepsilon; L^2(\Omega^\varepsilon)\|$ is bounded uniformly with respect to $\varepsilon$, the identity (3.4) yields

$$\begin{aligned} \|\partial_z U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 = \|\mathcal{E}_B \partial_z u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \leq \lambda_k^\varepsilon \|\mathcal{E}_B u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \| u_k^\varepsilon \nabla_y \mathcal{E}_B; L^2(\Omega^\varepsilon)\|^2 \leq \\ \leq c_k \varepsilon^{-2}\|\mathcal{E}_B(1 + |y|^2) u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \leq C_k \varepsilon^{-2}. \end{aligned}$$

which completes the proof. ⊠

The following assertion could be proven by a proper choice of test functions in the max-min principle (see, e.g., [4, Theorem 10.2.2]). However, we will only prove it as a consequence of the calculations in Section 5 (see Remark 5.1).

**Lemma 3.2.** *The eigenvalues* (1.10) *can be estimated by*

$$0 \leq \lambda_k^\varepsilon - \frac{\pi^2}{\varepsilon^2 h^2} \leq \frac{\Lambda_k}{\varepsilon} \tag{3.15}$$

*where the numbers $\Lambda_k$ do not depend on $\varepsilon$, although $\Lambda_k \to +\infty$ as $k \to +\infty$.*

**3.2. Calculations with the eigenfunctions.** The aim of this section is to present weighted estimates for some averages of the eigenfunctions $u_k^\varepsilon$. These functions will be needed in the treatment of the problem (2.5) in Section 3.3. We assume in the following that (3.1) holds for the eigenvalues $\lambda_k^\varepsilon$. Recalling the original ansatz (2.2), we set

$$\overline{u_k^\varepsilon}(y) = \frac{2}{\varepsilon H(y)} \int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} S_\varepsilon(y,z) u_k^\varepsilon(y,z)\, dz, \quad S_\varepsilon(y,z) = \sin\Big(\pi \frac{z + \varepsilon H_-(y)}{\varepsilon H(y)}\Big), \tag{3.16}$$

and also define $\overline{\nabla_y u_k^\varepsilon}$ in the same way, replacing $u_k^\varepsilon$ by $\nabla_y u_k^\varepsilon$ in (3.16). To clarify the role of the denominator $\varepsilon H(y)$ in front of the integral in (3.16), we remark that

$$\frac{1}{2}\varepsilon H(y) = \int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} \Big( \sin\Big(\pi \frac{z + \varepsilon H_-(y)}{\varepsilon H(y)}\Big)\Big)^2 dz. \tag{3.17}$$

Let us consider the following estimates.

**Lemma 3.3.** *We have*

$$\int_\omega \mathcal{E}_B(y)^2 \left|\nabla_y \overline{u_k^\varepsilon}(y) - \overline{\nabla_y u_k^\varepsilon}(y)\right|^2 dy \le \frac{c_k}{\varepsilon},$$

$$\int_\omega \mathcal{E}_B(y)^2 \left|\nabla_y \overline{u_k^\varepsilon}(y)\right|^2 dy \le \frac{c_k}{\varepsilon^2}. \tag{3.18}$$

**Proof.** The Cauchy-Schwartz-Bunyakowski inequality and the weighted estimate (3.2) yield

$$\int_\omega \mathcal{E}_B(y)^2 r^2 \left|\overline{u_k^\varepsilon}(y)\right|^2 dy$$

$$\le \frac{4}{\varepsilon^2}\int_\omega \mathcal{E}_B(y)^2 r^2 \Bigg(\int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} S_\varepsilon(y,z)^2\, dz\Bigg) \frac{1}{H(y)^2} \int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} |u_k^\varepsilon(y,z)|^2 dz dy$$

$$\le \frac{1}{2\varepsilon H_0}\int_{\Omega^\varepsilon} \mathcal{E}_B(y)^2 r^2 |u_k^\varepsilon(x)|^2 dx \le c_k \tag{3.19}$$

where $H_0 = \min\{H(y) \,\big|\, y \in \overline{\omega}\} > 0$.

Furthermore, differentiating the first equality in (3.16) and using the boundary condition (1.8), we obtain

$$\nabla_y \overline{u_k^\varepsilon}(y) = \overline{\nabla_y u_k^\varepsilon}(y) - \overline{u_k^\varepsilon}(y) H(y)^{-1}\nabla_y H(y)$$

$$+ \frac{\pi}{2\varepsilon^2 H(y)^3}\int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} \Big(\varepsilon H(y)\nabla_y H_-(y)$$

$$- \big(z+\varepsilon H_-(y)\big)\nabla_y H(y)\Big) \cos\Big(\pi\frac{z+\varepsilon H_-(y)}{\varepsilon H(y)}\Big) u_k^\varepsilon(y,z) dz$$

$$=: \overline{\nabla_y u_k^\varepsilon}(y) - \overline{u_k^\varepsilon}(y) H(y)^{-1}\nabla_y H(y) + I_k^\varepsilon(y). \tag{3.20}$$

By (1.2) we have $|\nabla_y H(y)| \le C_H |y|$ and $H(y)^{-1} \ge h^{-1}$ , hence, we can rewrite (3.19) as

$$\int_\omega \mathcal{E}_B(y)^2 \Big|\frac{\nabla_y H(y)}{H(y)}\Big|^2 \left|\overline{u_k^\varepsilon}(y)\right|^2 dy \le c_k. \tag{3.21}$$

We repeat the calculation (3.19) for $\overline{\nabla_y u_k^\varepsilon}$ by omitting the factor $r^2 = |y|^2$, and use Theorem 3.1, to get the estimate

$$\int_\omega \mathcal{E}_B(y)^2 \left|\overline{\nabla_y u_k^\varepsilon}(y)\right|^2 dy \le \frac{1}{2\varepsilon H_0}\int_{\Omega^\varepsilon} \mathcal{E}_B(y)^2 \left|\nabla_y u_k^\varepsilon(x)\right|^2 dx \le \frac{c_k}{\varepsilon^2}. \tag{3.22}$$

Again by (3.2), the last term $I_k^\varepsilon(y)$ in (3.20) satisfies the estimate

$$\int_\omega \mathcal{E}_B(y)^2 I_k^\varepsilon(y)^2\, dy \le \frac{c}{\varepsilon^4}\int_\omega \mathcal{E}_B(y)^2 \varepsilon^2 r^2 \Bigg(\int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} |u_k^\varepsilon(y,z)| dz\Bigg)^2 \le \frac{c_k}{\varepsilon}; \tag{3.23}$$

here, the first factor of the integrand of $I_k^\varepsilon(y,z)$ was bounded by $c\varepsilon r$.

The first inequality (3.18) follows now by solving $\left|\nabla_y \overline{u_k^\varepsilon}(y) - \overline{\nabla_y u_k^\varepsilon}(y)\right|^2$ from (3.20), multiplying by the weight, integrating, and using the bounds (3.21) and (3.23). The second comes from the first and (3.22). ⊠

Recalling now the choice of the rapid variables (1.15), we set

$$\overline{w_k^\varepsilon}(\eta) = \varepsilon\alpha_k(\varepsilon)\overline{u_k^\varepsilon}(\varepsilon^{1/2}\eta)\chi(\varepsilon^{1/2}\eta), \tag{3.24}$$

where $\alpha_k(\varepsilon)$ is a normalization factor such that $\|\overline{w_k^\varepsilon}; L^1(\mathbb{R}^2)\| = 1$ and $\chi$ is a smooth cut-off function with support inside the domain $\omega$ such that

$$0 \le \chi \le 1, \quad \chi(y) = 1 \text{ for } r < R \quad \text{and} \quad \chi(y) = 0 \text{ for } r > 2R > 0. \tag{3.25}$$

The functions (3.24) will appear in the construction of the eigenfunctions of the problem (2.5) in Section 3.3. The rest of this section is devoted to showing that the normalization factor $\alpha_k(\varepsilon)$ is bounded and bounded away of 0 uniformly in $\varepsilon$.

**Lemma 3.4.** *There are numbers $\alpha_k^\pm > 0$ such that*

$$0 < \alpha_k^- \le \alpha_k(\varepsilon) \le \alpha_k^+ \quad \forall\, \varepsilon \in (0, \varepsilon_k]. \tag{3.26}$$

**Proof.** We need additional estimates for the functions $u_k^\varepsilon(\eta)$, so we proceed by setting

$$u_k^{\varepsilon\perp}(y,z) = u_k^\varepsilon(y,z) - S_\varepsilon(y,z)\overline{u_k^\varepsilon}(y). \tag{3.27}$$

By (3.16) and (3.17), the orthogonality condition

$$\int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} S_\varepsilon(y,z) u_k^{\varepsilon\perp}(y,z)\, dz = 0 \tag{3.28}$$

holds true in the domain $\omega \ni y$. We write the minimum principle (see, e.g. [4, Thm 10.2.1])

$$\frac{4\pi^2}{\varepsilon^2 H(y)^2} = \min \frac{\|\partial_z U; L^2(-\varepsilon H_-(y), \varepsilon H_+(y))\|^2}{\|U; L^2(-\varepsilon H_-(y), \varepsilon H_+(y))\|^2}, \tag{3.29}$$

where minimum is computed over all non-zero functions $U \in H_0^1(-\varepsilon H_-(y), \varepsilon H_+(y))$ satisfying to the orthogonality condition (3.28). We emphasize that $4\pi^2\varepsilon^{-2}H(y)^{-2}$ is nothing but the second eigenvalue of the Dirichlet problem for the differential operator $-\partial_z^2$ in the interval $(-\varepsilon H_-(y), \varepsilon H_+(y))$. The identity (3.29) yields

$$\frac{4\pi^2}{\varepsilon^2}\|h^{-1}u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|^2 \le \|\partial_z u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|^2.$$

Furthermore,

$$\begin{aligned} \lambda_k^\varepsilon &= \|\nabla_x u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 = \|\partial_z u_k^{\varepsilon\perp} + \overline{u_k^\varepsilon}\partial_z S_\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \|\nabla_y u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 = \\ &= \|\partial_z u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|^2 + \|\overline{u_k^\varepsilon}\partial_z S_\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \|\nabla_y u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + 2I_k^\varepsilon. \end{aligned}$$

Since $\mathcal{E}_B(y) \ge 1$ in (3.2), we deduce that

$$\|\nabla_y u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \le c_k\varepsilon^{-1}.$$

The orthogonality condition (3.28) and the estimate (3.19), again with $B = 0$, yield

$$\begin{aligned} I_k^\varepsilon &= \int_{\Omega^\varepsilon} \overline{u_k^\varepsilon}(y)\partial_z S_\varepsilon(x)\partial_z u_k^{\varepsilon\perp}(x)\, dx = -\int_{\Omega^\varepsilon} \overline{u_k^\varepsilon}(y) u_k^{\varepsilon\perp}(x)\partial_z^2 S_\varepsilon(x)\, dx \\ &= \frac{\pi^2}{\varepsilon^2}\int_{\Omega^\varepsilon} \frac{1}{H(y)^2}\overline{u_k^\varepsilon}(y)u_k^{\varepsilon\perp}(x)S_\varepsilon(x)\, dx \\ &= \frac{\pi^2}{\varepsilon^2}\int_{\Omega^\varepsilon} \Big(\frac{1}{H(y)^2} - \frac{1}{h^2}\Big)\overline{u_k^\varepsilon}(y)u_k^{\varepsilon\perp}(x)S_\varepsilon(x)\, dx \\ &\le c\varepsilon^{-2}\|r\overline{u_k^\varepsilon}; L^2(\Omega^\varepsilon)\|\, \|u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\| \le c_k\varepsilon^{-1}\|u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|. \end{aligned} \tag{3.30}$$

We have used here that the integral of $h^{-2}\overline{u_k^\varepsilon}(y)u_k^{\varepsilon\perp}(x)S_\varepsilon(x)$ over $\Omega^\varepsilon$ vanishes. Moreover,

$$\begin{aligned}
&\|\overline{u_k^\varepsilon}\partial_z S_\varepsilon; L^2(\Omega^\varepsilon)\|^2 \\
&= \frac{\pi^2}{\varepsilon^2}\int\limits_\omega \frac{1}{H(y)^2}|\overline{u_k^\varepsilon}(y)|^2 \int\limits_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} \left(\cos\left(\pi\frac{z+\varepsilon H_-(y)}{\varepsilon H(y)}\right)\right)^2 dzdy \\
&= 2\frac{\pi^2}{\varepsilon^2}\int\limits_\omega H(y)^{-1}\,|\overline{u_k^\varepsilon}(y)|^2\,dy = \frac{\pi^2}{\varepsilon^2}\int\limits_{\Omega^\varepsilon} H(y)^{-2}\,|S_\varepsilon(x)\overline{u_k^\varepsilon}(y)|^2\,dx \\
&= \frac{\pi^2}{\varepsilon^2 h^2}\int\limits_{\Omega^\varepsilon}|S_\varepsilon(x)\overline{u_k^\varepsilon}(y)|^2\,dy + \frac{\pi^2}{\varepsilon^2}\int\limits_{\Omega^\varepsilon}\Big(\frac{1}{H(y)^2}-\frac{1}{h^2}\Big)\,|S_\varepsilon(x)\overline{u_k^\varepsilon}(y)|^2\,dx. \qquad (3.31)
\end{aligned}$$

The last term does not exceed $c\varepsilon^{-1}\|r\overline{u_k^\varepsilon}; L^2(\omega)\|^2 \le c_k\varepsilon^{-1}$ (see (1.2) and (3.19)).

Putting the above formulas together gives

$$\begin{aligned}
&\frac{3\pi^2}{\varepsilon^2 h^2}\,\|u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|^2 \\
&\le\ \lambda_k^\varepsilon - \frac{\pi^2}{\varepsilon^2 h^2}\Big(\|u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|^2 + \|S_\varepsilon\overline{u_k^\varepsilon}; L^2(\Omega^\varepsilon)\|^2\Big) \\
&\quad + c_k\varepsilon^{-1}(1+\|u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|). \qquad (3.32)
\end{aligned}$$

Since $u_k^\varepsilon = u_k^{\varepsilon\perp} + S_\varepsilon\overline{u_k^\varepsilon}$ is normalized and the summands are mutually orthogonal in $L^2(\Omega^\varepsilon)$, we have

$$\|S_\varepsilon\overline{u_k^\varepsilon}; L^2(\Omega^\varepsilon)\|^2 + \|u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|^2 = 1. \qquad (3.33)$$

We recall Lemma 3.2 and derive from (3.32) and (3.1) the estimate

$$\|u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|^2 \le c_k\varepsilon. \qquad (3.34)$$

Formulas (3.17), (1.2) and (3.28), (3.33) yield

$$\begin{aligned}
&\left|\|S_\varepsilon\overline{u_k^\varepsilon}; L^2(\Omega^\varepsilon)\|^2 - \frac{\varepsilon h}{2}\int\limits_\omega |\overline{u_k^\varepsilon}(y)|^2 dy\right| \\
&= \frac{\varepsilon}{2}\left|\int\limits_\omega (H(y)-h)\,|\overline{u_k^\varepsilon}(y)|^2 dy\right| \le C\varepsilon\||y|\overline{u_k^\varepsilon}; L^2(\omega)\|^2 \le c_k\varepsilon \qquad (3.35)
\end{aligned}$$

and, therefore,

$$\frac{\varepsilon h}{2}\|\overline{u_k^\varepsilon}; L^2(\omega)\|^2 \ge 1 - \|u_k^{\varepsilon\perp}; L^2(\Omega^\varepsilon)\|^2 - c_k\varepsilon \ge 1 - C_k\varepsilon. \qquad (3.36)$$

Notice that, by (3.19),

$$\begin{aligned}
&\int\limits_\omega (1-\chi(y)^2)\,|\overline{u_k^\varepsilon}(y)|^2 dy \\
&\le c_k \sup_{|y|>R}\Big\{|y|^{-2}\mathcal{E}_B(y)^{-2}\Big\}\int\limits_\omega |y|^2\mathcal{E}_B(y)^2\,|\overline{u_k^\varepsilon}(y)|^2 dy \\
&\le c_k\exp(-b_B/\varepsilon) \quad \text{with} \quad b_B > 0 \qquad (3.37)
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_\omega \chi(y)^2\,|\overline{u_k^\varepsilon}(y)|^2 dy \le \frac{2}{\varepsilon}\int\limits_{\Omega^\varepsilon} H(y)^{-1}\chi(y)^2\,|u_k^\varepsilon(x)|^2 dx \\
&\le \frac{c}{\varepsilon}\int\limits_{\Omega^\varepsilon}|u_k^\varepsilon(x)|^2 dx = \frac{c}{\varepsilon}. \qquad (3.38)
\end{aligned}$$

By definitions, the normalization factor $\alpha_k(\varepsilon)$ satisfies

$$1 = \int_{\mathbb{R}^2} |\overline{w}_k^\varepsilon(\eta)|^2 d\eta = \alpha_k(\varepsilon)^2 \varepsilon \int_\omega \chi(y)^2 \, |\overline{u_k^\varepsilon}(y)|^2 dy,$$

hence, (3.26) follows from (3.36), (3.37) and (3.38). ⊠

**3.3. Deriving the limit equation once more.** In this section we will derive the limit equation (2.5) from the original integral identity (1.9) by putting the eigenfunction $u_k^\varepsilon$ and a partially specified test function (3.39) into (1.9) and passing to the limit $\varepsilon \to +0$. Combining this process with the estimates of the previous section yields the eigenvalues and eigenfunctions of the limit problem, which will be denoted by $M_k^0$ and $W_k^0$ in the sequel.

We fix an arbitrary infinitely differentiable and compactly supported function $V \in C_c^\infty(\mathbb{R}^2)$, and set

$$v^\varepsilon(x) = S_\varepsilon(y,z) V(\varepsilon^{-1/2} y)\chi(y) \tag{3.39}$$

where $\chi \in C_c^\infty(\omega)$ is the cut-off function (3.25). Writing the integral identity (1.9) for the eigenpair $\{\lambda_k^\varepsilon, u_k^\varepsilon\}$ and using the test function (3.39), we have

$$\lambda_k^\varepsilon (u_k^\varepsilon, v^\varepsilon)_{\Omega^\varepsilon} - (\partial_z u_k^\varepsilon, \partial_z v^\varepsilon)_{\Omega^\varepsilon} = (\nabla_y u_k^\varepsilon, \nabla_y v^\varepsilon)_{\Omega^\varepsilon}. \tag{3.40}$$

We rewrite the left-hand side $I_l^\varepsilon$ of (3.40) by using the definition (3.16) and integrating by parts in $z$:

$$\begin{aligned} I_l^\varepsilon :=& \lambda_k^\varepsilon \int_\omega \chi(y) V(\varepsilon^{-1/2} y) \int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} S_\varepsilon(y,z) u_k^\varepsilon(y,z) dz dy \\ &- \frac{\pi^2}{\varepsilon^2} \int_\omega \chi(y) V(\varepsilon^{-1/2} y) H(y)^{-2} \int_{-\varepsilon H_-(y)}^{\varepsilon H_+(y)} S_\varepsilon(y,z) u_k^\varepsilon(y,z) dz dy \\ =& \frac{\varepsilon}{2} \int_\omega V(\varepsilon^{-1/2} y) \Big( \lambda_k^\varepsilon H(y) - \frac{\pi^2}{\varepsilon^2} H(y)^{-1} \Big) \chi(y) \overline{u_k^\varepsilon}(y) dy. \end{aligned} \tag{3.41}$$

The right-hand side $I_r^\varepsilon$ of (3.40) equals

$$\begin{aligned} I_r^\varepsilon =& (S_\varepsilon \nabla_y (\chi u_k^\varepsilon), \nabla_y V)_{\Omega^\varepsilon} + (\nabla_y u_k^\varepsilon, \chi V \nabla_y S_\varepsilon)_{\Omega^\varepsilon} \\ &+ \big( (\nabla_y u_k^\varepsilon, S_\varepsilon V \nabla_y \chi)_{\Omega^\varepsilon} - (u_k^\varepsilon \nabla_y \chi, S_\varepsilon \nabla_y V)_{\Omega^\varepsilon} \big) =: I_r^{\varepsilon 1} + I_r^{\varepsilon 2} + I_r^{\varepsilon 3}. \end{aligned} \tag{3.42}$$

Aiming to pass to the limit $\varepsilon \to 0^+$ in (3.40), we set

$$M_k^\varepsilon = \varepsilon \Big( \lambda_k^\varepsilon - \frac{\pi^2}{\varepsilon^2 h^2} \Big), \quad W_k^\varepsilon(\eta) = \varepsilon \chi(\varepsilon^{1/2}\eta) \overline{u_k^\varepsilon}(\varepsilon^{1/2}\eta). \tag{3.43}$$

From (3.15), (3.18), (3.24), (3.26), and (3.38) we derive the formulas

$$\begin{aligned} & 0 \le M_k^\varepsilon \le \Lambda_k, \\ & \|W_k^\varepsilon; \mathbf{H}\|^2 = \|\nabla_\eta W_k^\varepsilon; L^2(\mathbb{R}^2)\|^2 + \|(1+|\eta|) W_k^\varepsilon; L^2(\mathbb{R}^2)\|^2 \\ =& \int_\omega \Big( \varepsilon^2 \left| \nabla_y (\chi \overline{u_k^\varepsilon}) \right|^2 + (\varepsilon + \varepsilon^2 r^2) \left| \chi \overline{u_k^\varepsilon} \right|^2 \Big) dy \le C_k. \end{aligned} \tag{3.44}$$

By weak completeness of the space $\mathbf{H}$, we thus find number $M_k^0$, a function $W_k^0 \in \mathbf{H}$ and a positive null sequence $\{\varepsilon_q\}_{q\in\mathbb{N}}$ such that

$$M_k^{\varepsilon_q} \to M_k^0 \ , \quad W_k^{\varepsilon_q} \to W_k^0 \quad \text{weakly in } \mathbf{H} \ ; \tag{3.45}$$

consequently, the sequence $\{W_k^{\varepsilon_q}\}$ convergences to $W_k^0$ strongly in $L^2(\mathbb{R}^2)$. Moreover, since $W_k^\varepsilon = \alpha_k(\varepsilon)^{-1}\overline{w}_k^\varepsilon$, the normalization of $\overline{w}_k^\varepsilon$ in $L^2(\mathbb{R}^2)$, see (3.24), and (3.26) yield

$$\|W_k^\varepsilon; L^2(\mathbb{R}^2)\| \geq c_k > 0, \quad \|W_k^0; L^2(\mathbb{R}^2)\| \geq c_k. \tag{3.46}$$

Let us now write the expression (3.41) as follows:

$$\begin{aligned} I_l^\varepsilon &:= \frac{\varepsilon}{2}\int_{\mathbb{R}^2} V(\eta)W_k^\varepsilon(\eta)H(\varepsilon^{1/2}\eta)\Big(\frac{\pi^2}{\varepsilon^2 h^2} + \frac{M_k^\varepsilon}{\varepsilon} - \frac{\pi^2}{\varepsilon^2}H(\varepsilon^{1/2}\eta)^{-2}\Big)\, d\eta \\ &=: -\frac{h}{2}\big(I_l^{\varepsilon 0} + I_l^{\varepsilon 1}\big), \end{aligned} \tag{3.47}$$

$$I_l^{\varepsilon 0} = \int_{\mathbb{R}^2} V(\eta)W_k^\varepsilon(\eta)\big(M_k^\varepsilon - A(\eta)\big)\, d\eta,$$

$$\begin{aligned} I_l^{\varepsilon 1} &:= \frac{1}{h}\int_{\mathbb{R}^2} V(\eta)W_k^\varepsilon(\eta)\Big(\frac{\pi^2}{\varepsilon h^2} + M_k^\varepsilon - \frac{\pi^2}{\varepsilon}H(\varepsilon^{1/2}\eta)^{-2}\Big)\big(H(\varepsilon^{1/2}\eta) - h\big)\, d\eta \\ &+ \frac{\pi^2}{\varepsilon}\int_{\mathbb{R}^2} V(\eta)W_k^\varepsilon(\eta)\Big(\frac{1}{h^2} - \frac{1}{H(\varepsilon^{1/2}\eta)^2} - \frac{\varepsilon}{\pi^2}A(\eta)\Big)\, d\eta. \end{aligned} \tag{3.48}$$

Clearly, by virtue of (3.45),

$$I_k^{\varepsilon_q 0} \to M_k^0(W_k^0, V)_{\mathbb{R}^2} - (AW_k^0, V)_{\mathbb{R}^2}.$$

Notice that by (1.2) and (1.17),

$$|H(y) - h| \leq cr^2, \quad |H(y)^{-2} - h^{-2}| \leq cr^2, \quad |h^{-2} - H(y)^{-2} - \pi^{-2}A(y)| \leq cr^3.$$

Hence, we can apply the weighted estimate (3.19) with the exponential multiplier (3.3) and obtain

$$\begin{aligned} \big|I_l^{\varepsilon 1}\big| &\leq c\|V; L^2(\mathbb{R}^2)\| \sup_{y\in\omega}\big\{\mathcal{E}_B(y)^{-1}\big((1+\varepsilon^{-1}r^2)r + \varepsilon^{-1}r^2\big)\big\}\ \varepsilon^{1/2}\|r\mathcal{E}_B\overline{u_k^\varepsilon}; L^2(\omega)\| \\ &\leq C_k(V)\varepsilon^{1/2}. \end{aligned} \tag{3.49}$$

Note that the supremum is of the order $O(\varepsilon^0)$ and that the factor $\varepsilon^{1/2}$ comes from the coordinate compression $\eta \mapsto y = \eta\varepsilon^{1/2}$.

We take into account $\nabla_y\chi(y) = 0$ for $|y| > R$ and the exponential weight

$$e_B(\varepsilon^{-1/2}y)^{-1} \leq \exp(-\varepsilon^{-1}b_B r^2)$$

in the estimate (3.2), and thus obtain

$$\begin{aligned} \big|I_r^{\varepsilon 3}\big| &\leq c\exp(-\varepsilon^{-1}b_B R)\varepsilon^{1/2}\big(\|\nabla_\eta V; L^2(\mathbb{R}^2)\|\,\|\mathcal{E}_B u_k^\varepsilon; L^2(\Omega^\varepsilon)\| \\ &+ \varepsilon^{1/2}\|V; L^2(\mathbb{R}^2)\|\,\|\mathcal{E}_B\nabla_y u_k^\varepsilon; L^2(\Omega^\varepsilon)\|\big) \leq c_k(V)\varepsilon^{1/2}\exp(-\varepsilon^{-1}b_B R). \end{aligned}$$

Moreover, the estimate (3.2) with the weight $\mathcal{E}_B(y) \geq 1$ implies the inequality

$$\|\nabla_y u_k^\varepsilon; L^2(\Omega^\varepsilon)\| \leq c_k\varepsilon^{-1/2},$$

so, this and the two simple formulas

$$|\nabla_y S_\varepsilon(x)| \leq c_S, \quad \|\chi V; L^2(\Omega^\varepsilon)\| \leq c\varepsilon\|V; L^2(\mathbb{R}^2)\| \tag{3.50}$$

yield

$$\big|I_r^{\varepsilon 2}\big| \leq \|\nabla_y u_k^\varepsilon; L^2(\Omega^\varepsilon)\|\,\|\chi V; L^2(\Omega^\varepsilon)\| \leq C_k(V)\varepsilon^{1/2}.$$

We finally write the term $I_r^{\varepsilon 1}$ in (3.42) as follows:

$$I_r^{\varepsilon 1} = I_r^{\varepsilon 0} + I_r^{\varepsilon 4} + I_r^{\varepsilon 5},$$

$$I_r^{\varepsilon 0} = \frac{\varepsilon h}{2}(\nabla_y(\chi\overline{u_k^\varepsilon}), \nabla_y V)_\omega = \frac{h}{2}(\nabla_\eta W_k^\varepsilon, \nabla_\eta V)_{\mathbb{R}^2},$$

$$I_r^{\varepsilon 4} = \frac{\varepsilon}{2}((H-h)\overline{\nabla_y \chi u_k^\varepsilon}, \nabla_y V)_\omega, \quad I_r^{\varepsilon 5} = \frac{\varepsilon h}{2}(\overline{\nabla_y \chi u_k^\varepsilon} - \nabla_y(\chi \overline{u_k^\varepsilon}), \nabla_y V)_\omega. \tag{3.51}$$

Applying the estimates (3.22), (3.18) and taking into account that $\|\nabla_y V; L^2(\omega)\| \leq \|\nabla_\eta V; L^2(\mathbb{R}^2)\|$, we obtain

$$|I_r^{\varepsilon 4}| \leq c_k \varepsilon \sup_{y\in\omega} (r^2 \mathcal{E}_B(y)^{-1}) \|\mathcal{E}_B \overline{\nabla_y \chi u_k^\varepsilon}; L^2(\omega)\| \, \|\nabla_\eta V; L^2(\mathbb{R}^2)\| \leq C_k(V)\varepsilon,$$

$$|I_r^{\varepsilon 5}| \leq c_k \varepsilon \|\overline{\nabla_y \chi u_k^\varepsilon} - \nabla_y(\chi \overline{u_k^\varepsilon}); L^2(\omega)\| \, \|\nabla_y V; L^2(\omega)\| \leq C_k(V)\varepsilon^{1/2}.$$

Collecting the above presented information on the terms of (3.40), we see that only $I_l^{\varepsilon 0}$ and $I_r^{\varepsilon 0}$ have nontrivial limits when $\varepsilon = \varepsilon_q \to 0^+$; all other terms $I_l^{\varepsilon j}$, $I_r^{\varepsilon j}$ vanish in the limit. Although the bounds $C_k(V)$ in the above estimates depend on the test function $V \in C_c^\infty(\mathbb{R}^2)$, we can pass to the limit $\varepsilon_q \to 0^+$ by using (3.45). As a consequence of (3.48) and (3.51) we get the integral identity

$$(\nabla_\eta W_k^0, \nabla_\eta V)_{\mathbb{R}^2} + (AW_k^0, V)_{\mathbb{R}^2} = M_k^0 (W_k^0, V)_{\mathbb{R}^2}. \tag{3.52}$$

By a completion argument, (3.52) holds true for all $V \in \mathbf{H}$. Hence, $M_k^0$ is an eigenvalue of the problem (2.5) and $W_k^0$ is a corresponding eigenfunction, since is not zero, see (3.46).

**3.4. First result on asymptotics of eigenvalues.** In the following result we prove the asymptotics (1.20) of the eigenvalues $\lambda_l^\varepsilon$. Information on the relation of the order of the eigenvalues in the original and limit problems will be clarified only in later sections.

**Theorem 3.5.** *The eigenvalues* $\lambda_l^\varepsilon$, (1.10), *of the problem* (1.9) *have the asymptotic behavior* (1.20), *i.e., for all indices* $l$

$$\varepsilon(\lambda_l^\varepsilon(x) - \pi^2 \varepsilon^{-2} h^2) \to \mu_{J(l)} \quad as \quad \varepsilon \to 0^+,$$

*where* $J(l) \geq l$ *and* $\mu_{J(l)}$ *is an eigenvalue of the limit problem* (2.5).

Notice that (3.15) was used in (3.44), hence, the following proof assumes Lemma 3.2 to be proven.

**Proof.** Let $l \in \mathbb{N}$ be given, and consider the eigenvalues $\lambda_1^\varepsilon, \ldots, \lambda_l^\varepsilon$ and the corresponding eigenfunctions $u_1^\varepsilon, \ldots, u_l^\varepsilon \in H_0^1(\Omega^\varepsilon)$, cf. (1.10), (1.11). We select the null sequence $\{\varepsilon_q\}$ such that the convergence (3.45) occurs for any $k = 1, \ldots, l$, and consider the numbers $M_1^0, \ldots, M_l^0$ and the functions $W_1^0, \ldots, W_l^0 \in \mathbf{H}$ defined by (3.45). We have

$$\begin{aligned}(W_j^0, W_k^0)_{\mathbb{R}^2} &= \varepsilon \int_\omega \chi(y)^2\, \overline{u_j^\varepsilon}(y) \overline{u_k^\varepsilon}(y)\, dy \\ &= 2 \int_{\Omega^\varepsilon} H(y)^{-1} S_\varepsilon(y,z)^2\, \overline{u_j^\varepsilon}(y) \overline{u_k^\varepsilon}(y) \chi(y)^2\, dx = J_{jk}^{\varepsilon 1} + J_{jk}^{\varepsilon 2}\end{aligned} \tag{3.53}$$

where

$$J_{jk}^{\varepsilon 1} = \frac{2}{h} \int_{\Omega^\varepsilon} S_\varepsilon(y,z)^2\, \overline{u_j^\varepsilon}(y) \overline{u_k^\varepsilon}(y)\, dx = \frac{2}{h} \int_{\Omega^\varepsilon} (u_j^\varepsilon(x) - u_j^{\varepsilon\perp}(x))(u_k^\varepsilon(x) - u_k^{\varepsilon\perp}(x))\, dx$$

$$= \frac{2}{h} \int_{\Omega^\varepsilon} u_j^\varepsilon(x) u_k^\varepsilon(x)\, dx + J_{jk}^{\varepsilon 3} = \frac{2}{h}\, \delta_{j,k} + J_{jk}^{\varepsilon 3}$$

and the functions $u_j^{\varepsilon\perp}$ are as in (3.41). To estimate the integrals

$$\begin{aligned}J_{jk}^{\varepsilon 2} &= 2 \int_{\Omega^\varepsilon} S_\varepsilon(y,z)^2 \overline{u_j^\varepsilon}(y) \overline{u_k^\varepsilon}(y) (H(y)^{-1}\chi(y)^2 - h^{-1})\, dx \\ &= \int_\omega (\overline{u_j^\varepsilon}(y) \overline{u_k^\varepsilon}(y) (\chi(y)^2 - h^{-1} H(y))\, dy,\end{aligned}$$

$$J^{\varepsilon 3}_{jk} = \frac{2}{h} \int\limits_{\Omega^\varepsilon} \big( u^{\varepsilon\perp}_j(x) u^{\varepsilon\perp}_k(x) - u^{\varepsilon\perp}_j(x) u^{\varepsilon}_k(x) - u^{\varepsilon}_j(x) u^{\varepsilon\perp}_k(x) \big)\, dx,$$

we apply the inequalities (3.19), (3.34) and obtain

$$|J^{\varepsilon 2}_{jk}| \le c\varepsilon \sup_{y\in\omega} \big( r^{-2} |\chi(y)^2 - h^{-1} H(y)| \big) \; \||y|\overline{u^\varepsilon_j}; L^2(\omega)\| \; \||y|\overline{u^\varepsilon_k}; L^2(\omega)\| \le c\varepsilon,$$

$$|J^{\varepsilon 3}_{jk}| \le c\big( \|u^{\varepsilon\perp}_j; L^2(\Omega^\varepsilon)\| + \|u^{\varepsilon\perp}_k; L^2(\Omega^\varepsilon)\| \big) \big( 1 + \|u^\varepsilon_j; L^2(\Omega^\varepsilon)\| + \|u^\varepsilon_k; L^2(\Omega^\varepsilon)\| \big) \le c\varepsilon^{1/2}.$$

Hence, by (3.45), taking the limit $\varepsilon = \varepsilon_q \to 0^+$ turns the equality (3.53) into

$$(W^0_j, W^0_k)_{\mathbb{R}^2} = \frac{2}{h} \delta_{j,k}, \quad j,k = 1,\dots,l.$$

This means that the limit eigenfunctions $W^0_1,\dots,W^0_l$ are linearly independent in $\mathbf{H}$, and thus they correspond to $l$ limit eigenvalues $M^0_1,\dots,M^0_l$ in the spectrum (2.6) of the problem (2.5). This proves the theorem modulo Lemma 3.2. ⊠

## 4. Asymptotics of eigenvalues.

**4.1. Abstract formulation of the problem in $\Omega^\varepsilon$.** Our main result of the asymptotics of the eigenvalues $\lambda^\varepsilon_k$ will be presented in Theorem 4.2. To prepare the proof we give in this section the abstract operator theoretic formulation of the problem (1.9) and cite a basic result from general spectral theory.

In the same way as in Section 2.2 we introduce the Hilbert space $\mathbf{H}^\varepsilon = H^1_0(\Omega^\varepsilon)$ with the scalar product

$$\langle u^\varepsilon, v^\varepsilon \rangle_\varepsilon = (\nabla_x u^\varepsilon, \nabla_x v^\varepsilon)_{\Omega^\varepsilon} + \frac{1}{\varepsilon}\left( 1 - \frac{\pi^2}{\varepsilon h^2} \right) (u^\varepsilon, v^\varepsilon)_{\Omega^\varepsilon} \tag{4.1}$$

and the operator $\mathcal{T}^\varepsilon : \mathbf{H}^\varepsilon \to \mathbf{H}^\varepsilon$,

$$\langle \mathcal{T}^\varepsilon u^\varepsilon, v^\varepsilon \rangle_\varepsilon = (u^\varepsilon, v^\varepsilon)_{\Omega^\varepsilon}, \quad u^\varepsilon, v^\varepsilon \in \mathbf{H}^\varepsilon.$$

As a consequence of the Friedrichs inequality (3.9) and the first formula in (1.2), the properties of the bilinear form (4.1) imply that the operator $\mathcal{T}^\varepsilon$ is continuous, positive, self-adjoint and compact, hence, its spectrum is discrete. Problem (1.9) is equivalent to the abstract spectral equation

$$\mathcal{T}^\varepsilon \varphi^\varepsilon = \tau^\varepsilon \varphi^\varepsilon. \tag{4.2}$$

Eigenvalues of $\mathcal{T}^\varepsilon$ are

$$\tau^\varepsilon_k = \left( \lambda^\varepsilon_k - \pi^2 \varepsilon^{-2} h^{-2} + \varepsilon^{-1} \right)^{-1}, \tag{4.3}$$

and form a positive null sequence. The corresponding eigenfunctions $\varphi^\varepsilon_k = (\tau^\varepsilon_k)^{1/2} u^\varepsilon_k$ satisfy (cf. (1.11))

$$\begin{aligned} \langle \varphi^\varepsilon_j, \varphi^\varepsilon_k \rangle_\varepsilon &= (\tau^\varepsilon_j)^{1/2} (\tau^\varepsilon_k)^{1/2} \left( (\nabla_x u^\varepsilon_j, \nabla_x u^\varepsilon_k)_{\Omega^\varepsilon} + \frac{1}{\varepsilon}\left( 1 - \frac{\pi^2}{\varepsilon h^2} \right) (u^\varepsilon_j, u^\varepsilon_k)_{\Omega^\varepsilon} \right) \\ &= (u^\varepsilon_j, u^\varepsilon_k)_{\Omega^\varepsilon} = \delta_{j,k}. \end{aligned} \tag{4.4}$$

The following basic assertion is known as the lemma on “near eigenvalues and eigenvectors”. A proof can be found in [43] (see also [4, Chapter 6]).

**Lemma 4.1.** *Let the function $\Phi^\varepsilon \in \mathbf{H}^\varepsilon$ and the positive number $T^\varepsilon$ be such that*

$$\|\Phi^\varepsilon; \mathbf{H}^\varepsilon\| = 1, \quad \|\mathcal{T}^\varepsilon \Phi^\varepsilon - T^\varepsilon \Phi^\varepsilon; \mathbf{H}^\varepsilon\| = t \in (0, T^\varepsilon). \tag{4.5}$$

*Then the segment $[T^\varepsilon - t, T^\varepsilon + t]$ contains an eigenvalue of the operator $\mathcal{T}^\varepsilon$. Moreover, for any $t_1 \in (t, T^\varepsilon)$ there exist coefficients $a_J(\varepsilon), \dots, a_{J+K-1}(\varepsilon)$ such that*

$$\|\Phi^\varepsilon - \sum_{j=J}^{J+K-1} a_j(\varepsilon)\varphi_j^\varepsilon; \mathbf{H}^\varepsilon\| \le 2\frac{t}{t_1}, \qquad \sum_{j=J}^{J+K-1} |a_j(\varepsilon)|^2 = 1 \tag{4.6}$$

*where $\tau_J^\varepsilon(\varepsilon), \dots, \tau_{J+K-1}^\varepsilon$ are all of the eigenvalues of $\mathcal{T}^\varepsilon$ contained in the segment $[T^\varepsilon - t_1, T^\varepsilon + t_1]$ and $\varphi_J^\varepsilon, \dots, \varphi_{J+K-1}^\varepsilon$ are the corresponding eigenvectors normalized by*

$$\langle \varphi_j^\varepsilon, \varphi_k^\varepsilon \rangle_\varepsilon = \delta_{j,k}.$$

**4.2. Approximate solutions of the abstract equation and asymptotics of eigenvalues.** We construct approximate eigenfunctions related to the equation (4.2) from the eigenfunctions $w_p$, and use these functions and Lemma 4.1. to derive the asymptotic formula (4.24).

Let $\mu_k$ be an eigenvalue of the problem (2.5) of multiplicity $\varkappa_k$, so that

$$\mu_{k-1} < \mu_k = \dots = \mu_{k+\varkappa_k-1} < \mu_{k+\varkappa_k}\,; \tag{4.7}$$

the case $\varkappa_k = 1$ is not excluded. We set $T^\varepsilon = \varepsilon(1+\mu_k)^{-1}$. Using the eigenfunctions $w_k, \dots, w_{k+\varkappa_k-1}$, we construct $\varkappa_k$ approximate solutions for the equation (4.2) by

$$\Phi_p^\varepsilon(x) = \|\Psi_p^\varepsilon; \mathbf{H}^\varepsilon\|^{-1}\Psi_p^\varepsilon(x), \quad p = k, \dots, k+\varkappa_k - 1, \tag{4.8}$$

where $\chi$ is the cut-off function (3.25) and

$$\Psi_p^\varepsilon(x) = \chi(y) w_p(\varepsilon^{-1/2}y)\sin\Big(\pi\frac{z+\varepsilon H_-(y)}{\varepsilon H(y)}\Big). \tag{4.9}$$

We proceed by calculating the scalar products $\langle \Psi_p^\varepsilon, \Psi_q^\varepsilon\rangle_\varepsilon$ and computing the numbers $t := t_p$ in (4.5). The exponentially decaying estimates of Proposition 2.3 make this task quite easy, since we obtain

$$\begin{aligned}
&\int_{\Omega^\varepsilon} |y|^{2t}|w_p(\varepsilon^{-1/2}y)|^2 dx \le c\max_{r\in\mathbb{R}_+}\Big\{r^{2t}\frac{\exp(-2B(\varepsilon^{-1/2}r)}{(1+\varepsilon^{1/2}r)^2}\Big\}\times \\
&\times\int_{\Omega^\varepsilon}(1+\varepsilon^{1/2}r)^2\exp(2B(\varepsilon^{-1/2}r))|w_p(\varepsilon^{-1/2}y)|^2dx \le c\varepsilon^{t+2}, \quad t\ge 0, \\
&\int_{\Omega^\varepsilon}(1-\chi(y)^2)|w_p(\varepsilon^{-1/2}y)|^2dx + \int_{\Omega^\varepsilon}|\nabla\chi(y)|^2|w_p(\varepsilon^{-1/2}y)|^2dx \\
&\le c\exp(-\delta_\chi\varepsilon^{-1/2})
\end{aligned} \tag{4.10}$$

for some constant $\delta_\chi > 0$. Moreover,

$$\begin{aligned}
&|\nabla_x\Psi_p^\varepsilon(x) - \chi(y)S_\varepsilon(x)\varepsilon^{-1/2}\nabla_\eta w_p(\varepsilon^{-1/2}y)| \le c(1+|\nabla_y\chi((y)|)|w_p(\varepsilon^{-1/2}y)|, \\
&\partial_z\Psi_p^\varepsilon(x) = \chi(y)\frac{\pi C_\varepsilon(x)}{\varepsilon H(y)}w_p(\varepsilon^{-1/2}y), \quad C_\varepsilon(x) = \cos\Big(\pi\frac{z+\varepsilon H_-(y)}{\varepsilon H(y)}\Big).
\end{aligned}$$

We also recall the integral in (3.17) and note that replacing the integrand by $C_\varepsilon(x)$ does not change it. Then we use the relations (2.5), (2.7), (3.7), and (3.50) to obtain

$$\begin{aligned}
\langle\Psi_p^\varepsilon, \Psi_q^\varepsilon\rangle_\varepsilon &= \int_{\Omega^\varepsilon}\nabla_y\Psi_p^\varepsilon(y,z)\cdot\nabla_y\Psi_q^\varepsilon(y,z)\,dydz + \int_{\Omega^\varepsilon}\partial_z\Psi_p^\varepsilon(y,z)\cdot\partial_z\Psi_q^\varepsilon(y,z)\,dydz \\
&\quad + \frac{1}{\varepsilon}\Big(1-\frac{\pi^2}{\varepsilon h^2}\Big)\int_{\Omega^\varepsilon}\Psi_p^\varepsilon(y,z)\Psi_q^\varepsilon(y,z)\,dydz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int\limits_{\mathbb{R}^2} \chi(y)^2 H(y)\nabla_\eta w_p(\eta)\cdot\nabla_\eta w_q(\eta)\,dy + \frac{\pi^2}{2\varepsilon}\int\limits_{\mathbb{R}^2}\chi(y)^2\frac{1}{H(y)}w_p(\eta)w_q(\eta)\,dy \\
&+ \frac{1}{2}\Big(1-\frac{\pi^2}{\varepsilon h^2}\Big)\int\limits_{\mathbb{R}^2}\chi(y)^2 H(y) w_p(\eta)w_q(\eta)\,dy \\
&+ O(\exp(-\delta_\chi\varepsilon^{-1/2}) + O(\varepsilon^2) \\
&= \varepsilon\frac{h}{2}\left((\nabla_\eta w_p,\nabla_\eta w_q)_{\mathbb{R}^2} + (w_p,w_q)_{\mathbb{R}^2} + (Aw_p,w_q)_{\mathbb{R}^2}\right) + O(\varepsilon^{3/2}) \\
&= \varepsilon\frac{h}{2}(1+\mu_k)\delta_{p,q} + O(\varepsilon^{3/2}),
\end{aligned}
\tag{4.11}
$$

where $p,q=k,\dots,k+\varkappa_k-1$. In particular, this formula shows that for small $\varepsilon>0$,

$$
\|\Psi_p^\varepsilon;\mathbf{H}^\varepsilon\| \geq \frac{1}{2}\varepsilon^{1/2}h^{1/2}(1+\mu_k)^{1/2}\ ,\quad \left|\langle\Phi_p^\varepsilon,\Phi_q^\varepsilon\rangle_\varepsilon - \delta_{p,q}\right| \leq C_k\varepsilon^{1/2}. \tag{4.12}
$$

Now we write

$$
\begin{aligned}
\delta_p &= \|\mathcal{T}^\varepsilon\Phi_p^\varepsilon - T_k^\varepsilon\Phi_p^\varepsilon;\mathbf{H}^\varepsilon\| = \|\Psi_p^\varepsilon;\mathbf{H}^\varepsilon\|^{-1}T_k^\varepsilon\|\Psi_p^\varepsilon - (T_k^\varepsilon)^{-1}\mathcal{T}^\varepsilon\Psi_p^\varepsilon;\mathbf{H}^\varepsilon\| \\
&= \|\Psi_p^\varepsilon;\mathbf{H}^\varepsilon\|^{-1}\frac{\varepsilon}{1+\mu_k}\sup\Big|(\nabla_x\Psi_p^\varepsilon,\nabla_x v^\varepsilon)_{\Omega^\varepsilon} \\
&+ \Big(\frac{1}{\varepsilon}-\frac{\pi^2}{\varepsilon^2h^2}\Big)(\Psi_p^\varepsilon,v^\varepsilon)_{\Omega^\varepsilon} - \frac{1}{\varepsilon}(1+\mu_k)(\Psi_p^\varepsilon,v^\varepsilon)_{\Omega^\varepsilon}\Big| \\
&= \|\Psi_p^\varepsilon;\mathbf{H}^\varepsilon\|^{-1}\frac{\varepsilon}{1+\mu_k}\sup\Big|(\nabla_x\Psi_p^\varepsilon,\nabla_x v^\varepsilon)_{\Omega^\varepsilon} - \Big(\frac{\pi^2}{\varepsilon^2h^2}+\frac{\mu_k}{\varepsilon}\Big)(\Psi_p^\varepsilon,v^\varepsilon)_{\Omega^\varepsilon}\Big| \\
&= \|\Psi_p^\varepsilon;\mathbf{H}^\varepsilon\|^{-1}\frac{\varepsilon}{1+\mu_k}\sup\Big|\frac{\mu_k}{\varepsilon}(\Psi_p^\varepsilon,v^\varepsilon)_{\Omega^\varepsilon} + \Big(\frac{\pi^2}{\varepsilon^2h^2}\Psi_p^\varepsilon \\
&+ \partial_z^2\Psi_p^\varepsilon, v^\varepsilon\Big)_{\Omega^\varepsilon} - (\nabla_y\Psi_p^\varepsilon,\nabla_y v^\varepsilon)_{\Omega^\varepsilon}\Big|,
\end{aligned}
\tag{4.13}
$$

where the supremum is computed over all $v^\varepsilon\in\mathbf{H}^\varepsilon$ such that $\|v^\varepsilon;\mathbf{H}^\varepsilon\|=1$. This normalization condition and inequalities (3.8), (3.19) provide the estimate

$$
\begin{aligned}
&\varepsilon^{-1}\|r\overline{v^\varepsilon};L^2(\omega)\|^2 + \|\overline{v^\varepsilon};L^2(\omega)\|^2 \\
&\leq c(\varepsilon^{-2}\|rv^\varepsilon;L^2(\Omega^\varepsilon)\|^2 + \varepsilon^{-1}\|v^\varepsilon;L^2(\Omega^\varepsilon)\|^2) \leq C\|v^\varepsilon;\mathbf{H}^\varepsilon\|^2 = C.
\end{aligned}
\tag{4.14}
$$

The first scalar product $I_1^\varepsilon$ on the right-hand side of (4.13) can be written as

$$
\begin{aligned}
I_1^\varepsilon &= \frac{\mu_k}{\varepsilon}\int\limits_\omega w_p(\eta)\chi(y)\int\limits_{-\varepsilon H_-(y)}^{+\varepsilon H_+(y)} S_\varepsilon(y,z)v^\varepsilon(y,z)dzdy \\
&= \frac{\mu_k}{\varepsilon}\int\limits_\omega w_p(\eta)\frac{H(y)}{h}V^\varepsilon(\eta)dy.
\end{aligned}
\tag{4.15}
$$

Here, $\eta$ is the fast variable (1.15) and

$$
V^\varepsilon(\eta) = \frac{h}{2}\varepsilon\chi(\varepsilon^{1/2}\eta)\overline{v^\varepsilon}(\varepsilon^{1/2}\eta),
$$

cf.(3.24). Hence, in view of (1.2) and (4.14),

$$
\begin{aligned}
&|I_1^\varepsilon - \mu_k(w_p,V^\varepsilon)_{\mathbb{R}^2}| \leq c\varepsilon^{-1}\int\limits_\omega r^2|w_p(\eta)|\,|V^\varepsilon(\eta)|dy \\
&\leq c\varepsilon^{-1}\|\chi|y|w_p;L^2(\omega)\|\,\varepsilon\,\|r\overline{v^\varepsilon};L^2(\omega)\| \leq c\varepsilon^{3/2}\||\eta|w_p;L^2(\mathbb{R}^2)\| = c_p\varepsilon^{3/2}.
\end{aligned}
\tag{4.16}
$$

For the second scalar product $I_2^\varepsilon$ in (4.13), we use (3.7) and write

$$I_2^\varepsilon = \frac{\pi^2}{\varepsilon^2 h^2} \int\limits_\omega \frac{H(y)}{h}\Big(1 - \frac{h^2}{H(y)^2}\Big) w_p(\eta) V^\varepsilon(\eta) dy,$$

$$|I_2^\varepsilon + (Aw_p, V^\varepsilon)_{\mathbb{R}^2}| \le c\varepsilon^{-2} \int\limits_\omega r^3 |w_p(\eta)|\, |V^\varepsilon(\eta)| dy$$

$$\le c\varepsilon \|\rho^2 w_p; L^2(\mathbb{R}^2)\| = c_p \varepsilon. \tag{4.17}$$

The last scalar product $I_3^\varepsilon$ in (4.13) requires a quite involved argument using all the tricks introduced above. Indeed, we claim that the formula

$$|I_3^\varepsilon + (\nabla_y w_p, \nabla_y V^\varepsilon)_{\mathbb{R}^2}| = |I_3^\varepsilon - (\Delta_y w_p, V^\varepsilon)_{\mathbb{R}^2}| \le c_p \varepsilon \tag{4.18}$$

holds true for $I_3^\varepsilon$. To see this we start by calculating $\nabla_y \Psi_p^\varepsilon$ with the help of (4.9) and get three terms, out of which the one with the derivative of $w_p$ equals

$$(\chi S_\varepsilon \Delta_y w_p, v^\varepsilon)_{\Omega^\varepsilon} = \frac{1}{\varepsilon} \int\limits_\omega \frac{H(y)}{h} V^\varepsilon(\eta) \Delta_\eta w_p(\eta) dy, \tag{4.19}$$

after integration by parts. The additional terms with derivatives of the factors $\chi(y)$ and $S_\varepsilon(y,z)$ are of small order $O(\varepsilon)$ as a consequence of the following: the integral with $|\nabla_y \chi(y)|$ has been estimated in (4.10), and the uniform estimate (3.50) holds for the gradient $\nabla_y S_\varepsilon(y,z)$, cf. (3.20). Finally, the term (4.19) can be processed in the same way as in (4.15)–(4.17), and it turns into $(\Delta_y w_p, V^\varepsilon)_{\mathbb{R}^2}$ (the subtrahend in (4.18)) plus terms of order $O(\varepsilon)$. This yields (4.18).

Since the eigenpairs $\{\mu_k, w_p\}$, $p = k, \dots, k + \varkappa_k - 1$, satisfy the equation (1.16), we have

$$-\mu_k (w_p, V^\varepsilon)_{\mathbb{R}^2} + (Aw_p, V^\varepsilon)_{\mathbb{R}^2} - (\Delta_y w_p, V^\varepsilon)_{\mathbb{R}^2} = 0.$$

Hence, the estimates (4.16), (4.17) and (4.18) together with the bound (4.12) yield for (4.13)

$$\delta_p = \|\Psi_p^\varepsilon; \mathbf{H}^\varepsilon\|^{-1} \frac{\varepsilon}{1+\mu_k} \sup_{v^\varepsilon} \big(I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon\big) \le c_k \varepsilon^{3/2}. \tag{4.20}$$

By Lemma 4.1, the operator $\mathcal{T}^\varepsilon$ has the eigenvalues $\tau^\varepsilon_{J^\varepsilon(p)}$, $p = k, \dots, k + \varkappa_k - 1$, such that

$$\left| \tau^\varepsilon_{J^\varepsilon(p)} - \frac{\varepsilon}{1+\mu_k} \right| \le c_k \varepsilon^{3/2}. \tag{4.21}$$

Combining this estimate with (4.3), we get for $\varepsilon \le \varepsilon_k := (2c_k(1+\mu_k))^{-2}$

$$\left| \lambda^\varepsilon_{J^\varepsilon(p)} - \frac{\pi^2}{\varepsilon^2 h^2} - \frac{\mu_k}{\varepsilon} \right| \le c_k (1+\mu_k) \varepsilon^{1/2} \left( \lambda^\varepsilon_{J^\varepsilon(p)} - \frac{\pi^2}{\varepsilon^2 h^2} + \frac{1}{\varepsilon} \right)$$

$$\Rightarrow \ \lambda^\varepsilon_{J^\varepsilon(p)} - \frac{\pi^2}{\varepsilon^2 h^2} + \frac{1}{\varepsilon} \le c_k (1+\mu_k) \varepsilon^{1/2} \left( \lambda^\varepsilon_{J^\varepsilon(p)} - \frac{\pi^2}{\varepsilon^2 h^2} + \frac{1}{\varepsilon} \right) + \frac{1}{\varepsilon} + \frac{\mu_k}{\varepsilon}$$

$$\Rightarrow \ \lambda^\varepsilon_{J^\varepsilon(p)} - \frac{\pi^2}{\varepsilon^2 h^2} + \frac{1}{\varepsilon} \le 2 \frac{1+\mu_k}{\varepsilon} \tag{4.22}$$

and thus

$$\left| \lambda^\varepsilon_{J^\varepsilon(p)} - \frac{\pi^2}{\varepsilon^2 h^2} - \frac{\mu_k}{\varepsilon} \right| \le 2c_k (1+\mu_k)^2 \varepsilon^{-1/2}. \tag{4.23}$$

Let us formulate a theorem based on the above considerations.

**Theorem 4.2.** *For all $k \in \mathbb{N}$ there exist positive numbers $\varepsilon_k$ and $C_k$ such that the estimate*

$$\left| \lambda_k^\varepsilon - \frac{\pi^2}{\varepsilon^2 h^2} - \frac{\mu_k}{\varepsilon} \right| \le \frac{C_k}{\sqrt{\varepsilon}} \tag{4.24}$$

*holds for the eigenvalues* $\lambda_k^\varepsilon$ *and* $\mu_k$ *(cf.* (1.10), (2.6)*) of the problems* (1.9) *and* (2.5), *respectively, and for all* $\varepsilon \in (0, \varepsilon_k]$.

In the case all eigenvalues of the limit problem are simple, the proof is given by the arguments in this section (formula (4.23)) and Section 3.3.. However, in the general case, the subindex $J^\varepsilon(p)$ of $\lambda^\varepsilon_{J^\varepsilon(p)}$ is not yet shown to be equal to $k$. This gap will be removed and the proof thus completed in the next section, see Remark 5.1.

## 5. Asymptotics and localization of eigenfunctions.

**5.1. Treating multiple eigenvalues.** We first complete the proof of the previous theorem by additional remarks on possible multiple eigenvalues of the limit problem. Thus, we assume first that $\mu_k$ is as in (4.7) an eigenvalue of multiplicity $\varkappa_k \geq 1$ and prove that there exist a least $\varkappa_k$ eigenvalues $\tau^\varepsilon_{J^\varepsilon(k)}, \dots, \tau^\varepsilon_{J^\varepsilon(k)+\varkappa_k-1}$ of the operator $\mathcal{T}^\varepsilon$ satisfying the estimate

$$\left|\tau_p^\varepsilon - \frac{\varepsilon}{1+\mu_k}\right| \leq \mathbf{C}_k \varepsilon^{3/2} \tag{5.1}$$

for some constant $\mathbf{C}_k$, which may be large but still independent of $\varepsilon$ . To this end we apply the second part of Lemma 4.1 with $t_1 = \mathbf{C}_k\varepsilon^{3/2}$ and get the coefficient vector

$$a^\varepsilon_{(p)} = (a^\varepsilon_{p,J^\varepsilon(k)}, \dots, a^\varepsilon_{p,J^\varepsilon(k)+K^\varepsilon(k)-1}) \in \mathbb{R}^{K^\varepsilon(k)}, \quad p = k, \dots, k+\varkappa_k - 1, \tag{5.2}$$

such that

$$\|\Phi_p^\varepsilon - \mathbf{S}_p^\varepsilon; \mathbf{H}^\varepsilon\| \leq \frac{2\delta_p}{\mathbf{C}_k\varepsilon^{3/2}} \leq \frac{2c_k}{\mathbf{C}_k}, \tag{5.3}$$

$$\mathbf{S}_p^\varepsilon = \sum_{j=J^\varepsilon(k)}^{J^\varepsilon(k)+K^\varepsilon(k)-1} a^\varepsilon_{p,j}\varphi^\varepsilon_j, \qquad |a^\varepsilon_{(p)}| = 1. \tag{5.4}$$

Here, $\tau^\varepsilon_{J^\varepsilon(k)}, \dots, \tau^\varepsilon_{J^\varepsilon(k)+K(k)^\varepsilon-1}$ are all of the eigenvalues of the operator $\mathcal{T}^\varepsilon$ in the segment

$$[\varepsilon(1+\mu_k)^{-1} - \mathbf{C}_k\varepsilon^{3/2}, \varepsilon(1+\mu_k)^{-1} + \mathbf{C}_k\varepsilon^{3/2}], \tag{5.5}$$

and at the end of the inequality (5.3) we used the estimate (4.20). Now (5.3), (4.12) imply, for $p, q = k, \dots, k+\varkappa_k-1$,

$$\begin{aligned}&\left|(a^\varepsilon_{(p)}, a^\varepsilon_{(q)})_{\mathbb{R}^{K^\varepsilon(k)}} - \delta_{p,q}\right| = \left|\langle \mathbf{S}_p^\varepsilon, \mathbf{S}_q^\varepsilon\rangle_\varepsilon - \delta_{p,q}\right| \\ &= \left|\langle \mathbf{S}_p^\varepsilon, \mathbf{S}_q^\varepsilon - \Phi_q^\varepsilon\rangle_\varepsilon + \langle \mathbf{S}_p^\varepsilon - \Phi_p^\varepsilon, \Phi_q^\varepsilon\rangle_\varepsilon + \langle \Phi_p^\varepsilon, \Phi_q^\varepsilon\rangle_\varepsilon - \delta_{p,q}\right| \leq \frac{4c_k}{\mathbf{C}_k} + C_k\varepsilon^{1/2}.\end{aligned} \tag{5.6}$$

Thus, in the case of a small enough $\varepsilon$ and a large enough $\mathbf{C}_k$ the vectors (5.2) are mutually "almost orthogonal" and thus at least linearly independent (cf. (4.6) for the normalization in $\mathbb{R}^{K^\varepsilon(k)}$). This is possible only, if $K^\varepsilon(k) \geq \varkappa_k$.

**Remark 5.1.** Combining (5.1) and (4.3) shows that every eigenvalue (1.10) of the problem (1.9) is in a $C_k\varepsilon^{-1/2}$-neighborhood of some point $\pi^2\varepsilon^{-2}h^{-2} + \varepsilon^{-1}\mu_k$. This proves Lemma 3.2.

The relation $J^\varepsilon(k) \geq k$ and Theorem 3.5 imply the equality $J^\varepsilon(k) = k$, hence, the proof of Theorem 4.2 is completed, too.

**5.2. Asymptotics of eigenfunctions.** In this section we formulate a result on the asymptotic behavior of the eigenfunctions $\varphi_p^\varepsilon = (\tau_k^\varepsilon)^{1/2}u_k^\varepsilon$, see (4.4). Recall that the functions $\Psi_p^\varepsilon$ are localized and satisfy exponential decay estimates as a consequence of (4.9) and Proposition 2.4.

**Theorem 5.2.** *Let $\mu_k$ be an eigenvalue of the problem* (2.5) *with multiplicity $\varkappa_k \geq 1$, cf.* (2.4), (4.7). *There exist positive numbers $\varepsilon_k$, $c_k$ and an orthonormal sequence of vectors $\{b_k^\varepsilon, \dots, b_{k+\varkappa_k-1}^\varepsilon\} \subset \mathbb{R}^{\varkappa_k}$, such that the eigenfunctions $\varphi_k^\varepsilon, \dots, \varphi_{k+\varkappa_k-1}^\varepsilon$ admit the estimates*

$$\Big\| \varphi_p^\varepsilon - \frac{2}{\sqrt{h\varepsilon}}(1+\mu_k)^{-1/2} \sum_{q=k}^{k+\varkappa_k-1} b_{(q)}^\varepsilon \Psi_p^\varepsilon; \mathbf{H}^\varepsilon \Big\| \leq c_k \sqrt{\varepsilon} \quad \forall\, \varepsilon \in (0, \varepsilon_k]. \tag{5.7}$$

*Here, $p \in \{k, \dots, k+\varkappa_k-1\}$, the norm of $\mathbf{H}^\varepsilon$ is as in* (4.1), *the eigenfunctions $\varphi_p^\varepsilon$ are orthonormalized in $\mathbf{H}^\varepsilon$, the functions $\Psi_p^\varepsilon$ are defined in* (4.9) *and $w_k, \dots, w_{k+\varkappa_k-1}$ denote the eigenfunctions of the limit equation* (1.16) *subject to the orthogonality and normalization conditions* (2.7).

**Proof.** In view of the proof in Section 5.1. and Theorem 3.5, we can choose for every $k$ the number $t_1 = \mathbf{c}_k\varepsilon$ in Lemma 4.1 such that the segment

$$[\varepsilon(1+\mu_k)^{-1} - \mathbf{c}_k\varepsilon, \varepsilon(1+\mu_k)^{-1} + \mathbf{c}_k\varepsilon], \tag{5.8}$$

contains the interval (5.5) and the eigenvalues $\tau_k^\varepsilon, \dots, \tau_{k+\varkappa_k-1}^\varepsilon$ but no other eigenvalues of the operator $\mathcal{T}^\varepsilon$. In this way we obtain $J^\varepsilon(k) = k$ and $K^\varepsilon(k) = \varkappa_k$ in (5.4), while the estimates (5.3) and (5.6) now take the form

$$\|\Phi_p^\varepsilon - \mathbf{S}_p^\varepsilon; \mathbf{H}^\varepsilon\| \leq 2\frac{c_k}{\mathbf{c}_k}\sqrt{\varepsilon} \tag{5.9}$$

and

$$\Big| (a_{(p)}^\varepsilon, a_{(q)}^\varepsilon)_{\mathbb{R}^{\varkappa_k}} - \delta_{p,q} \Big| \leq C_k \sqrt{\varepsilon}, \tag{5.10}$$

hence, these bounds vanish as $\varepsilon \to 0^+$. We now use them to estimate the remainder terms in the asymptotic presentation of eigenfunctions, below.

First, by (4.8), (4.9) and (4.11), we can rewrite (5.9) as

$$\Big\| \frac{2}{\sqrt{h\varepsilon}}(1+\mu_k)^{-1/2}\Psi_p^\varepsilon - \mathbf{S}_p^\varepsilon; \mathbf{H}^\varepsilon \Big\| \leq 2\frac{c_k}{\mathbf{c}_k}\sqrt{\varepsilon}. \tag{5.11}$$

Second, the definition (4.1), (3.7), and the Friedrichs inequality (3.9) yield for the function (4.9)

$$\begin{aligned} &\|\nabla_y \Psi_p^\varepsilon; L^2(\Omega_\varepsilon)\|^2 + \varepsilon \|\partial_z \Psi_p^\varepsilon; L^2(\Omega_\varepsilon)\|^2 \\ &+ \frac{1}{\varepsilon}\|\Psi_p^\varepsilon; L^2(\Omega_\varepsilon)\|^2 + \frac{1}{\varepsilon^2}\|r\Psi_p^\varepsilon; L^2(\Omega_\varepsilon)\|^2 \leq c\|\Psi_p^\varepsilon; \mathbf{H}^\varepsilon\|^2. \end{aligned} \tag{5.12}$$

Third, direct calculations together with (4.9) and the exponential decay of $w_p(\eta)$ as $\rho \to +\infty$ show that each term on the left-hand side of (5.12) is of order $\varepsilon$. We also remark that

$$\|\mathbf{S}_p^\varepsilon; \mathbf{H}^\varepsilon\| = O(1) \tag{5.13}$$

due to (4.4) and (5.4). Finally, we observe that the $\varkappa_k \times \varkappa_k$-matrix

$$a^\varepsilon = \Big( a_{(k)}^\varepsilon, \dots, a_{(k+\varkappa_k-1)}^\varepsilon \Big)$$

is "almost orthogonal" due to (5.10), and thus there exists an orthogonal matrix $b^\varepsilon$ such that $b^\varepsilon a^\varepsilon$ differs from the unit $\varkappa_k \times \varkappa_k$-matrix by $O(\sqrt{\varepsilon})$ in the standard matrix norm (see, e.g., [37, Sect. 7.1] and [26, Lemma 1.5]). Taking for $b_{(q)}^\varepsilon$ the columns of $b^\varepsilon$ and putting together (5.4), (5.9)–(5.13) yields (5.7). ⊠

**5.3. Localization effect revisited.** Theorem 5.2 can be written for the eigenfunctions $u_k^\varepsilon$, (1.11), by using the estimate (5.12). However, although the estimate (5.7) is in a sense even asymptotically sharp, it does not yet prove the desired localization effect: recall that we expect the eigenfunctions $u_k^\varepsilon$ to be exponentially small as a function of the distance to the maximum thickness point. The bound (5.7) would only tell that the difference of $u_k^\varepsilon$

from a function with such a decay property would be small in some norm. This kind of result is already contained in Theorem 3.1 because of the exponentially growing weight on the left-hand side of (3.2).

In the next theorem we assume that the profile functions $H_\pm$ are smooth and derive estimates for the eigenfunctions in a weighted parameter-dependent Hölder norm

$$\begin{aligned} \|u; C^{l,\alpha}_\varepsilon(\Omega_\varepsilon)\| &= \sum_{j=0}^{l} \varepsilon^j \sup_{x\in\Omega_\varepsilon} |\nabla^j_x u(x)| \\ &+ \varepsilon^{l+\alpha} \sup_{x\in\Omega_\varepsilon} \sup_{\mathbf{x}\in\Omega_\varepsilon:\, |x-\mathbf{x}|\le\varepsilon} \left(|x-\mathbf{x}|^{-\alpha}|\nabla^j_x u(x) - \nabla^j_{\mathbf{x}} u(\mathbf{x})|\right), \end{aligned} \tag{5.14}$$

cf. (2.18). Since the edges $\upsilon_\pm = \{x : y \in \partial\omega, z = \pm\varepsilon H_\pm(y)\}$ of the plate (1.5) still cause boundary irregularities, we will give the estimates only in a subset of $\Omega_\varepsilon$ not including the the edge $\partial\omega \times \{0\}$. We mention that weighted Hölder estimates were extended up to the edges of an elastic cylindrical plate in the paper [34], but for simplicity we do not repeat that consideration here.

**Theorem 5.3.** *Assume that the profile functions $H_\pm$ are $C^\infty$-smooth. For any $l \in \mathbb{N}$, $\alpha \in (0,1)$ and $d > 0$, there exist positive numbers $\varepsilon_k$ and $c_k$ such that for all $\varepsilon \in (0,\varepsilon_k]$ the eigenfunction $u^\varepsilon_k$, (1.11), satisfies*

$$\|\exp(\varepsilon^{-1}br^2)u^\varepsilon_k; C^{l+1,\alpha}_\varepsilon(\Omega^\varepsilon(d))\| \le c_k\varepsilon^{-3/2}. \tag{5.15}$$

*Here $b$ is a positive constant, $\Omega_\varepsilon(d) := \{x \in \Omega_\varepsilon : y \in \omega_{\varepsilon d}\}$ and $\omega_{\varepsilon d} := \{y \in \omega : \operatorname{dist}(y, \partial\omega) > \varepsilon d\}$.*

**Proof.** We fix a point $y^0 \in \omega_{\varepsilon d}$ and the discs $B^p = \{y : |y - y^0| < p\varepsilon d/2\}$, $p = 1, 2$. The change of variables

$$x \mapsto (\eta, \zeta) = (\varepsilon^{-1}(y - y^0), \varepsilon^{-1}z)$$

transforms the small cells $\Xi^p_\varepsilon = \{x \in \Omega_\varepsilon : y \in B^p\} \subset \Omega_\varepsilon$ into the cells of unit size

$$\widehat{\Xi}^p_\varepsilon = \{(\eta,\zeta) : |\eta| \le pd, -H_-(y^0 + \varepsilon\eta) < \zeta < H_+(y^0 + \varepsilon\eta)\}, \quad p = 1, 2.$$

Furthermore, the function

$$\widehat{\Xi}^2_1 \ni (\eta,\zeta) \mapsto U^\varepsilon_k(\eta,\zeta) = u^\varepsilon_k(y^0 + \varepsilon\eta, \varepsilon\zeta)$$

vanishes at the surfaces $\widehat{\xi}^{2\pm}_\varepsilon(y^0) = \{(\eta,\zeta) : |\eta| \le 2\varepsilon d, \zeta = \pm H_\pm(y^0 + \varepsilon\eta)\}$ and satisfies the equation

$$\Delta_{(\eta,\zeta)}U^\varepsilon_k(\eta,\zeta) + \varepsilon^2\lambda^\varepsilon_k U^\varepsilon_k(\eta,\zeta) = F^\varepsilon_k(\eta,\zeta) := 0, \quad (\eta,\zeta) \in \widehat{\Xi}^2_\varepsilon.$$

Local elliptic estimates [2] for solutions of boundary-value problems in domains with smooth boundaries show that

$$\begin{aligned} \|U^\varepsilon; C^{l+1,\alpha}(\widehat{\Xi}^1_\varepsilon(y^0))\| &\le C\left(\|F^\varepsilon; C^{l-1,\alpha}(\widehat{\Xi}^2_\varepsilon(y^0))\| + \|U^\varepsilon; L^2(\widehat{\Xi}^2_\varepsilon(y^0))\|\right) \\ &= C\|U^\varepsilon; L^2(\widehat{\Xi}^2_\varepsilon(y^0))\|. \end{aligned} \tag{5.16}$$

The constant $C$ in (5.16) can be chosen independently of $\lambda^\varepsilon_k$, $\varepsilon$ and $y^0$, because $\varepsilon^2\lambda^\varepsilon_k > 0$ is bounded, see (3.15), and the bases $\widehat{\xi}^{2\pm}_\varepsilon(y^0)$ of the cell $\widehat{\Xi}^2_\varepsilon(y^0)$ are gently sloping and dependent smoothly on $y^0$. Defining the weight function

$$E^\varepsilon_B(\eta) = \exp(\varepsilon^{-1}b|y^0 + \varepsilon\eta|^2)$$

as in Remark 2.2 with $b = b_1 = b_2$, its derivatives are uniformly bounded for all $\eta \in \widehat{\Xi}^2_\varepsilon(y^0)$ and $\varepsilon \in (0,\varepsilon_k]$. This weight can thus be inserted into all norms in the estimate (5.16), and we obtain

$$\|E^\varepsilon_B U^\varepsilon; C^{l+1,\alpha}(\widehat{\Xi}^1_\varepsilon)\| \le C_B\|E^\varepsilon_B U^\varepsilon; L^2(\widehat{\Xi}^2_\varepsilon)\|. \tag{5.17}$$

It suffices to change back to the original coordinates $x = (y^0 + \varepsilon\eta, \varepsilon\zeta)$ and observe the following facts. First, this coordinate change turns the standard Hölder norms in (5.17) into the $\varepsilon$-dependent Hölder norm of (5.14). Second, applying the inequality (3.2) with the weight function (3.3) and $B(y) = b|y|^2$ yields

$$\|E_B^\varepsilon U^\varepsilon; L^2(\widehat{\Xi}^2_\varepsilon)\| \le \varepsilon^{-3/2}\|\mathcal{E}_B u^\varepsilon; L^2(\Xi^2_\varepsilon)\| \le c_k \varepsilon^{-3/2},$$

where the coefficient $\varepsilon^{-3/2}$ comes from the coordinate dilation. ⊠

The exponential factor $\exp(\varepsilon^{-1}br^2)$ in (5.15) shows that the eigenfunction $u_k^\varepsilon(x)$ is exponentially small outside any neighborhood of the coordinate origin. The large coefficient $\varepsilon^{-3/2}$ on the right hand side in (5.15) is caused by the $L^2(\Omega_\varepsilon)$-normalization. Derivatives of $j$th order gain the additional large factor $\varepsilon^{-j}$ (cf. (5.14)) as a consequence of scaling, or, the use of the intrinsic stretched coordinates.

## 6. Generalizations and open questions

**6.1. Mixed boundary value problem.** In this section we consider a problem related to (1.7), (1.8) but having different, namely, mixed boundary conditions, see (6.1)–(6.3). The asymptotic behavior of the eigenvalues differs very much from the problem (1.7)–(1.8), as shown by Theorem 6.1. In addition, the case of a mirror symmetric plate is considered in Theorem 6.2 with the help of an auxiliary spectral problem (6.13) with artificial boundary conditions. We include a discussion on the results and the existing literature in Remark 6.3.

So, we consider in the thin domain (1.5) the problem

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Omega^\varepsilon, \tag{6.1}$$

$$\partial_\nu u^\varepsilon(x) = 0, \quad x \in \Sigma^\varepsilon_\pm, \tag{6.2}$$

$$u^\varepsilon(x) = 0, \quad x \in \Sigma^\varepsilon_0 := \partial\Omega^\varepsilon \setminus (\Sigma^\varepsilon_- \cup \Sigma^\varepsilon_+), \tag{6.3}$$

where $\Sigma^\varepsilon_\pm$ denotes the plate bases (1.6), $\Sigma^\varepsilon_0 = \partial\Omega^\varepsilon \setminus (\Sigma^\varepsilon_- \cup \Sigma^\varepsilon_+)$ the lateral sides of the plate (1.12), and $\partial_\nu$ is the directional derivative along the outward normal on $\Sigma^\varepsilon_\pm$. The variational formulation of the problem (6.1)–(6.3) reads as

$$(\nabla_x u^\varepsilon, \nabla_x v^\varepsilon)_{\Omega^\varepsilon} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Omega^\varepsilon} \qquad \forall \quad v^\varepsilon \in H^1_0(\Omega^\varepsilon; \Sigma^\varepsilon_0), \tag{6.4}$$

where $H^1_0(\Omega^\varepsilon; \Sigma^\varepsilon_0)$ is a subspace of functions in the Sobolev space $H^1(\Omega^\varepsilon)$ which satisfy the Dirichlet condition (6.3) on $\Sigma^\varepsilon_0$. We use in this section the notation $\lambda^\varepsilon_k$, $k = 1, 2, \dots$, also for the eigenvalues of the problem (6.4); they form a positive monotone unbounded sequence as in (1.10). The corresponding eigenfunctions, still denoted by $u^\varepsilon_k$, can be subject to the same normalization and orthogonality conditions as in (1.11).

It is well known that the eigenvalues convergence,

$$\lambda^\varepsilon_k \to \beta_k \quad \text{as } \varepsilon \to 0,$$

where $\beta_k$ is the $k$th eigenvalue in the spectrum

$$0 < \beta_1 < \beta_2 \le \beta_3 \le \dots \le \beta_k \le \dots \le\to +\infty \tag{6.5}$$

of the two-dimensional Dirichlet limit problem

$$-\nabla_y \cdot H(y)\nabla_y \varphi(y) = \beta H(y)\varphi(y), \; y \in \omega, \quad w(y) = 0, \; y \in \partial\omega.$$

Notice that the variational form of this problem is

$$(H\nabla_y \varphi, \nabla_y \psi)_\omega = \beta(H\varphi, \psi)_\omega \quad \forall \; \psi \in H^1_0(\omega; \partial\omega).$$

The normalization and orthogonality conditions for the eigenfunctions are

$$(H\varphi_j, \varphi_k)_\omega = \delta_{j,k}, \quad j, k \in \mathbb{N}.$$

Let us formulate the following theorem from [37, Ch. 7], the proof of which uses the procedures of direct and inverse reduction. The proof given in [37] requires smoothness of the profiles $H_\pm$, but only minor modifications are needed to cover the case of piecewise smooth, continuous $H_\pm$ in (1.2), (1.3). (For more details, see also [36, 35].) This result gives estimates for the asymptotic remainders in the expansions of the eigenvalues $\lambda_k^\varepsilon$ and the eigenfunctions $u_k^\varepsilon$. The dependence on the eigenvalue $\beta_k$, its multiplicity $\varkappa_k$ and other attributes of the limit spectrum (6.5) is explicitly presented in these estimates. We also mention the papers [32, 26, 20], where this procedure has been applied to other singularly perturbed problems.

If $\beta_k$ is an eigenvalue of multiplicity $\varkappa_k$, i.e.,

$$\beta_{k-1} < \beta_k = \ldots = \beta_{k+\varkappa_k-1} < \beta_{k+\varkappa_k}, \tag{6.6}$$

then the *relative distance* $d_k$ to other eigenvalues is defined by

$$d_k = \min\left\{\frac{\beta_k}{\beta_{k-1}} - 1, 1 - \frac{\beta_k}{\beta_{k+\varkappa_k}}\right\}.$$

**Theorem 6.1.** *There exist positive numbers $\varepsilon_0$ and $c_0$, $C_0$ depending only on $\omega$ and $H$ such that the following statements hold true for the eigenvalue sequence $\{\lambda_p^\varepsilon\}_{p=1}^\infty$ of the problem* (6.4)*.*

*Let us consider a $k \in \mathbb{N}$ and the corresponding eigenvalue $\beta_k$.*

1) *If $\varepsilon$ satisfies*

$$0 < \varepsilon \le \varepsilon_0 \varkappa_k^{-1} \beta_k^{-1}, \tag{6.7}$$

*then there are elements $\lambda_j^\varepsilon, \ldots, \lambda_{j+\varkappa_k-1}^\varepsilon$ such that*

$$|\lambda_p^\varepsilon - \beta_k| \le c_0 \varepsilon^{1/2} \varkappa_k \beta_k^{3/2}, \quad p = j, \ldots, j + \varkappa_k - 1. \tag{6.8}$$

2) *If $\varepsilon$ is so small that*

$$0 < \varepsilon \le \varepsilon_0 \varkappa_k^{-2} \left(1 + \frac{1}{d_k}\right)^{-2} \beta_k^{-1}, \tag{6.9}$$

*then the interval*

$$\left[\beta_k - c_0 \varepsilon^{1/2} \beta_k^{3/2}, \beta_k + c_0 \varepsilon^{1/2} \beta_k^{3/2}\right] \tag{6.10}$$

*contains the eigenvalues $\lambda_k^\varepsilon, \ldots, \lambda_{k+\varkappa_k-1}^\varepsilon$ and no other eigenvalues. Hence, $j = k$ in* (6.8)*.*

3) *If the slightly stronger assumption*

$$0 < \varepsilon \le \varepsilon_0 \varkappa_k^{-2} \left(1 + \frac{1}{d_k}\right)^{-2} (\beta_k + \beta_{k+\varkappa_k})^{-1}, \tag{6.11}$$

*holds, then the interval*

$$\left[\frac{1}{2}(\beta_k + \beta_{k-1}), \frac{1}{2}(\beta_k + \beta_{k+\varkappa_k})\right] \tag{6.12}$$

*does not include any eigenvalue $\lambda_p^\varepsilon$ with $p < k$ or $p \ge k + \varkappa_k$ either.*

4) *If* (6.11) *holds, then there exist an orthonormal sequence $\left(b^j(\varepsilon)\right)_{j=k}^{k+\varkappa_k-1}$ of vectors in $\mathbb{R}^{\varkappa_k}$, $b^j(\varepsilon) = (b_k^j(\varepsilon), \ldots, b_{k+\varkappa_k-1}^j(\varepsilon))$, such that the following estimates are valid:*

$$\left\| u_j^\varepsilon - \varepsilon^{-1/2} \sum_{p=k}^{k+\varkappa_k-1} b_p^j(\varepsilon) \varphi_p ; H^1(\Omega^\varepsilon) \right\|$$
$$\le C_0 \varepsilon^{1/2} \varkappa_k \left(1 + \frac{1}{d_k}\right) \beta_k, \quad j = k, \ldots, k + \varkappa_k - 1.$$

We comment this result in Remark 6.3, below.

We next consider the eigenvalues of the problem (6.1)–(6.3) in the special case of a plate $\Omega^\varepsilon$ which is mirror symmetric with respect to the plane $\{z = 0\}$, that is, $H_+ = H_-$ in (1.5). In this case it is natural to pose artificial boundary conditions, [9], on the central cross-section $\Gamma_0^\varepsilon = \{x \in \Omega^\varepsilon : z = 0\}$, and we are thus led to the auxiliary spectral problem

$$\begin{aligned} -\Delta_x u_\wedge^\varepsilon(x) &= \lambda_\wedge^\varepsilon u_\wedge^\varepsilon(x), \quad x \in \Omega_\wedge^\varepsilon, \\ \partial_\nu u_\wedge^\varepsilon(x) &= 0, \quad x \in \Sigma_+^\varepsilon, \\ u_\wedge^\varepsilon(x) &= 0, \quad x \in \Sigma_\wedge^\varepsilon \cup \Gamma_0^\varepsilon, \end{aligned} \tag{6.13}$$

where $\Omega_\wedge^\varepsilon = \{x \in \Omega^\varepsilon : z > 0\}$ and $\Sigma_\wedge^\varepsilon = \{x \in \Sigma_0^\varepsilon : z > 0\}$. Since every function $v_\wedge \in H_0^1(\Omega_\wedge^\varepsilon; \Sigma_\wedge^\varepsilon \cup \Gamma_0^\varepsilon)$ vanishes on the central plane, one can extend such a $v_\wedge$ as an odd function $v_\Diamond$ in $z$ to the entire plate $\Omega^\varepsilon$, and as a consequence $v_\Diamond \in H_0^1(\Omega^\varepsilon; \Sigma_0^\varepsilon)$. Thus, any eigenpair $\{\lambda_\wedge^\varepsilon, u_\wedge^\varepsilon\}$ of the problem (6.13) gives rise to an eigenpair $\{\lambda_\wedge^\varepsilon, u_\Diamond^\varepsilon\}$ of the problem (6.1)–(6.3).

The mixed boundary value problem (6.13) has all the properties of the Dirichlet problem (1.7)–(1.8), which are necessary for the arguments and results in Sections 2–5, except for some minor modification; we return to this in the next section, when treating a straightforward generalization of (6.13). So, the following result concerning the eigenvalues $\lambda_k^\varepsilon$ of the problem (6.1)–(6.3) can be proven in the same way as Theorem 4.2 (see also Remark 6.5).

**Theorem 6.2.** *Assume that $H_+ = H_-$ holds in* (1.1) *and that the thickness function $H = 2H_+$ satisfies the conditions* (1.2) *and* (1.3). *Then, for every $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that for all $\varepsilon \in (0, \varepsilon_k]$ the eigenvalue sequence $\big(\lambda_p^\varepsilon\big)_{p=1}^\infty$ of the problem* (6.1)–(6.3) *has entries $\lambda_{K(\varepsilon)}^\varepsilon, \dots, \lambda_{K(\varepsilon)+\varkappa_k-1}^\varepsilon$ satisfying the relationship*

$$\left| \lambda_j^\varepsilon - \frac{\pi^2}{4\varepsilon^2 h^2} - \sqrt{2}\varepsilon^{-1}\mu_k \right| \le c_k \varepsilon^{-1/2}, \tag{6.14}$$

*where $j = K(\varepsilon), \dots, K(\varepsilon) + \varkappa_k - 1$, $\mu_k$ is an eigenvalue of problem* (2.5) *with multiplicity $\varkappa_k$ and the factor $c_k > 0$ is independent of $\varepsilon \in (0, \varepsilon_k]$.*

We emphasize that the eigenvalue index $K(\varepsilon)$ in (6.14) in general depends on $\varepsilon$, because the interval $(0, \frac{1}{4}\pi^2\varepsilon^{-2}h^2)$ contains indefinitely many eigenvalues $\beta_m$ when $\varepsilon \to 0^+$. Also, it is not possible to guarantee that the number of eigenvalues satisfying (6.14) equals $\varkappa_k$. Hence, part of the information on eigenfunctions in Theorem 5.2 is lost, when applied to the eigenfunctions of the problem (6.4).

**Remark 6.3.** Let us return to Theorem 6.1. There, the weakest of the assumptions for $\varepsilon$ is the first one (6.7), however, the conclusion does not ensure that $p = k$ in the estimate (6.8), and, moreover, there may still be other eigenvalues satisfying (6.8). To guarantee the localization and isolation of $\lambda_k^\varepsilon, \dots, \lambda_{k+\varkappa_k-1}^\varepsilon$ into the segments (6.10) and (6.12), respectively, one has to accept the much smaller bounds (6.9) and (6.11) for $\varepsilon$. In the last case we also have the asymptotic description of the eigenfunctions $u_k^\varepsilon, \dots, u_{k+\varkappa_k-1}^\varepsilon$. These observations suggest that the traditional asymptotic ansätze

$$\lambda_k^\varepsilon \sim \beta_k, \qquad u_k^\varepsilon(x) \sim \varepsilon^{-1/2}\alpha_k(\varepsilon)\varphi_k(y)$$

for the eigenpairs of the mixed boundary value problem (6.1)–(6.3) only work for a wide but certainly restricted range of the spectrum.

The above observed difference of the asymptotic behavior of lower and higher eigenvalues actually defines the higher frequency range of the spectrum, and in spite of the failure of the asymptotic expansions in the high-frequency range one may find eigenvalue sequences with other types of stable asymptotics. This phenomenon is discussed in some special cases in [41, 29, 37, 31, 32, 26, 27, 18].

We still mention other types of problems, namely, models arising from reinforcement problems, where the spectral parameter is related with the size of reinforced bands and the physical properties of the materials such as stiffness or density. In [18] and [21], the low, middle and high frequencies have been considered in very different situations. Depending on the problem and the geometry of the band, the middle frequencies can give rise to vibrations with energy localized along the interface between the media (cf. [18]), or they can give rise to vibrations localized near points of local maxima of the function defining the geometry of the reinforcing band. The latter type of localization is consider in [21], and it is similar to the one appearing in the present paper: see Section 1.4. for more details.

**6.2. Another mixed boundary value problem.** Let us extend the Dirichlet conditions over the lower base $\Sigma_-^\varepsilon$ and consider the problem

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Omega^\varepsilon, \tag{6.15}$$

$$\partial_\nu u^\varepsilon(x) = 0, \quad x \in \Sigma_+^\varepsilon, \tag{6.16}$$

$$u^\varepsilon(x) = 0, \quad x \in \Sigma_\sqcup^\varepsilon := \Sigma_0^\varepsilon \cup \Sigma_-^\varepsilon, \tag{6.17}$$

with the variational formulation

$$(\nabla_x u^\varepsilon, \nabla_x v^\varepsilon)_{\Omega^\varepsilon} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Omega^\varepsilon} \quad \forall\ v^\varepsilon \in H_0^1(\Omega^\varepsilon; \Sigma_\sqcup^\varepsilon).$$

Then the asymptotic ansätze (2.1), (2.2) are replaced by

$$\lambda^\varepsilon = \varepsilon^{-2}\frac{\pi^2}{4h^2} + \varepsilon^{-1}\mu + \ldots, \quad u^\varepsilon(x) = \sin\left(\frac{\pi}{2}\,\frac{z+\varepsilon H_-(y)}{\varepsilon H(y)}\right) w(\eta) + \ldots \tag{6.18}$$

**Remark 6.4.** Note that $\sin(\ldots)$ in (2.2) vanishes both for $z = \varepsilon H_+(y)$ and $z = -\varepsilon H_-(y)$, whereas $\sin(\ldots)$ in (6.18) vanishes only for $z = -\varepsilon H_-(y)$ and $\partial_z \sin(\ldots) = 0$ for $z = \varepsilon H_+(y)$. The last fact imitates the Neumann boundary conditions (6.16), which, in view of (1.6), can be written in the form

$$\begin{aligned}&\left(1+\varepsilon|\nabla_y H_+(y)|^2\right)^{-1/2}\big(\partial_z u^\varepsilon(y,z)\\ &-\varepsilon\nabla_y H_+(y)\cdot\nabla_y u^\varepsilon(y,z)\big)\big|_{z=\varepsilon H_+(y)} = 0.\end{aligned} \tag{6.19}$$

Since the stretching coefficient in (1.15) is equal to $\varepsilon^{-1/2}$, the subtrahend in (6.19) is of higher order than the derivative $\partial_z u^\varepsilon(y, \varepsilon H_+(y))$ in local variables. ⊠

Repeating the calculations leading to (1.16) with minor modifications and using the notation (1.17), yields the following limit differential equation

$$-\Delta_\eta w(\eta) + \frac{1}{4}A(\eta)w(\eta) = \mu w(\eta), \quad \eta \in \mathbb{R}^2. \tag{6.20}$$

The coordinate dilation $\eta \mapsto 2^{-1/2}\eta$ and the parameter change $\mu \mapsto 2\mu$ reduce (6.20) to the spectral equation (1.16). Then we use the eigenvalue sequence (2.6) to obtain the eigenvalue sequence of the spectral problem (6.20),

$$0 < \frac{1}{2}\mu_1 < \frac{1}{2}\mu_2 \le \frac{1}{2}\mu_3 \le \ldots \le \frac{1}{2}\mu_k \le \ldots \le \ldots \to +\infty.$$

**Remark 6.5.** In the artificial problem (6.13) we also need to replace $h$ by $h_\wedge = h/2$. Hence, the second term on the left of (6.20) becomes $2A(\eta)w(\eta)$. This explains the factor $\sqrt{2}$ in (6.14). ⊠

Asymptotics of eigenvalues and eigenfunctions of problem (6.15)–(6.17) can now be formulated as in Theorems 4.2 and 5.2 with obvious modifications.

**6.3. Concluding geometric remarks.** *(i) Continuity of the profile functions.* Since the eigenfunctions are localized near the maximum point $0 \in \omega$ of $H$, the considerations

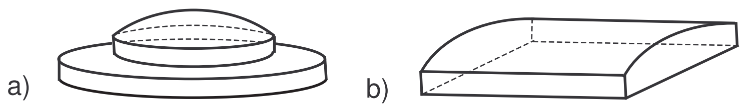


FIGURE 6.1. a) Thin domain permitting generalization, b) thin domain not suitable for generalization

in Sections 3–5 require the inclusion $H_\pm \in H^{1,\infty}$ only in some neighborhood of 0. Outside this neighborhood the functions $H_\pm$ may have jumps as depicted in Fig. 6.1, a.

*(ii) Local maxima.* Let us return to the Dirichlet problem (1.7), (1.8) and assume that the thickness function (1.1) has a local strict maximum at a point $y^\natural \in \omega \setminus \{0\}$, that is

$$H(y) = h_\natural - r_\natural^2 \mathcal{H}_\natural(\varphi_\natural) + O(r_\natural^3), \quad h_\natural \in (0, h),$$

where $(r_\natural, \varphi_\natural) \in \mathbb{R}_+ \times \mathbb{S}^1$ is the polar coordinate system centered at $y^\natural$ and, as in (1.3), $\mathcal{H}_\natural \in H^{1,\infty}(\mathbb{S}^1)$, $\mathcal{H}_\natural > 0$. Using the same argument as in Section 2.1, one can perform the formal asymptotic analysis and derive the following limit differential equation

$$-\Delta_{\eta^\natural} w^\natural(\eta^\natural) + A^\natural(\eta^\natural) w^\natural(\eta^\natural) = \mu^\natural w^\natural(\eta^\natural), \quad \eta^\natural \in \mathbb{R}^2, \tag{6.21}$$

where, similarly to (1.17),

$$A^\natural(\eta^\natural) = 2\pi^2 h_\natural^{-3} \rho_\natural^2 \mathcal{H}_\natural(\varphi_\natural)$$

and $\eta^\natural = \varepsilon^{-1/2}(y - y^\natural)$ are the stretched coordinates centered at $y^\natural$. By an appropriate affine transform, equation (6.21) reduces to the differential equation (1.16). We write the spectrum of (6.21) as

$$0 < \mu_1^\natural < \mu_2^\natural \le \mu_3^\natural \le \ldots \le \mu_k^\natural \le \ldots \le \ldots \to +\infty.$$

Note that in the special case $\mathcal{H}_\natural = c_\natural \mathcal{H}$ the eigenvalue $\mu_k^\natural$ is an explicit function of the numbers $\mu_k$, $h_\natural$, $a_{pq}^\natural$ and $c_\natural$. The eigenvalues have the expansions

$$\lambda_{K(\varepsilon)}^\varepsilon = \varepsilon^{-2} h_\natural^{-2} \pi^2 + \varepsilon^{-1} \mu_k^\natural + \widetilde{\lambda}_{K(\varepsilon)}^\varepsilon, \tag{6.22}$$

which look quite similar to (1.18), and they show that eigenvalue sequences have stable asymptotics in the high-frequency range of the spectrum. The justification procedure from Section 4 can be applied to derive the estimate $|\widetilde{\lambda}_{K(\varepsilon)}^\varepsilon| \le c_k \varepsilon^{3/2}$ for the remainder in (6.22), and assertions similar to Theorems 4.2 and 6.2 can also be proven. We recall that other eigenvalue sequences of this type have been discussed in Section 6.1.

*(iii) Limit problem in the half-plane.* The global maximum $h$ of the function $H$ may occur at a point $y^0 \in \partial\omega$. To treat this case let us assume that $H$ is smooth enough, for example, of class $H^{3,\infty}$, and that $y^0 = 0$ and the $y_1$-axis is tangent to the contour $\partial\omega$. Moreover, the relations (1.2), (1.3) are supposed to hold, when the circle $\mathbb{S}^1$ is replaced by the semi-circle $\mathbb{S}_+^1$. The limit problem then reads as

$$-\Delta_\eta w(\eta) + A(\eta) w(\eta) = \mu w(\eta), \quad \eta \in \mathbb{R}_+^2 = (0, \infty) \times \mathbb{R}, \tag{6.23}$$

$$w(0, \eta_2) = 0, \quad \eta_2 \in \mathbb{R}, \tag{6.24}$$

where the positive function $A$ is still given by (1.17). Actually all eigenvalues of the problem (6.23), (6.24) have already been listed in Proposition 2.1, because the even extension of the function

$$\mathbb{R}^2_+ \ni \eta \mapsto A(\eta) \tag{6.25}$$

to the entire plane $\mathbb{R}^2$ and the simultaneous odd extension of a solution $w$ convert the Dirichlet problem in $\mathbb{R}^2_+$ into the single differential equation (1.16). In other words, the spectrum of the problem (6.23), (6.24) is composed of those eigenvalues (2.6), which are associated with eigenfunctions that are odd in $\eta_2$.

The boundary condition (6.24) is inherited from the Dirichlet condition (1.8) on the lateral side $\Gamma^\varepsilon$ of the plate $\Omega^\varepsilon$. To treat the corresponding Neumann condition instead of (6.24), one starts with the mixed boundary value problem composed of the differential equation (1.7) and the boundary conditions

$$u^\varepsilon(x) = 0,\ x \in \Sigma^\varepsilon_\pm, \qquad \partial_\nu u^\varepsilon(x) = 0,\ x \in \Sigma^\varepsilon_0. \tag{6.26}$$

Then, the Neumann boundary condition in (6.26) gives rise to the Neumann condition

$$\frac{\partial w}{\partial \eta_1}(0, \eta_2) = 0, \quad \eta_2 \in \mathbb{R},$$

to be combined with the equation (6.23). Again, we have already seen the spectrum of this limit Neumann problem: by a consideration analogous to the one in the previous paragraph, it consists of those eigenvalues $\mu_k$ in (2.6) for which the corresponding eigenfunctions are even in $\eta_2$.

We emphasize that the curvature $\kappa$ of the contour $\partial\omega$ has no effect on the asymptotic and justification procedures in Sections 2–5. Indeed, the Laplace operator $\Delta_y$ reads in the curvilinear coordinates $(n, s)$ as

$$(1 + n\kappa(s))^{-1}\frac{\partial}{\partial n}(1 + n\kappa(s))\frac{\partial}{\partial n} + (1 + n\kappa(s))^{-1}\frac{\partial}{\partial n}(1 + n\kappa(s))^{-1}\frac{\partial}{\partial n}. \tag{6.27}$$

Here, $n$ is the oriented distance from the contour $\partial\omega$, $n > 0$ inside $\omega$, and $s$ is the arc length calculated from the point $y^0 = (0, 0)$ along $\partial\omega$ anticlockwise. After the change of variables

$$(n, s) \mapsto \eta = (\eta_1, \eta_2) = (\varepsilon^{-1/2}n, \varepsilon^{-1/2}s),$$

the main asymptotic part $\varepsilon^{-1}\Delta_\eta$ of the differential operator (6.27) appears on the left-hand side of the limit equation (1.16). Moreover, the next term $\varepsilon^{-1/2}\kappa(0)\partial/\partial\eta_1$ in the asymptotic decomposition of (6.27) is small in comparison and thus does not exist in limit problem. The eigenfunctions $w_k$ in Proposition 2.4 still have an exponential discrepancy

$$|(\Delta_x - \varepsilon^{-1}\Delta_\eta)w_k(\eta)| \le c_k\varepsilon^{-1/2}\exp(-\varepsilon^{-1}(B_1^{1/2}n^2 + B_2^{1/2}s^2)),$$

which can be estimated along the scheme in Section 4.

If the boundary $\partial\omega$ is piecewise smooth and the maximum of $H$ occurs at a corner point, then a limit equation similar to (6.23) is to be posed in the corresponding unbounded corner domain. However, this generalization is quite straightforward and we skip a detailed discussion of it.

*(iv) A different type of global maximum.* Let

$$\begin{aligned} &H(y) < H(0) =: h \quad \text{for} \quad y \in \overline{\omega} \setminus \{0\}, \\ &H(y) = h - r^\kappa \mathcal{H}(\varphi) + O(r^{\kappa+1}). \end{aligned} \tag{6.28}$$

Then, the coordinate dilation (1.15) has to be replaced by

$$y \mapsto \varepsilon^{-\alpha}y \quad \text{with} \quad \alpha = \frac{2}{2 + \kappa}.$$

To study the eigenvalues $\lambda_k^\varepsilon$ and the corresponding eigenfunctions $u_k^\varepsilon(x)$ of the problem (1.9), we first notice that a calculation similar to that in Section 2.1 yields the limit equation

$$-\Delta_\eta w(\eta) + 2\pi^2 h^{-3} \rho^\kappa \mathcal{H}(\varphi) w(\eta) = \mu w(\eta), \quad \eta \in \mathbb{R}^2.$$

This can be used to determine the perturbation term in the expansion

$$\lambda_k^\varepsilon = \varepsilon^{-2} \frac{\pi^2}{h^2} + \varepsilon^{-4/(2+\kappa)} \mu_k + \widetilde{\lambda}_k^\varepsilon$$

and also the last multiplier $w_k(\eta)$ in the asymptotic ansatz (2.2) for $u_k^\varepsilon(x)$. We predict the following estimate for the eigenvalues

$$\left| \widetilde{\lambda}_k^\varepsilon \right| \leq c_k \varepsilon^{-2/(2+\kappa)},$$

yet, we refrain to formulate this as a rigorous result for the case (6.28), since the approach of our paper would inevitably lead to many additional cumbersome formulas requiring some further arguments. We also refer to papers [5, 6] containing the case (1.24) with a positive homogeneous polynomial of degree $\kappa = 2m$.

*(v) Open questions.* We finally mention two cases where even the formal asymptotic ansätze for the eigenpairs of the problem (1.7), (1.8) remain unclear.

In the first case we assume that

$$H(y) = h - a_1 y_1^2 - a_2 y_2^{2m} + O(r^{2m+1}) \quad \text{with} \quad a_1, a_2 > 0, \ m > 1.$$

In the formal asymptotic procedures in Section 2.1 the term $a_2 y_2^{2m}$ would be ignored, and we would be lead to the following limit differential equation,

$$-\Delta_\eta w(\eta) + 4\pi^2 h^{-3} a_1 \eta_1^2 w(\eta) = \mu w(\eta), \quad \eta \in \mathbb{R}^2, \tag{6.29}$$

which looks quite similar to (1.16). Thus, it would be natural to introduce the Hilbert space $\mathbf{H}$ with the weighted norm

$$\|w; \mathbf{H}\| = (\|\nabla_\eta w; L^2(\mathbb{R}^2)\|^2 + \|(1 + |\eta_1|^2)^{1/2} w; L^2(\mathbb{R}^2)\|^2.$$

However, the spectrum of the operator of problem (6.29) cannot be discrete due to the following observation: If $\chi \in C_c^\infty(\mathbb{R}^2)$ is a function with a support in the unit square $(-1/2, 1/2)^2$, the functions $\eta \mapsto \chi_q(\eta_1, \eta_2 - q)$, $q \in \mathbb{N}$, have the properties

$$\|\chi_q; L^2(\mathbb{R}^2)\| = c_\chi \neq 0, \tag{6.30}$$

$$\|\chi_q; \mathbf{H}\| \leq C_\chi, \qquad \mathrm{supp}\chi_q \cap \mathrm{supp}\chi_p = \varnothing \quad \text{for} \quad q \neq p, \ q, p \in \mathbb{N}. \tag{6.31}$$

Using (6.31) one can find a subsequence $\{\chi_{q_j}\}$ which converges to null weakly in $\mathbf{H}$, but this subsequence cannot converge to null in the norm of $L^2(\mathbb{R}^2)$, due to (6.30). In other words, the embedding $\mathbf{H} \subset L^2(\mathbb{R}^2)$ cannot be compact, and the spectrum is thus not discrete (cf. [4, Theorem 10.5.1])

Second, let us assume that

$$\begin{aligned} &H(y) < h \quad \text{for} \quad y \in \omega \setminus \gamma, \\ &H(y) = h - \frac{1}{2} a_0 \, \mathrm{dist}(y, \gamma)^2 + O(\mathrm{dist}(y, \gamma)^3), \quad a_0 > 0, \end{aligned}$$

i.e. the maximum $h$ is attained by the thickness function (1.1) along a simple smooth closed contour $\gamma$ inside the domain $\omega$. It is quite probable that a modification of the ansätze developed in [10, 33, 36] could be used to describe the asymptotic behavior ($\varepsilon \to 0^+$) of the eigenpairs of the spectral problem (1.7), (1.8), however, a much more elaborate analysis is needed to confirm these hypothesis. We finally mention that asymptotic ansätze become incomprehensible, if $\gamma$ is a smooth open curve with ends in $\partial\omega$, see Fig. 6.1, b, or even a criss-cross curve.

**6.4. Spectral gaps for the Dirichlet problem in a thin infinite layer.** Let $\omega$ be the rectangle $\{y : -l_m < y_m < l_m, m = 1, 2\}$ with sides of length $2l_m > 0$. We assume that the profile functions $H_\pm$ satisfy the requirements (1.1), (1.2) and (1.3) in $\omega$, and in addition that the periodic extensions of $H_\pm$ to the entire plane $\mathbb{R}^2$ are continuous, for simplicity (cf. Section 6.3 (i) and papers [36, 35]). We consider the spectral Dirichlet problem

$$-\Delta U^\varepsilon(x) = \Lambda^\varepsilon U^\varepsilon(x), \quad x \in \Pi^\varepsilon, \tag{6.32}$$

$$U^\varepsilon(x) = 0, \quad x \in \partial\Pi^\varepsilon, \tag{6.33}$$

in the thin layer

$$\Pi^\varepsilon = \left\{(y, z) : y \in \mathbb{R}^2, -\varepsilon H_-(y) < z < \varepsilon H_+(y)\right\}. \tag{6.34}$$

This problem is associated with a positive definite self-adjoint unbounded operator $\mathbf{A}^\varepsilon$, see, e.g., [4, Ch. 10]. By e.g. [13, 42, 14] it is known that the spectrum $\sigma^\varepsilon = \sigma(\mathbf{A}^\varepsilon)$ of problem (6.33) has the band-gap structure

$$\sigma^\varepsilon = \bigcup_{k\in\mathbb{N}} \sigma_k^\varepsilon, \quad \sigma_k^\varepsilon = \{\Lambda = \lambda_k^\varepsilon(\xi) : \xi_m \in [0, \pi/l_m),\ m = 1, 2\}, \tag{6.35}$$

where $\xi = (\xi_1, \xi_2)$ denotes the dual variable of the Gelfand transform [17] and the numbers $\lambda_k^\varepsilon(\xi)$ are entries of the eigenvalue sequence

$$0 < \lambda_1^\varepsilon(\xi) \le \lambda_2^\varepsilon(\xi) \le \cdots \le \lambda_k^\varepsilon(\xi) \le \cdots \to +\infty \tag{6.36}$$

of the following model problem in the periodicity cell, or, the curved prism $\Omega^\varepsilon$, (1.5):

$$-\Delta u^\varepsilon(x;\xi) = \lambda^\varepsilon u^\varepsilon(x;\xi), \quad x \in \Omega^\varepsilon, \tag{6.37}$$

$$u^\varepsilon(y, \pm\varepsilon H_\pm(y);\xi) = 0, \quad y \in \omega = (-l_1, l_1) \times (-l_2, l_2), \tag{6.38}$$

$$\begin{aligned} u^\varepsilon(l_1, y_2, z;\xi) &= e^{2i\xi_1 l_1} u^\varepsilon(-l_1, y_2, z;\xi), \quad y_2 \in (-l_2, l_2),\\ u^\varepsilon(y_1, l_2, z;\xi) &= e^{2i\xi_2 l_2} u^\varepsilon(y_1, -l_2, z;\xi), \quad y_1 \in (-l_1, l_1), \end{aligned} \tag{6.39}$$

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial y_1}(l_1, y_2, z;\xi) &= e^{2i\xi_1 l_1} \frac{\partial u^\varepsilon}{\partial y_1}(-l_1, y_2, z;\xi), \quad y_2 \in (-l_2, l_2),\\ \frac{\partial u^\varepsilon}{\partial y_2}(y_1, l_2, z;\xi) &= e^{2i\xi_2 l_2} \frac{\partial u^\varepsilon}{\partial y_2}(y_1, -l_2, z;\xi), \quad y_1 \in (-l_1, l_1). \end{aligned} \tag{6.40}$$

The problem is formally self-adjoint due to the quasi-periodicity conditions (6.39), (6.40) with the real parameters $\xi_1$ and $\xi_2$, and its variational formulation is

$$(\nabla u^\varepsilon(\cdot;\xi), \nabla v^\varepsilon(\cdot;\xi))_{\Omega^\varepsilon} = \lambda^\varepsilon (u^\varepsilon(\cdot;\xi), v^\varepsilon(\cdot;\xi))_{\Omega^\varepsilon}\ \forall\ v^\varepsilon(\cdot;\xi) \in \mathbf{H}^\varepsilon(\Omega^\varepsilon;\xi), \tag{6.41}$$

where $\mathbf{H}^\varepsilon(\Omega^\varepsilon;\xi)$ is the subspace of functions $v^\varepsilon(\cdot;\xi) \in H_0^1(\Omega^\varepsilon;\Sigma_\pm^\varepsilon)$ satisfying the Dirichlet conditions (6.38) and the stable quasi-periodicity conditions (6.39). The variational problem (6.41) is associated with a positive definite operator $\mathbf{A}^\varepsilon(\xi)$ in $L^2(\Omega^\varepsilon)$. Since the embedding $H^1(\Omega^\varepsilon) \subset L^2(\Omega^\varepsilon)$ is compact, the spectrum of $\mathbf{A}^\varepsilon(\xi)$ is discrete and forms the eigenvalue sequence (6.37), where the eigenvalues are listed according to their multiplicities (see, e.g., [4, Theorems 10.1.5 and 10.2.2]). Furthermore, each of the functions

$$[0, \pi/l_1) \times [0, \pi/l_2) \ni \xi \mapsto \lambda_k^\varepsilon(\xi)$$

is evidently continuous and $(\pi/l_m)$-periodic in $\xi_m$ so that the spectral bands $\sigma_k^\varepsilon$ in (6.35) are closed connected bounded intervals.

The eigenvalues (6.36) of the problem (6.41) can be investigated in the same way as the eigenvalues (1.10) of the problem (1.9). Replacing the Dirichlet condition by the quasi-periodicity conditions (6.39), (6.40) on the lateral side $\Sigma_0^\varepsilon$ of $\Omega^\varepsilon$, (1.12), does not have an effect on the formal asymptotic analysis (Sections 2 and 6.2) and the justification scheme (Sections 3–5). Even more importantly, all bounds in the estimates of the previous sections can be proven independently of $\xi \in [0, \pi/l_1) \times [0, \pi/l_2)$. We refrain to formulate an assertion on the asymptotics of the eigenvalues $\lambda_k^\varepsilon(\xi)$ of the problem (6.41), but just refer to the result (4.24), Theorem 4.2, which can easily be adapted to this case as well.

We emphasize that the asymptotic terms $\pi^2\varepsilon^{-2}h^{-2}$ and $\mu_k\varepsilon^{-1}$ in (4.24) are independent of $\xi$. Moreover, the quasi-periodicity conditions (6.39), (6.40) are imposed at a fixed distance from the point $y^0$, i.e., in a set where the eigenfunctions do not localize. Hence, the parameter $\xi$ has only a very small effect on the eigenvalues so that the inequalities

$$\left|\lambda_k^\varepsilon(\xi) - \lambda_k^\varepsilon(\xi')\right| \le c_k \exp(-\varepsilon^{-1}\delta), \tag{6.42}$$

are valid for some $\delta > 0$, for all $\xi, \xi' \in [0,\pi/l_1) \times [0,\pi/l_2)$. The inequality (6.42) can be proven from the estimates of this paper by a rather simple argument found e.g. in [36]. Due to (6.42) and (4.24), the length $|\sigma_k^\varepsilon|$ of the spectral band $\sigma_k^\varepsilon$ in (6.35) can be estimated by $c_k \exp(-\varepsilon^{-1/2}\delta)$, and the band on the other hand satisfies

$$\sigma_k^\varepsilon \subset \left(\frac{\pi^2}{\varepsilon^2 h^2} + \frac{\mu_k}{\varepsilon} - \frac{c_k}{\sqrt{\varepsilon}}, \frac{\pi^2}{\varepsilon^2 h^2} + \frac{\mu_k}{\varepsilon} + \frac{c_k}{\sqrt{\varepsilon}}\right). \tag{6.43}$$

These two observations together with the unboudedness and monotonicity of the eigenvalue sequence (2.6) imply that the number of open spectral gaps grows to the infinity as $\varepsilon \to 0^+$. (By a spectral gap we mean an interval in $\mathbb{R}_+$ which is free of the spectrum $\sigma^\varepsilon$, but has endpoints in it.)

The above observation is based on the the original idea of [15], where the two-dimensional periodic thin strip was considered and the sequence (2.6) consisted of the simple eigenvalues of the harmonic oscillator (1.25). In our case the eigenvalues of the problem (2.5) may become multiple, which complicates the asymptotic description of the band-gap structure (6.35). Indeed, if $\mu_k$ has multiplicity $\varkappa_k$ so that

$$\mu_{k-1} < \mu_k = \ldots = \mu_{k+\varkappa-1} < \mu_{k+\varkappa},$$

cf. (6.6), then the inclusions (6.43) guarantee the existence of two gaps: one gap of length $\varepsilon^{-1}(\mu_k - \mu_{k-1}) + O(\varepsilon^{-1/2})$ between the bands $\sigma_{k-1}^\varepsilon$ and $\sigma_k^\varepsilon$, and another one of length $\varepsilon^{-1}(\mu_{k+\varkappa_k} - \mu_k) + O(\varepsilon^{-1/2})$ between the bands $\sigma_{k+\varkappa_k-1}^\varepsilon$ and $\sigma_{k+\varkappa_k}^\varepsilon$. However, our asymptotic formulas do not suffice to make conclusions on the existence of non-empty gaps between the bands $\sigma_k^\varepsilon, \ldots, \sigma_{k+\varkappa_k-1}^\varepsilon$. The existence of these gaps are a matter of higher order terms in the asymptotics of $\lambda_k^\varepsilon(\xi), \ldots, \lambda_{k+\varkappa_k-1}^\varepsilon(\xi)$, as was demonstrated in paper [36] dealing with a different geometric situation, see fig. 2,a.. We mention that Theorem 1 of the paper [6] presents several explicit asymptotic terms of those eigenvalues, and this result could be used to detect the gaps in the case (1.4) holds for the thickness function $H$; the justification of the asymptotics remains to be worked out.

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