SINGULAR PERTURBATION DIRICHLET PROBLEM IN A DOUBLE-PERIODIC PERFORATED PLANE

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Abstract. We show that the spectrum of the Dirichlet problem for the Laplace operator $-\Delta$ in the plane $\mathbb{R}^2$ perforated by a double-periodic family of holes contains any a priori number of gaps, for sufficiently large holes. While this result was already known in the case of circular holes, we consider here a more general geometric setting with holes of the shape $\{|x_1|^\mu + |x_2|^\mu \le r\}$, $1 < \mu < \infty$.

1. Introduction

This work is a generalization of the results contained in the paper [7], which deals with a singular perturbation problem for the spectrum of the Dirichlet-Laplacian in the plane $\mathbb{R}^2$ with doubly periodic perforation. The motivation of the problem arises from a physical example modelled by the linear water wave equation, namely the propagation of surface waves over a layer of an ideal fluid with a double-periodic family of obstacles, which was studied in the article [5]. It has turned out that the structure of the spectrum may be interesting from both the mathematical and physical points of view, since in some geometric situations spectral gaps may occur. As is well known, existence of spectral gaps has immediate effects on the wave's propagation in the given medium: waves can not propagate in the frequency range corresponding to gaps. A similar effect of the geometry to the spectrum may also arise for example in the framework of acoustic waves.

The reference [7] only concentrates on circular holes. We aim to extend the results to more general geometric settings, in which more complicated singularities can arise at the boundary of the limit domain (terminology explained in the sequel). We refer to [7] for a more thorough exposition on the background and literature.

We consider the Dirichlet problem for the Helmholtz equation

$$(D) \qquad \begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Pi_R^\mu, \\ u(x) = 0, & x \in \partial\Pi_R^\mu, \end{cases}$$

where Δ is the Laplace operator with respect to the variable $x \in \Pi_R^\mu \subset \mathbb{R}^2$ and $\lambda \in \mathbb{C}$ is the spectral parameter. The domain Π_R^μ is a doubly periodic perforated plane defined as follows. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ be a multi-index, $1 \le \mu < +\infty$ and $\mathcal{B}_R^\mu$ be the disk in $\mathbb{R}^2$ defined by

$$\mathcal{B}_R^\mu = \{x = (x_1, x_2) \in \mathbb{R}^2 : (x_1^\mu + x_2^\mu)^{1/\mu} = R\}.$$

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We set

$$\mathcal{B}_R^\mu\Big(\alpha+\frac{1}{2}\Big)=\Big\{x=(x_1,x_2):\Big(x_1-\alpha_1-\frac{1}{2},x_2-\alpha_2-\frac{1}{2}\Big)\in\mathcal{B}_R^\mu\Big\},$$

and let Π_R^μ be the plane $\mathbb{R}^2$ perforated by the periodic family of μ-depending holes $(\mathcal{B}_R^\mu(\alpha+1/2))_{\alpha\in\mathbb{Z}^2}$, i.e.

$$\Pi_R^\mu=\mathbb{R}^2\setminus\bigcup_{\alpha\in\mathbb{Z}^2}\overline{\mathcal{B}_R^\mu\Big(\alpha+\frac{1}{2}\Big)}.$$

Let Q be the unit square in $\mathbb{R}^2$ with center in the origin. We define, for $j=1,2:$

$$S_R^{j+}=\{x\in\partial Q:x_j=\frac{1}{2},|x_{3-j}|<\frac{1}{2}-R\},$$

and

$$S_R^{j-}=\{x\in\partial Q:x_j=-\frac{1}{2},|x_{3-j}|<\frac{1}{2}-R\}.$$

Finally we set

$$T_R\equiv\partial Q_R\setminus\bigcup_{j=1,2}(\overline{S_R^{j+}}\cup\overline{S_R^{j-}}).$$

To underline the singular perturbation features of our problem we switch from the parameter $R\in]0,1/2[$ to the parameter

$$\epsilon=\frac{1}{2}-R>0,$$

and we define the periodicity cell:

$$Q^\mu:=Q_\epsilon^\mu=\Bigg\{x\in Q:\Big(\Big|x_1-\frac{1}{2}\Big|^\mu+\Big|x_2-\frac{1}{2}\Big|^\mu\Big)^{1/\mu}>\frac{1}{2}-\epsilon\Bigg\}.$$

In the limit case of $\epsilon=0$, we get the bell-flower domain

$$Q_0^\mu=\Bigg\{x\in Q:\Big(\Big|x_1-\frac{1}{2}\Big|^\mu+\Big|x_2-\frac{1}{2}\Big|^\mu\Big)^{1/\mu}>\frac{1}{2}\Bigg\}.$$

Obviously, the domains Q^μ are Lipschitz but Q_0^μ is not. The main result we prove is the following

Theorem 1.1. *Given $N\in\mathbb{N}$, there exists $\epsilon(N)>0$ such that for $0<\epsilon\le\epsilon(N)$ the spectrum σ of the problem (D) has at least N gaps $[a_j,b_j]$, $j=1,\dots,N$, where $a_j<b_j$, $\sigma\cap[a_j,b_j]=\emptyset$ and $\sigma\cap[b_{j-1},a_j]\ne\emptyset$ for all j.*

The theorem will be a corollary of the more detailed statements of Theorem 3.2 and Corollary 3.3. The general band-gap structure of the spectrum of (D) is explained in the sequel, see (2) and (3).

Notation. By c, C, C' etc. (respectively, C_n) we denote positive constants independent of the variables, parameters or functions (resp. depending on the parameter n) in the given expressions. However, most of the quantities in the following depend also on the number μ, but this will usually not be displayed. By $(f,g)_\Omega$ we denote the $L^2(\Omega)$-inner product of functions $f,g\in L^2(\Omega)$. We use standard notation for Sobolev spaces, in particular $H^1(\Omega)$ denotes the Sobolev-Hilbert space of $L^2(\Omega)$-functions with weak derivatives in $L^2(\Omega)$, and $H_0^1(\Omega)$ is its subspace consisting of functions with vanishing traces on the boundary.

2. Problems on bounded domains.

The Floquet-Bloch transform, also known as the Gelfand transform, is defined for functions on Π_R^μ by

$$u \mapsto U(y,\eta) = \frac{1}{2\pi} \sum_{\alpha\in\mathbb{Z}^2} e^{-i\eta\cdot(x+\alpha)} u(x+\alpha)\ ,\ y \in Q^\mu.$$

For more information about this, see e.g. [7], [6]. This transforms the problem (D) into the following parameter $\eta \in [-\pi,\pi[\times[-\pi,\pi[$ -dependent problem in the cell Q^μ,

$$(D') \qquad \begin{cases} \Delta U(y,\eta) = \Lambda(\eta)U(y,\eta), & y \in Q_\epsilon, \eta \in [-\pi,\pi[^2, \\ U(y,\eta) = 0, & y \in T_\epsilon, \\ U|_{S_\epsilon^{j+}} = e^{i\eta_j} U|_{S_\epsilon^{j-}}, & j = 1,2, \end{cases}$$

where $\Lambda(\eta)$ is a spectral parameter and its connection with λ is explained below. Let $\mathcal{H}^\epsilon(\eta)$ be the subspace of $H^1(Q_\epsilon)$ satisfying the η-dependent boundary conditions on T_ϵ and $S_\epsilon^{j\pm}$ of the problem (D'). Clearly this is a closed subspace of $H^1(Q_\epsilon)$ for the topology induced by the standard inner product of $H^1(Q_\epsilon)$, and thus it is itself a Hilbert space, for all η. The weak solutions of (D') are the solutions in $\mathcal{H}^\epsilon(\eta)$ of the variational identity

$$(1) \qquad (\nabla U(\cdot,\eta), \nabla V)_{Q_\epsilon} = \Lambda(\eta)(U(\cdot,\eta),V)_{Q_\epsilon} \quad \forall\ V \in \mathcal{H}^\epsilon(\eta).$$

Note that the quadratic form on the left-hand side is closed and non-negative with dense domain $\mathcal{H}^\epsilon(\eta) \subset L^2(Q_\epsilon)$. By the quadratic form theory for semibounded differential operators (see e.g. [9]), the problem (1) can be associated with a self-adjoint operator $A(\eta)$ on $\mathcal{H}^\epsilon(\eta)$, the spectrum of which is discrete due to the compactness of the embedding $\mathcal{H}^\epsilon(\eta) \to L^2(Q_\epsilon)$. Hence, the spectrum of the problem (D') is discrete for each fixed η, and consists of the increasing sequence

$$(2) \qquad 0 \le \Lambda_1^\epsilon(\eta) \le \Lambda_2^\epsilon(\eta) \le \Lambda_3^\epsilon(\eta) \le \cdots \to \infty$$

of eigenvalues, multiplicities taken into account. We can also assume that the set of eigenfunctions $(U_n^\epsilon(\eta))_{n\in\mathbb{N}}$ associated with the identity (1) is a complete orthonormal set in $L^2(Q_\epsilon)$.

It is also known that the functions

$$\eta \mapsto \Lambda_n^\epsilon(\eta)$$

are continuous (see e.g. [10, Chapter 6]), so that the sets $\Upsilon_n = \{\Lambda_n^\epsilon(\eta) : \eta \in [-\pi,\pi[^2\}$ are (possibly overlapping) closed real intervals. The Floquet-Bloch-Gelfand theory (see e.g. [11], [12]) states that the spectrum σ of the original problem (D) and the spectra of the problems (D') are related by

$$(3) \qquad \sigma = \bigcup_{n\in\mathbb{N}} \Upsilon_n.$$

Our main result will follow by proving estimates for the endpoints of the intervals Υ_n.

We next consider the limit problem, which formally corresponds to the case $\epsilon = 0$. Note that the domain Q_0^μ is no more Lipschitz, since four μ-power outer cusps appear at its boundary. We identify the vertices of these cusps with θ^{j+}, θ^{j-}, $j = 1,2$, where

$$\theta^{1\pm} = \Big(\pm\frac{1}{2}, 0\Big)\ , \quad \theta^{2\pm} = \Big(0, \pm\frac{1}{2}\Big).$$

We define also the Hilbert space $\mathcal{H}_0$ as the subspace of $H^1(Q_0)$ such that

$$V(x) = 0 \quad \text{in the trace sense, for all } x \in T^0 = \partial Q_0 \setminus \{\theta^{j\pm} : j = 1, 2\}.$$

The variational setting for the limit problem (D_0') in Q_0 is

$$(\nabla U^0, \nabla V)_{Q_0} = \Lambda^0 (U^0, V)_{Q_0} \quad \forall V \in \mathcal{H}_0 \tag{4}$$

where U^0 is to be found in $\mathcal{H}_0$, too. We want to show that the non-negative self-adjoint operator A^0 on $\mathcal{H}_0$ associated with the closed quadratic form on the left hand-side of (4) has discrete spectrum. According to the theory of densely defined and symmetric quadratic forms (see e.g. [9, Chp.10]), it is sufficient to show that $\mathcal{H}_0$ is compactly embedded in $L^2(Q_0)$. This is not trivial, because the functions in $\mathcal{H}_0$ might be unbounded in a neighbourhood of $\theta^{j\pm}$ in ϖ_0.

In the following we define, for a given $\delta > 0$, the Lipschitz domain

$$Q_0(\delta) = \{x \in Q_0 : |x_j| < \frac{1}{2} - \delta,\ j = 1, 2\},$$

and $R_0(\delta) = Q_0 \setminus \overline{Q_0(\delta)}$

Proposition 2.1. *The embedding operator*

$$\iota : \mathcal{H}_0 \hookrightarrow L^2(Q_0)$$

can be represented for all $\delta > 0$ as the sum of the compact operator $K : H_0^1(Q_0(\delta)) \to L^2(Q_0)$ and a bounded operator $L : H^1(R_0(\delta)) \to L^2(R_0(\delta))$ with the estimate $\|J\| = O(\delta^\mu)$ for its operator norm. In particular ι is a compact operator.

Proof. We use here the ideas and tricks developed in [7]. We divide this proof to two steps.
Step 1. We first derive a weighted estimate (6). Let us write the Cartesian coordinates $y = (y_1, y_2)$ in such a way that the peak θ^{1+} is locally given by the relations

$$y_1 > 0, \qquad |y_2| < Y_0(y_1) \equiv \frac{1}{2} - \left|\frac{1}{2^\mu} - |y_1|^\mu\right|^{\frac{1}{\mu}}.$$

Using the Taylor expansion for $y_1 = 0$, we get

$$Y_0(y_1) = c(\mu) 2^{\mu-1} y_1^\mu + O(y_1^{2\mu}), \qquad Y_0(y_1) \geq y_1^\mu. \tag{5}$$

Note that if $U \in \mathcal{H}_0$, then $U(y_1, \cdot)$ lies in $H_0^1([-Y_0(y_1), Y_0(y_1)])$ for almost all $y_1 > 0$. Thus we can use the one-dimensional Poincaré inequality to get

$$\frac{1}{Y_0(y_1)^2} \int_{-Y_0(y_1)}^{Y_0(y_1)} |U(y)|^2 \,\mathrm{d}y_2 \leq c \int_{-Y_0(y_1)}^{Y_0(y_1)} \left|\frac{\partial U}{\partial y_2}(y)\right| \mathrm{d}y_2, \quad U \in \mathcal{H}_0$$

We now introduce a smooth positive function ρ on the set $Q_0 \setminus \{\theta^{j\pm} : j = 1, 2\}$ such that ρ coincides with $\mathrm{dist}(x, \partial Q_0)$ for all x in a neighbourhood of $\{\theta^{j\pm} : j = 1, 2\}$ in Q_0. We then integrate the former equality with respect to y_1 to get

$$\left\|\frac{U}{\rho^\mu}\right\|^2_{L^2(Q_0)} \leq c_1 \int_0^1 \frac{1}{Y_0(y_1)^2} \int_{-Y_0(y_1)}^{Y_0(y_1)} |U(y)|^2 \,\mathrm{d}y_2 \mathrm{d}y_1 \leq 4cc_1 \|\nabla U\|^2_{L^2(Q_0)}, \tag{6}$$

where $U \in \mathcal{H}_0$, and the first inequality is trivial for big y_1, whereas for $y_1 \to 0$ we have to use the Taylor expansion (5).

Step 2. To complete the proof we assume $\delta > 0$ is given and we define for all the non-negative cut-off function $\chi_\delta \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that

$$\chi_\delta(x) = \begin{cases} 1, \text{ if } |x - \theta^{j\pm}| \le \delta, \; j = 1, 2, \\ 0, \text{ if } |x - \theta^{j\pm}| > 2\delta, \; j = 1, 2. \end{cases}$$

Let $U \in H^1(Q_0)$. We write

$$U = \chi_\delta U + (1 - \chi_\delta)U \equiv \chi_\delta U + g$$

where $g \in H^1(Q_0)$ has support in Q_δ, which is a Lipschitz domain. By the Rellich-Kondrachov theorem we already know that the embedding of $H^1(Q_\delta)$ in $L^2(Q_0)$ is compact. Thus we only need to prove that $\chi_\delta U$ has L^2-norm of order δ^μ. Note that

$$\begin{aligned} \|\chi_\delta U\|^2_{L^2(Q_0)} &= \int_{Q_0} |\chi_\delta U|^2 \, \mathrm{d}x \le \int_{R_0(2\delta)} \rho^{2\mu} |\rho^{-\mu} U|^2 \, \mathrm{d}x \\ &\le (2\delta)^{2\mu} \int_{R_0(2\delta)} |\rho^{-\mu} U|^2 \mathrm{d}x \le c\delta^{2\mu} \|U\|^2_{H^1(Q_0)} \le C\delta^{2\mu}, \end{aligned}$$

which concludes the proof. □

Consequently, the spectrum of the self-adjoint operator associated with the limit problem (4) is discrete. We denote the unbounded sequence of eigenvalues and the corresponding eigenfunctions by

$$(\Lambda_n^0)_{n\in\mathbb{N}} \quad , \quad (U_n)_{n\in\mathbb{N}}, \tag{7}$$

and keep in mind the orthonormalization $(U_m, U_n)_{Q_0} = \delta_{m,n}$.

The following result can be proven as Proposition 2.1 in [7].

Proposition 2.2. *For any $n \in \mathbb{N}$ and $h \in \mathbb{R}$, $h \ge 0$, the weighted norms*

$$\|\rho^{-h} \nabla U_n^0\|_{L^2(Q_0)}, \qquad \|\rho^{-h-\mu} U_n^0\|_{L^2(Q_0)},$$

are finite. Here ρ is the smooth function introduced in the proof of Proposition 2.1.

At the end of this section we give an argument showing that the eigenfunctions U_n^0 actually vanish at the vertices $\theta^{j\pm}$. A proof for this can be constructed by following the references [1], [2], [3], [8], but for the sake of simplicity we collect the details here. Notice that under our geometric assumptions, the cusps of the limit domain can become very sharp for large μ. Thus the Sobolev embedding $H^2(Q_0) \hookrightarrow C(Q_0)$ can fail, as pointed out in [4, Thm 8.2.1]. Nevertheless by standard elliptic regularity theory we know that the eigenfunctions lie in all the spaces $H^k(Q_0)$, for all $k \in \mathbb{N}$. This implies that for each $\mu > 1$ there exists a sufficiently big $k \in \mathbb{N}$ such that the embedding $H^k(Q_0) \hookrightarrow C(Q_0)$ holds. From this and Corollary 8.2.1 in [4] we deduce that the eigenfunctions are also in $C(\overline{Q_0})$.

We continue by the following exponential decay argument. Let the numbers ϵ and δ be fixed such that $0 < \epsilon < 2\epsilon < \delta < 1$, let $r(x) = \min_{j\pm} \operatorname{dist}(x, \theta^{j\pm})$ and let $E(x)$ be the regularized distance function defined by

$$E(x) = \begin{cases} e^{\beta/\epsilon}, & \text{if } r(x) < \epsilon, \\ e^{\beta/r(x)}, & \text{if } \epsilon \le r(x) < \delta, \\ e^{\beta/\delta}, & \text{if } r(x) \ge \delta, \end{cases}$$

for all $x \in Q_0$, $\beta > 0$. If u is an eigenfunction of the limit problem (4), it is clear that the function $E^2 u$ is an admissible test function in the equality (4) as well. Setting also $U = Eu$ there we get that

$$\Lambda \int_{Q_0} E^2 u^2 \, \mathrm{d}x = \int_{Q_0} (\nabla u) \cdot \nabla (E^2 u) \, \mathrm{d}x = \int_{Q_0} (\nabla u) \cdot \nabla (EU) \, \mathrm{d}x$$

$$= \int_{Q_0} E(\nabla u) \cdot \nabla U \, \mathrm{d}x + \int_{Q_0} U(\nabla u) \cdot \nabla E \, \mathrm{d}x = \int_{Q_0} |\nabla U|^2 \, \mathrm{d}x - \int_{Q_0} E(\nabla u) \cdot \nabla U \, \mathrm{d}x$$

$$+ \int_{Q_0} E(\nabla u) \cdot \nabla U \, \mathrm{d}x - \int_{Q_0} |\nabla E|^2 u^2 \, \mathrm{d}x,$$

Hence,

$$\int_{Q_0} |\nabla U|^2 \, \mathrm{d}x = \int_{Q_0} |\nabla E|^2 \frac{U^2}{E^2} \, \mathrm{d}x + \Lambda \int_{Q_0} E^2 u^2 \, \mathrm{d}x.$$

Now we use the easy inequalities $|\nabla E(x)|^2 \leq \beta^2 r(x)^{-4} E(x)^2$ and $e^{\beta/\delta} \leq E(x) \leq e^{\beta/\epsilon}$ for all $x \in Q_0$, to get

$$\int_{Q_0} |\nabla U|^2 \, \mathrm{d}x \leq \beta^2 \int_{Q_0} \frac{U^2}{r^4} \, \mathrm{d}x + \Lambda e^{2\beta/\epsilon} \int_{Q_0} u^2 \, \mathrm{d}x.$$

The weight inequality (6) and the boundedness of $r(x)$ yield

$$(c - \beta^2) \int_{Q_0} \frac{U^2}{r^4} \, \mathrm{d}x \leq \Lambda e^{2\beta/\epsilon} \int_{Q_0} u^2 \, \mathrm{d}x$$

for some constant $c > 0$ independent of ϵ. In particular, choosing β such that $0 < \beta^2 < c$ we obtain

$$e^{-\frac{2\beta}{\epsilon}} \int_{Q_0} \frac{U^2}{r^4} \, \mathrm{d}x \leq c_\lambda < \infty.$$

Hence, both of the integrals

$$e^{-\frac{2\beta}{\epsilon}} \int\limits_{Q_0 \cap \{x : r(x) \leq e^{-\beta/2\epsilon}\}} U^2 r^{-4} \, \mathrm{d}x \quad , \quad e^{-\frac{2\beta}{\epsilon}} \int\limits_{Q_0 \cap \{x : r(x) > e^{-\beta/2\epsilon}\}} U^2 r^{-4} \, \mathrm{d}x,$$

are bounded for all $\epsilon > 0$. The first one gives

$$\int\limits_{Q_0 \cap \{x : r(x) \leq e^{-\beta/2\epsilon}\}} U^2 \, \mathrm{d}x \leq e^{-\frac{2\beta}{\epsilon}} \int\limits_{Q_0 \cap \{x : r(x) \leq e^{-\beta/2\epsilon}\}} U^2 r^{-4} \, \mathrm{d}x < \infty,$$

for all $\epsilon > 0$. This implies that u has a very fast exponential decay in L^2 norm in a neighbourhood of the cusps, thus locally pointwise at the cusp vertices. Thus the eigenfunctions are also in $C^\infty(\overline{Q_0})$.

3. Opening gaps in the spectrum of the Dirichlet problem

We first prove upper and lower estimates, cf. (8) and (17) and Theorem 3.2, for the endpoints of the spectral bands of the problem (D') in Q_ϵ. These will be enough to verify the existence of the spectral gaps. The idea is to show that the eigenvalues $\Lambda_n^\epsilon(\eta) \in \Upsilon_n$ can be approximated by the limit problem eigenvalue Λ_n for small ϵ.

We start by stating an upper bound: the inequality

$$\Lambda_n^\epsilon(\eta) \le \Lambda_n^0 \tag{8}$$

holds for all $n \ge 1$, for all $\eta \in [-\pi, \pi[^2$. The proof is similar to the one in [7], but for the convenience of the reader we sketch the proof.

Recall the max-min formula for the eigenvalues of the ϵ-perturbed problem,

$$\Lambda_n^\epsilon(\eta) = \max_{\mathcal{H}_n^\epsilon(\eta)} \inf_U \frac{(\nabla U, \nabla U)_{Q_\epsilon}}{(U, U)_{Q_\epsilon}},$$

where the infimum is taken over non-zero functions $U \in \mathcal{H}_n^\epsilon(\eta)$ and $\mathcal{H}_n^\epsilon(\eta)$ is an arbitrary linear subspace of $\mathcal{H}^\epsilon(\eta)$ of codimension $n-1$.
In view of the remark at the end of the previous section it is possible to extend by zero the eigenfuctions $(U_k^0)_k$ of the limit problem in Q_0 to the larger domain Q_ϵ, and with a little abuse of notation we use the symbols U_k^0 also for these extensions. It is easy to verify that $U_k^0 \in \mathcal{H}^\epsilon(\eta)$ for all k. Since $\{U_k^0 : k \ge 1\}$ was a complete orthonormal set, the extended functions $U_1^0, \ldots, U_n^0$ are still linearly independent. Thus, each subspace $\mathcal{H}_n^\epsilon(\eta)$ in the max-min principle contains a non-trivial combination

$$\mathbf{U} = \sum_{k=1}^n a_k U_k^0, \quad \sum_{k=1}^n |a_k|^2 = 1,$$

where the coefficients a_k depend on the subspace $\mathcal{H}_n^\epsilon(\eta)$. We recall that the eigenvalues $(\Lambda_k^0)_{k=1}^n$ associated with the eigenfunctions $U_1^0, \ldots, U_n^0$ are taken in increasing order. Using this fact we substitute the function $\mathbf{U}$ in the max-min principle, getting the following chain of inequalities:

$$\begin{aligned}\Lambda_n^\epsilon(\eta) &\le \max_{\mathcal{H}_n^\epsilon(\eta)} \frac{(\nabla \mathbf{U}, \nabla \mathbf{U})_{Q_\epsilon}}{(\mathbf{U}, \mathbf{U})_{Q_\epsilon}} = \max_{\mathcal{H}_n^\epsilon(\eta)} \frac{\sum_{k,j=1}^n a_k \overline{a_j} (\nabla U_k^0, \nabla U_j^0)_{Q_0}}{\sum_{k=1}^n |a_k|^2 (U_k^0, U_k^0)_{Q_0}} \\ &= \max_{\mathcal{H}_n^\epsilon(\eta)} \frac{\sum_{k=1}^n |a_k|^2 \Lambda_k^0}{\sum_{k=1}^n |a_k|^2} \le \Lambda_n^0.\end{aligned}$$

Here we have used the normalization condition $(U_k^0, U_k^0)_{Q_0} = 1$ for all k, cf. (7). This yields the upper bound (8), and we note here that this reasoning is valid for all $\mu > 1$, according to the regularity remark at the end of the previous Section.

The proof of the lower estimate, see (17), is more complicated. We begin by writing the Rayleigh quotient for the eigenvalues of the limit problem in Q_0:

$$\Lambda_n^0 = \max_{\mathcal{H}_n^0} \inf_U \frac{(\nabla U, \nabla U)_{Q_0}}{(U^0, U^0)_{Q_0}}. \tag{9}$$

Again, the infimum is taken over non-trivial functions U in an arbitrary linear subspace $\mathcal{H}_n^0$ of $\mathcal{H}_0$ of codimension $n-1$.
We want to define appropriate test-functions with domain contained in Q_0. Defining

$$\hat{Q}_\epsilon \equiv \{x \in Q : (1 + 2\epsilon^{1/\mu})x \in Q_\epsilon\}$$

we note that $\hat{Q}_\epsilon \subset Q_0$. (The reader may notice that there is an error at this point in the definition of the corresponding dilated set in the reference [7]: the previous inclusion does not hold there. However, this can easily be corrected and it does not affect the validity of the result of the citation.) Hence, we can consider the function

$$x \mapsto U_k^\epsilon((1 + 2\epsilon^{1/\mu})x, \eta), \quad x \in \hat{Q}_\epsilon,$$

whose domain is now included in Q_0. Unfortunately this cannot be extended by zero to the whole of Q_0 because of the quasiperiodicity conditions on $S_\epsilon^{j\pm} \subset \partial Q_\epsilon$, although it can be extended by zero to

$$Q_0 \cap \left\{ x \in Q : |x_j| < \frac{1}{2} - \frac{\epsilon^{1/\mu}}{1+2\epsilon^{1/\mu}},\ j = 1,2 \right\},$$

due to the Dirichlet boundary condition on the curved part of $\partial \hat{Q}_\epsilon$. We use the same name $U_k^\epsilon(x,\eta)$ for this extension. We need to multiply these functions by an appropriate cut-off function. To this end we define the smooth real-valued function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(t) = 1$, if $t \le 1$, and $\varphi(t) = 0$, if $t \ge 2$, and set

$$\varphi_\epsilon(t) = \varphi\big(t\epsilon^{-1/\mu}(1+2\epsilon^{1/\mu})\big),$$

and

$$\chi(\epsilon, x) = \Big(1 - \varphi_\epsilon\Big(\frac{1}{2} - |x_1|\Big)\Big)\Big(1 - \varphi_\epsilon\Big(\frac{1}{2} - |x_2|\Big)\Big).$$

Notice that this cut-off function χ vanishes, if one of the coordinates x_j satisfies $|x_j| \ge 1/2 - s(\epsilon)$, whereas it takes the value 1, if $|x_j| < 1/2 - 2s(\epsilon)$ for all j. Here we have set

$$s(\epsilon) = \frac{\epsilon^{1/\mu}}{1+2\epsilon^{1/\mu}}.$$

In particular χ vanishes on the segments of the boundary of $\hat{Q}_\epsilon$, where the quasiperiodicity conditions for U_k^ϵ hold. Thus the new function $\mathbf{U}_k^\epsilon$ with domain $\hat{Q}_\epsilon$, defined by

$$\mathbf{U}_k^\epsilon(x,\eta) = \chi(\epsilon, x) U_k^\epsilon((1+2\epsilon^{1/\mu})x, \eta),$$

can be extended by zero to the Sobolev class $H_0^1(Q_0)$.

Let us state a weighted estimate for the eigenfunctions of the new problem; we skip the proof since the calculations are very similar to the proof of [7, Lemma 3.1].

Proposition 3.1. *For all $n \in \mathbb{N}$ there exist positive constants C_n and ϵ_n such that the eigenfuctions U_n^ϵ of the problem* (1) *satisfy the inequality*

$$\left\| \frac{\nabla U_n^\epsilon(\cdot,\eta)}{(\rho+\epsilon^{1/\mu})^\mu} \right\|_{L^2(Q_\epsilon)} + \left\| \frac{U_n^\epsilon(\cdot,\eta)}{(\rho+\epsilon^{1/\mu})^{2\mu}} \right\|_{L^2(Q_\epsilon)} \le C_n$$

for all $\epsilon \in]0, \epsilon_n]$ and $\eta \in [-\pi,\pi[^2$, where ρ is the weight in the estimate (6).

Turning back to the Rayleigh quotient (9), we want to verify that the functions $\mathbf{U}_1^\epsilon, \dots, \mathbf{U}_n^\epsilon$ are linearly independent for any $n > 1$, so that each subspace $\mathcal{H}_n$ appearing in the max-min principle (9) contains their linear combination

$$\mathbf{V}^\epsilon = \sum_{k=1}^n a_k^\epsilon \mathbf{U}_k^\epsilon, \quad \sum_{k=1}^n |a_k^\epsilon|^2 = 1.$$

We set

$$\chi_1(\epsilon, x) = \chi(\epsilon, (1+2\epsilon^{1/\mu})^{-1}x) \tag{10}$$

for all $x \in Q_0$, and for the sake of simplicity, we omit the explicit reference to ϵ for the functions in the following calculations. Recalling the orthonormality conditions on the eigenfunctions U_n, a change of the integration variable yields

$$
\begin{aligned}
(\mathbf{U}_k, \mathbf{U}_m)_{Q_0} &= \int_{\hat{Q}_\epsilon} \chi^2(x) U_k((1+2\epsilon^{1/\mu})x) U_m((1+2\epsilon^{1/\mu})x)\, \mathrm{d}x \\
&= \frac{1}{(1+2\epsilon^{1/\mu})^2} \int_{Q_\epsilon} \chi_1^2 U_k(\cdot,\eta) U_m(\cdot,\eta)\, \mathrm{d}x \\
&= \frac{1}{(1+2\epsilon^{1/\mu})^2} (\chi_1^2 U_k, U_m)_{Q_\epsilon} \\
&= \frac{1}{(1+2\epsilon^{1/\mu})^2} \delta_{k,m} - \frac{1}{(1+2\epsilon^{1/\mu})^2} ((1-\chi_1^2) U_k, U_m)_{Q_\epsilon}.
\end{aligned}
$$

From this we deduce that the functions $\mathbf{U}_1, \ldots, \mathbf{U}_n$ form an "almost orthonormal" set in $L^2(Q_0)$. Indeed, setting

$$Q_\epsilon^* = \{x \in Q_\epsilon : \rho(x) < cs(\epsilon)\},$$

we have $(1 - \chi_1(x)) \neq 0$ only, if $x \in Q_\epsilon^*$ and we deduce that

$$
\begin{aligned}
|((1-\chi_1^2)U_k, U_m)_{Q_\epsilon}| &\leq \|U_k\|_{L^2(Q_\epsilon^*)} \|U_m\|_{L^2(Q_\epsilon^*)} \\
&\leq (\|\rho\|_{L^\infty(Q_\epsilon^*)} + \epsilon^{1/\mu})^{4\mu} \left\| \frac{U_k}{(\rho+\epsilon^{1/\mu})^{2\mu}} \right\|_{L^2(Q_\epsilon^*)} \left\| \frac{U_m}{(\rho+\epsilon^{1/\mu})^{2\mu}} \right\|_{L^2(Q_\epsilon^*)} \\
&\leq C_n \epsilon^4, \qquad (11)
\end{aligned}
$$

where we used Proposition 3.1. Hence, for $k \neq m$ the estimate (11) implies

$$|(\mathbf{U}_k, \mathbf{U}_m)_{Q_0}| = (1+2\epsilon^{1/\mu})^{-2} |((1-\chi_1^2)U_k, U_m)_{Q_\epsilon}| \leq C_n \epsilon^4.$$

On the other hand, if $k = m$, then

$$
\begin{aligned}
|(\mathbf{U}_k, \mathbf{U}_k)_{Q_0}| &= (1+2\epsilon^{1/\mu})^{-2} |(1 - (1-\chi_1^2 U_k, U_k)_{L^2(Q_\epsilon)})| \\
&\leq (1+2\epsilon^{1/\mu})^{-2} (1 + C_n\epsilon^4) \leq (1 + C_n \epsilon^4).
\end{aligned}
$$

In particular, for ϵ small enough, the functions $\mathbf{U}_1^\epsilon, \ldots, \mathbf{U}_n^\epsilon$ are linearly independent for all $n > 1$.

Putting the test-function $\mathbf{V}$ in the Rayleigh quotient (9) we get

$$
\begin{aligned}
\Lambda_n^0 &\leq \sup_{\mathcal{H}_n^0} \frac{(\nabla \mathbf{V}, \nabla \mathbf{V})_{Q_0}}{(\mathbf{V}, \mathbf{V})_{Q_0}} \\
&= \sup_{\mathcal{H}_n^0} \frac{\sum_{k,j=1}^n a_k^\epsilon \overline{a}_j^\epsilon (\nabla \mathbf{U}_k^\epsilon, \nabla \mathbf{U}_j^\epsilon)_{Q_0}}{\sum_{k,j=1}^n a_k^\epsilon \overline{a}_j^\epsilon (\mathbf{U}_k^\epsilon, \mathbf{U}_j^\epsilon)_{Q_0}}. \qquad (12)
\end{aligned}
$$

The inner product in the numerator on the right-hand side is explicitly given by

$$(\nabla \mathbf{U}_k^\epsilon, \nabla \mathbf{U}_j^\epsilon)_{Q_0} = \big(\chi_1 \nabla U_k^\epsilon + (\nabla \chi_1) U_k^\epsilon,\ \chi_1 \nabla U_j^\epsilon + (\nabla \chi_1) U_j^\epsilon\big)_{Q_\epsilon}. \qquad (13)$$

Here, we first use an estimate like (11), where we replace U_k and U_m by ∇U_k and ∇U_m. This, together with the eigenvalue and orthogonality properties of the functions U_k, U_m, yield

$$
\begin{aligned}
&|(\chi_1 \nabla U_k^\epsilon, \chi_1 \nabla U_j^\epsilon)_{Q_\epsilon}| = |(\nabla U_k^\epsilon, \nabla U_j^\epsilon)_{Q_\epsilon} + ((1-\chi_1^2)\nabla U_k^\epsilon, \nabla U_j^\epsilon)_{Q_\epsilon}| \\
&\leq |\delta_{k,j} \Lambda_k^\epsilon(\eta) (U_k^\epsilon, U_j^\epsilon)_{Q_\epsilon}| + C_n \epsilon^4 = \delta_{k,j} \Lambda_k^\epsilon(\eta) + C_n \epsilon^4.
\end{aligned}
$$

Next we recall that

$$\|\chi_1 \nabla U_k\|_{L^2(Q_\epsilon)} \leq \|\nabla U_k\|_{L^2(Q_\epsilon)} \leq (\Lambda_k^\epsilon(\eta))^{1/2},$$

and also that $|\nabla\chi_1| \leq C\epsilon^{-1/\mu}$ (cf. (10)). Hence, using the boundedness of the weighted norms in Proposition 3.1 we get

$$\begin{aligned} \|(\nabla\chi_1)U_k\|_{L^2(Q_\epsilon)} &\leq \frac{c}{\epsilon^{1/\mu}}\|U_k\|_{L^2(Q_\epsilon^*)} = \frac{c}{\epsilon^{1/\mu}}\left\|\frac{(\rho+\epsilon^{1/\mu})^{2\mu}}{(\rho+\epsilon^{1/\mu})^{2\mu}}U_k\right\|_{L^2(Q_\epsilon^*)} \\ &\leq \frac{c'\epsilon^2}{\epsilon^{1/\mu}}\left\|\frac{U_k}{(\rho+\epsilon^{1/\mu})^{\mu}}\right\|_{L^2(Q_\epsilon^*)} \leq C_n\epsilon^{\frac{2\mu-1}{\mu}}. \end{aligned} \tag{14}$$

Thus, by the Cauchy-Schwartz inequality,

$$|(\chi_1\nabla U_k, (\nabla\chi_1)U_j)_{Q_\epsilon}| \leq \|\chi_1\nabla U_k\|_{L^2(Q_\epsilon)}\|U_j\nabla\chi_1\|_{L^2(Q_\epsilon)} \leq C_n\epsilon^{\frac{2\mu-1}{\mu}}. \tag{15}$$

Finally, (14) and the Cauchy-Schwartz equality imply

$$|((\nabla\chi_1)U_k, (\nabla\chi_1)U_j)_{Q_\epsilon}| \leq C_n\epsilon^{\frac{2(2\mu-1)}{\mu}}. \tag{16}$$

Substituting the estimates (14)–(16) to (13) and coming back to the Rayleigh quotient (12) we get the estimate

$$\begin{aligned} \Lambda_n^0 &\leq \sup_{\mathcal{H}_n^0}\frac{(\nabla\mathbf{V},\nabla\mathbf{V})_{Q_0}}{(\mathbf{V},\mathbf{V})_{Q_0}} \\ &\leq \sup_{\mathcal{H}_n^0}\frac{\sum_{k,j=1}^n a_k^\epsilon\overline{a}_j^\epsilon(\delta_{k,j}\Lambda_k^\epsilon(\eta)+C_n\epsilon^{\frac{2(2\mu-1)}{\mu}})}{(1+2\epsilon^{1/\mu})^{-2}\sum_{k,j=1}^n a_k^\epsilon\overline{a}_j^\epsilon(\delta_{k,j}-(1-(\chi_1^2U_k^\epsilon,U_j^\epsilon)_{Q_\epsilon}))} \\ &\leq \sup_{\mathcal{H}_n^0}(1+2\epsilon^{1/\mu})^2\frac{\sum_{k=1}^n|a_k^\epsilon|^2\Lambda_k^\epsilon(\eta)+C_n\epsilon^{\frac{2(2\mu-1)}{\mu}}}{(\sum_{k=1}^n|a_k^\epsilon|^2-c_n\epsilon^2)} \\ &\leq \sup_{\mathcal{H}_n^0}(1+2\epsilon^{1/\mu})^2\frac{\Lambda_n^\epsilon(\eta)\sum_{k=1}^n|a_k^\epsilon|^2+C_n\epsilon^{\frac{2(2\mu-1)}{\mu}}}{(\sum_{k=1}^n|a_k^\epsilon|^2-c_n\epsilon^2)} \\ &\leq (1+2\epsilon^{1/\mu})^2C_n\Lambda_n^\epsilon(\eta) \end{aligned} \tag{17}$$

for a sufficiently big constant $C_n > 1$ and sufficiently small ϵ.

The two bounds (8) and (17) yield the following result.

Theorem 3.2. *For any $n \in \mathbb{N}$ there exist numbers $\epsilon_n > 0$ and $c_n \in]0,1[$ such that for all $\epsilon \in]0,\epsilon_n]$ and $\eta \in [-\pi,\pi[^2$ the eigenvalues $\Lambda_n^\epsilon(\eta)$ of the Dirichlet problem (D') in Q_ϵ, see* (2)*, are bounded by*

$$c_n\frac{\Lambda_n^0}{(1+2\epsilon^{1/\mu})^2} \leq \Lambda_n^\epsilon(\eta) \leq \Lambda_n^0,$$

where Λ_n^0 are the eigenvalues of the limit problem, see (4), (7).

We can now summarize the main result of the paper in the following statement, which also includes Theorem 1.1.

Corollary 3.3 (Opening spectral gaps)**.** *Let $n \in \mathbb{N}$, $n \geq 2$, and let the numbers c_n and ϵ_n be as in the above theorem. Assume that the eigenvalue of the limit problem Λ_n^0, see* (4), (7)*, has multiplicity $\varkappa_n \geq 1$ so that*

$$\Lambda_{n-1}^0 < \Lambda_n^0 = \cdots = \Lambda_{n+\varkappa_n-1} < \Lambda_{n+\varkappa_n}^0. \tag{18}$$

Then, for ϵ so small that $\epsilon \leq \min\{\epsilon_{n-1}, \epsilon_n\}$ and

$$(1 + 2\epsilon^{1/\mu})^2 \Lambda_{n-1}^0 < c_n \Lambda_n^0,$$

the spectrum σ of the original problem (D) has a gap between the bands $\Upsilon_{n-1}^{\epsilon}$ and Υ_n^{ϵ}, see (3).

Consequently, given $N \in \mathbb{N}$, the spectrum σ has at least N gaps, if ϵ is sufficiently small.

The last statement follows by choosing N different eigenvalues Λ_n^0 as in (18).

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