Spectral gaps for periodic piezoelectric waveguides

Sergei A. Nazarov and Jari Taskinen

Abstract. We construct a family of periodic piezoelectric waveguides Π^ε, depending on a small geometrical parameter, with the following property: as $\varepsilon \to +0$, the number of gaps in the essential spectrum of the piezoelectricity problem on Π^ε grows unboundedly.

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1. Introduction.

1.1. Motivation.

The continuous spectrum of a cylindrical waveguide of any physical nature is always the entire ray $[\lambda_\dagger, +\infty)$, included in the positive closed real axis $\overline{\mathbb{R}^+}$ of the complex plane $\mathbb{C}$. This means that for the spectral parameter λ above the cut-off value $\lambda_\dagger$, wave processes occur and propagating waves may drive energy from the central part of the waveguide to the infinity and vice versa. However, the structure of the spectrum may be much more complicated in a periodic wavequide: spectral bands, "passing zones", allowing wave propagation, may be interleaved with spectral gaps, "stopping zones", which disable wave processes. This distinguishing feature of periodic waveguides is used in the present day engineering applications including the design of wave filters, dampers, and many devices of nano-technology. In this context, a most challenging question has become to find waveguides of simplest shape having a spectral gap with a prescribed position and width. We give in some sense a positive answer to this question in the case of piezoelectric waveguides, which are examples of "smart" materials and devices. Although we shall consider quite exotic geometric shapes, a shape optimization analysis is omitted here, and it is planned to be the subject of a forthcoming paper.

Examples of periodic quantum, acoustic and elastic waveguides with gaps in their essential spectra have been presented in the literature, see [11, 10, 23, 24, 5, 6, 29], [2, 3], and [22, 8, 28, 7], respectively. However, the approach of these papers, or the general approach of [21], cannot be applied to the piezoelectric waveguides of the present paper, because a direct weak formulation of the piezoelectricity problem does not correspond to a semi-bounded self-adjoint operator; the essential spectrum of such a problem would be non-physical, covering the whole plane $\mathbb{C}$ as shown in Section 1.5. In this paper we apply a reduction scheme from [20], which improves the situation, since it leads to self-adjoint Hilbert space operators. However, an integral operator introduced in the scheme is not local, a fact which again prevents the use of the above mentioned papers directly and which makes the study of the spectrum rather involved.

The approach in the above mentioned works is based on asymptotic methods in regularly or singularly perturbed domains, cf. [18, Ch.2,4,5]. The method requires the choice of an unperturbed

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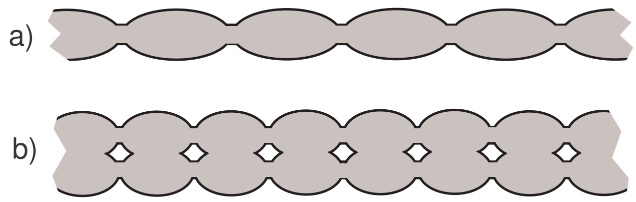

FIGURE 1. Periodic waveguides.

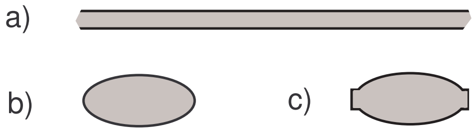

FIGURE 2. Thin cylinder and cells.

reference waveguide with explicitly known essential spectrum, like a straight cylinder or an infinite periodic row of identical bounded cells. In the former case the essential spectrum is the ray $[\lambda_\dagger, +\infty)$, and in the latter an unbounded monotone sequence of eigenvalues with infinite multiplicity. Perturbing the shape of the waveguide can either open small gaps, cf. [22, 2, 24, 5, 6], or create small bands, cf. [22, 8, 28, 3].

In this paper we use the second variant of asymptotic analysis and consider a piezoelectric waveguide Π_ε, which consists of "beads" $\varpi(j)$, $j \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ connected by a thin "needle" Ω^ε, Fig. 1.a. With the help of max-min-principle and elaborate estimates we prove that for any $J \in \mathbb{N} = \{1, 2, 3, , \ldots\}$ there exists $\varepsilon_J > 0$ such that the essential spectrum of the waveguide Π_ε with $\varepsilon \in (0, \varepsilon_J)$ contains at least J gaps, see Theorem 1.5. Multiple needles described in Fig. 1.b can be treated with the same tools. Notice that the connecting needles disappear as $\varepsilon \to +0$ and the waveguide Π_ε decomposes into a row of isolated bodies $\varpi(j)$, $j \in \mathbb{Z}$.

1.2. Notation for the piezoelectricity system.

Let us present the geometry of the waveguide Π^ε in detail. By ϖ we denote a bounded domain in $\mathbb{R}^3$,

$$\varpi \subset \{x = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, |z| < 1/2\}, \tag{1}$$

having Lipschitz boundary $\partial\varpi$, Fig. 2.b. We assume that the points $\mathcal{O}^\pm = (0, 0, \pm 1/2)$ are contained in the boundary and that, in some neighbourhood of them, $\partial\varpi$ is of smoothness C^3. Let Ω^ε denote the thin infinite straight cylinder $\Omega^\varepsilon = \omega^\varepsilon \times \mathbb{R}$, Fig. 2.a, where the boundary of the domain $\omega \subset \{y \in \mathbb{R}^2 : |y| < 1\}$ is smooth, and

$$\omega^\varepsilon := \varepsilon\omega = \{y \in \mathbb{R}^2 \ : \ \varepsilon^{-1} y \in \omega\}.$$

We consider the periodic waveguide Π^ε (Fig. 1)

$$\Pi^\varepsilon = \Omega^\varepsilon \cup \bigcup_{j \in \mathbb{Z}} \varpi(j) \tag{2}$$

consisting of Ω^ε and the periodic family of bodies

$$\varpi(j) = \{x : (y, z - j) \in \varpi\} \ , \ j \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}. \tag{3}$$

The set

$$\varpi^\varepsilon = \{x \in \Pi^\varepsilon : |z| < 1/2\} = \varpi \cup \big(\omega^\varepsilon \times (-1/2, 1/2)\big) \tag{4}$$

is called the periodicity cell of the quasi–cylinder Π^ε (see Fig. 2.c). We remark that for small enough ε the domains ϖ^ε are still Lipschitz with Lipschitz-constants having a bound independent of ε. This follows from the assumption that $\partial\varpi$ is C^3 in some neighbourhoods of $\mathcal{O}^\pm$: due to (1), the surface $\partial\varpi$ must be tangential to the planes $\{z = \pm 1/2\} \subset \mathbb{R}^3$ at the points $\mathcal{O}^\pm$.

At $\varepsilon = 0$ the set (2) turns into a union of disconnected domains (3).

The following general notation will be used in the sequel. The L^2-inner product on a domain Ω will be denoted by $(f, g)_\Omega$, $f, g \in L^2(\Omega)$, and the same notation will be used also in the case of $\mathbb{R}^n$-valued functions $n = 2, 3, \ldots$. We write the norm of a Banach function space X as $\|f; X\|$. The letters c, C, C' etc. denote positive constants, the value of which may change from place to place. The possible dependence of some parameter like ε is indicated by $c(\varepsilon)$ etc. The symbols ∇ and Δ will stand for the gradient and Laplacian with respect to the variable x, respectively. We write $B(a, r)$ for the open ball with center $a \in \mathbb{R}^3$ and radius $r > 0$.

As for the piezoelectricity system, the indices M and E will be used for mechanical and electric characteristics, respectively. We aim to use matrix formulation of the problem with the Mandel-Voigt-notation, so we denote by $M_{p\times q}$ the space of matrices of size $p \times q$ with real valued, possibly non-constant, entries, and by $\mathbb{O}_{p\times q}$ the null matrix of size $p \times q$. The unknown vector function u is written as the column $(u_1^\mathsf{M}, u_2^\mathsf{M}, u_3^\mathsf{M}, u^\mathsf{E})^\top$, where $u^\mathsf{M} = (u_1^\mathsf{M}, u_2^\mathsf{M}, u_3^\mathsf{M})^\top$ is the displacement vector and u^E is the electric potential and $\top$ stands for transposition. The three-dimensional elastic strain column is $\varepsilon^\mathsf{M} = (\varepsilon_{11}^\mathsf{M}, \varepsilon_{22}^\mathsf{M}, \varepsilon_{33}^\mathsf{M}, \sqrt{2}\varepsilon_{23}^\mathsf{M}, \sqrt{2}\varepsilon_{13}^\mathsf{M}, \sqrt{2}\varepsilon_{12}^\mathsf{M})^\top$, where

$$\varepsilon_{ij}^\mathsf{M} = \varepsilon_{ij}^\mathsf{M}(u) = \frac{1}{2}\Big(\frac{\partial u_i^\mathsf{M}}{\partial x_j} + \frac{\partial u_i^\mathsf{M}}{\partial x_j}\Big).$$

Hence, the vector $\varepsilon^\mathsf{M}(u^\mathsf{M})$ can be written as $D^\mathsf{M}(\nabla)u^\mathsf{M}$, where $D^\mathsf{M}(x) \in M_{3\times 6}$ is the matrix with linear dependence on $x = (x_1, x_2, x_3)^\top$,

$$D^\mathsf{M}(x) = \begin{pmatrix} x_1 & 0 & 0 & 0 & 2^{-1/2}x_3 & 2^{-1/2}x_2 \\ 0 & x_2 & 0 & 2^{-1/2}x_3 & 0 & 2^{-1/2}x_1 \\ 0 & 0 & x_3 & 2^{-1/2}x_2 & 2^{-1/2}x_1 & 0 \end{pmatrix}^\top .$$

Moreover, the electric field strength vector $\varepsilon^\mathsf{E} = (\varepsilon_1^\mathsf{E}, \varepsilon_2^\mathsf{E}, \varepsilon_3^\mathsf{E})$ is related to u^E by $\varepsilon^\mathsf{E}(u^\mathsf{E}) = -\nabla u^\mathsf{E}$. The piezoelasticity relations are formulated for the Hooke's tensor $(A_{pqjk}^\mathsf{MM})_{p,q,j,k}$, where $p, q, j, k = 1, 2, 3$, piezoelectric tensor $(A_{pqk}^\mathsf{ME})_{p,q,k}$ and dielectric tensor $(A_{pk}^\mathsf{EE})_{p,k}$ as

$$\sigma_{pq}^\mathsf{MM} = \sum_{j,k=1}^{3} A_{pqjk}^\mathsf{MM}\varepsilon_{jk}^\mathsf{M} - \sum_{k=1}^{3} A_{pqk}^\mathsf{ME}\varepsilon_k^\mathsf{E}\ , \quad \sigma_p^\mathsf{E} = \sum_{j,k=1}^{3} A_{pjk}^\mathsf{EM}\varepsilon_{jk}^\mathsf{M} + \sum_{k=1}^{3} A_{pk}^\mathsf{E}\varepsilon_k^\mathsf{E}.$$

The piezoelectricity system under investigation is the time-harmonic limit case of the standard time-dependent piezoelectrity equation. We refer to [32, 30] for the derivation of the system, using common assumptions like that the electro-magnetic field is controlled by the equations for the electro-static limit case. So, the formulation of our problem reads as

$$L(x,\nabla)u(x) = \lambda\varrho(x)Eu(x)\ ,\ x \in \Pi^\varepsilon, \tag{5}$$

$$B(x,\nabla)u(x) = 0 \quad \text{for almost every } x \in \partial\Pi^\varepsilon. \tag{6}$$

Here $\varrho > 0$ is the material density, which we assume to be 1-periodic with respect to z, continuous and positive: $\varrho(x) \geq \varrho_0$ for all $x \in \Pi^\varepsilon$ and some constant $\varrho_0 > 0$. The operators $L, B \in M_{4\times 4}$ are defined by

$$L(x,\nabla) = D(-\nabla)^\top A D(\nabla)\ , \quad B(x,\nabla) = D(n(x))^\top A(x) D(\nabla),$$

where $n(x)$ is the unit outward normal vector at $x \in \partial\Pi^\varepsilon$, and

$$D(x) = \begin{pmatrix} D^\mathsf{M}(x) & \mathbb{O}_{6\times 1} \\ \mathbb{O}_{3\times 3} & D^\mathsf{E}(x) \end{pmatrix}^\top ,\ D^\mathsf{E}(x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} ,\ E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The absence of the electric component u^{E} on the right of the system (5) means that we deal with the low and middle frequency range of the spectrum, where the high frequency oscillations of the electromagnetic field can be neglected, cf. [32, 30].

We shall use the abbreviated notation

$$D^{\mathsf{M}} := D^{\mathsf{M}}(\nabla) \quad , \quad D^{\mathsf{E}} := D^{\mathsf{E}}(\nabla).$$

Moreover, $A = A(x) \ \in M_{9\times 9}$ is defined by

$$A = \begin{pmatrix} A^{\mathsf{MM}} & A^{\mathsf{ME}} \\ -A^{\mathsf{EM}} & A^{\mathsf{EE}} \end{pmatrix} \tag{7}$$

where the matrices $A^{\mathsf{MM}} \in M_{6\times 6} = (A^{\mathsf{MM}}_{pqjk})_{p,q,j,k}$ and $A^{\mathsf{EE}} = (A^{\mathsf{EE}}_{pk})_{p,k} \in M_{3\times 3}$ are symmetric, while $A^{\mathsf{ME}} = (A^{\mathsf{ME}}_{pqk})_{p,q,k} = (A^{\mathsf{EM}})^\top \in M_{6\times 3}$. The minus sign at the bottom of the left-hand block of the matrix (7) reflects and implies the loose transform of the mechanical energy into the electric one and vice versa; this accounts for all the exceptional properties of smart materials and devices so useful for engineering applications. We assume that for some constant $C > 0$ there holds the positivity conditions

$$u^\top A^{\mathsf{MM}}(x) u \geq C|u|^2 \quad \text{and} \quad v^\top A^{\mathsf{EE}}(x) v \geq C|v|^2 \tag{8}$$

for all $x \in \Pi^\varepsilon$, $u \in \mathbb{R}^6$ and $v \in \mathbb{R}^3$.

Given a domain $\Omega \subset \mathbb{R}^3$, the bilinear form

$$\big(A^{\mathsf{MM}} D^{\mathsf{M}} u, D^{\mathsf{M}} v\big)_\Omega \ , \quad u, v \in H^1(\Omega)^3, \tag{9}$$

is nothing but the energy form associated with the linear elasticity problem in the domain Ω. As well known, if Ω is Lipschitz, the following Korn inequality holds true for $u \in H^1(\Omega)^3$ (see [13]):

$$\|u; H^1(\Omega)\|^2 \leq C(\Omega)\big(\big(A^{\mathsf{MM}} D^{\mathsf{M}} u, D^{\mathsf{M}} U\big)_\Omega + \|u; L^2(\Omega)\|^2\big). \tag{10}$$

We shall need to control the dependence of the constant $C(\Omega)$ on the domain as stated in the following lemma. Its proof follows from the comments to Proposition 2.2 and Lemma 2.3 of [26, § 2.3].

Lemma 1.1. *The constant $C(\varpi^\varepsilon)$ can be chosen in* (10) *independently of ε for all domains ϖ^ε.*

1.3. Reduction to a self-adjoint problem.

For the weak formulation of the problem (5)–(6) we first define the Sobolev-type space $H^{\mathsf{E}}(\Pi^\varepsilon)$ as the completion of $C_c^\infty(\overline{\Pi}^\varepsilon)$ with respect to the norm

$$\Big(\|f; L^2(\varpi)\|^2 + \|\nabla f; L^2(\Pi^\varepsilon)\|^2\Big)^{1/2}. \tag{11}$$

Since ϖ is bounded, this space contains the constant functions, contrary to $H^1(\Pi^\varepsilon)$, and that the norm (11) is equivalent to the weighted norm

$$\Big(\|(1+z^2)^{-1/2} f; L^2(\Pi^\varepsilon)\|^2 + \|\nabla f; L^2(\Pi^\varepsilon)\|^2\Big)^{1/2}$$

as a consequence of the one-dimensional Hardy inequality

$$\int\limits_{-\infty}^{\infty} \Big|\frac{dV(t)}{dt}\Big|^2 dt \geq C\Big(\int\limits_{-\infty}^{\infty} \frac{1}{1+t^2}|V(t)|^2 dt + \int\limits_{-1/2}^{1/2} |V(t)|^2 dt\Big).$$

The weak formulation of the problem (5)–(6) is to find $u \in H^1(\Pi^\varepsilon)^3 \times H^{\mathsf{E}}(\Pi^\varepsilon)$, $u \neq 0$, and λ such that

$$\big(AD(\nabla)u, D(\nabla)v\big)_{\Pi^\varepsilon} = \lambda(\varrho u^{\mathsf{M}}, v^{\mathsf{M}})_{\Pi^\varepsilon}, \tag{12}$$

for all $v = (v^{\mathsf{M}}, v^{\mathsf{E}}) \in H^1(\Pi^\varepsilon)^3 \times H^{\mathsf{E}}(\Pi^\varepsilon)$. However, by definition, (7), the matrix A is not symmetric and hence the operator of the problem (12) is not formally self-adjoint (cf. [9]). We shall reduce the problem into a standard spectral problem for a positive self-adjoint operator. To this end we write (12) componentwise as

$$\big(A^{\mathsf{MM}}D^{\mathsf{M}}u^{\mathsf{M}}, D^{\mathsf{M}}v^{\mathsf{M}}\big)_{\Pi^\varepsilon} + \big(A^{\mathsf{ME}}D^{\mathsf{E}}u^{\mathsf{E}}, D^{\mathsf{M}}v^{\mathsf{M}}\big)_{\Pi^\varepsilon}$$

$$= \lambda(\varrho u^{\mathsf{M}}, v^{\mathsf{M}})_{\Pi^\varepsilon} \quad \forall\, v^{\mathsf{M}} \in H^1(\Pi^\varepsilon)^3, \tag{13}$$

$$\big(A^{\mathsf{EE}}D^{\mathsf{E}}u^{\mathsf{E}}, D^{\mathsf{E}}v^{\mathsf{E}}\big)_{\Pi^\varepsilon} = \big(A^{\mathsf{EM}}D^{\mathsf{M}}u^{\mathsf{M}}, D^{\mathsf{E}}v^{\mathsf{E}}\big)_{\Pi^\varepsilon} \quad \forall\, v^{\mathsf{E}} \in H^{\mathsf{E}}(\Pi^\varepsilon). \tag{14}$$

Recall that $D^{\mathsf{E}}v^{\mathsf{E}} = \nabla v^{\mathsf{E}}$, hence, the equation (14) is just the weak formulation of the Neumann problem with homogeneous boundary conditions for the formally positive operator $-D^{\mathsf{E}}A^{\mathsf{EE}}D^{\mathsf{E}}$ (see (8)). We remark that given $F^\varepsilon \in H^{\mathsf{E}}(\Pi^\varepsilon)^*$, the compatibility condition for the problem

$$\big(A^{\mathsf{EE}}D^{\mathsf{E}}u^{\mathsf{E}}, D^{\mathsf{E}}v^{\mathsf{E}}\big)_{\Pi^\varepsilon} = F^\varepsilon(v^{\mathsf{E}}) \;\; \forall\, v^{\mathsf{E}} \in H^{\mathsf{E}}(\Pi^\varepsilon) \tag{15}$$

reads as $F^\varepsilon(1) = 0$, and the solution of (15) is defined up to a constant and becomes unique under the orthogonality condition

$$(u^{\mathsf{E}}, 1)_{\varpi^\varepsilon} = 0. \tag{16}$$

It is important that

$$\int\limits_{\Pi^\varepsilon} D^{\mathsf{E}}A^{\mathsf{EM}}D^{\mathsf{M}}u^{\mathsf{M}}dx = 0$$

holds true, see (6). In this way the problem (14) defines a bounded operator

$$T^\varepsilon : H^1(\Pi^\varepsilon)^3 \to H^{\mathsf{E}}(\Pi^\varepsilon)\ ,\ T^\varepsilon(u^{\mathsf{M}}) = u^{\mathsf{E}} \tag{17}$$

with u^{E} satisfying (16). By definition, T^ε is a non-local operator, which is the main qualitative difference in comparison with the elasticity system (cf. [20]) and also a source of technical difficulties for the rest of this work.

Denoting

$$\mathcal{R}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) := (A^{\mathsf{ME}}D^{\mathsf{E}}T^\varepsilon u^{\mathsf{M}}, D^{\mathsf{M}}v^{\mathsf{M}})_{\Pi_\varepsilon} \tag{18}$$

and inserting (17) turns (13) into the problem

$$\big(A^{\mathsf{MM}}D^{\mathsf{M}}u^{\mathsf{M}}, D^{\mathsf{M}}v^{\mathsf{M}}\big)_{\Pi^\varepsilon} + \mathcal{R}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) = \lambda(\varrho u^{\mathsf{M}}, v^{\mathsf{M}})_{\Pi^\varepsilon} \quad \forall\, v^{\mathsf{M}} \in H^1(\Pi^\varepsilon)^3. \tag{19}$$

The problem (19) is self-adjoint, which can be seen from the following fact.

Lemma 1.2. *The bilinear form $\mathcal{R}^\varepsilon$ is Hermitian and positive in $H^1(\Pi^\varepsilon)^3 \times H^1(\Pi^\varepsilon)^3$.*

Proof. Let $u^{\mathsf{M}}, v^{\mathsf{M}} \in H^1(\Pi^\varepsilon)^3$ be given and denote $u^{\mathsf{E}} := T^\varepsilon u^{\mathsf{M}}$ and $v^{\mathsf{E}} := T^\varepsilon v^{\mathsf{M}}$. Since (14) holds with the roles of the pairs $(v^{\mathsf{M}}, v^{\mathsf{E}})$ and $(u^{\mathsf{M}}, u^{\mathsf{E}})$ interchanged, we can write

$$\begin{aligned}\mathcal{R}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) &= (A^{\mathsf{ME}}D^{\mathsf{E}}u^{\mathsf{E}}, D^{\mathsf{M}}v^{\mathsf{M}})_{\Pi_\varepsilon} = \overline{(A^{\mathsf{EM}}D^{\mathsf{M}}v^{\mathsf{M}}, D^{\mathsf{E}}u^{\mathsf{E}})_{\Pi_\varepsilon}} \\ &= \overline{(A^{\mathsf{EE}}D^{\mathsf{E}}v^{\mathsf{E}}, D^{\mathsf{E}}u^{\mathsf{E}})_{\Pi_\varepsilon}} = (A^{\mathsf{EE}}D^{\mathsf{E}}u^{\mathsf{E}}, D^{\mathsf{E}}v^{\mathsf{E}})_{\Pi_\varepsilon}.\end{aligned} \tag{20}$$

The positivity of $\mathcal{R}^\varepsilon$ follows from the last line and the positivity of A^{EE}, see (8). Furthermore, since the matrix A^{EE} is symmetric, (20) equals $\overline{(A^{\mathsf{EE}}D^{\mathsf{E}}v^{\mathsf{E}}, D^{\mathsf{E}}u^{\mathsf{E}})_{\Pi_\varepsilon}}$ and since the pair $(u^{\mathsf{M}}, u^{\mathsf{E}})$ also satisfies (14), we can reverse the deduction chain (20) to see that $\mathcal{R}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) = \overline{\mathcal{R}^\varepsilon(v^{\mathsf{M}}, u^{\mathsf{M}})}$ holds. ⊠

Let us define the bilinear forms

$$\begin{aligned}\mathcal{A}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) &:= \big(A^{\mathsf{MM}}D^{\mathsf{M}}u^{\mathsf{M}}, D^{\mathsf{M}}v^{\mathsf{M}}\big)_{\Pi^\varepsilon}, \\ \mathcal{B}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) &:= \mathcal{A}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) + \mathcal{R}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) + (\varrho u^{\mathsf{M}}, v^{\mathsf{M}})_{\Pi^\varepsilon}.\end{aligned} \tag{21}$$

The following coercivity property holds true for $\mathcal{B}^\varepsilon$, as a direct consequence of (10), Lemma 1.2 and the assumption $\varrho \geq \varrho_0 > 0$.

Corollary 1.3. *The form $\mathcal{B}^\varepsilon$ satisfies, for some constant $C > 0$,*

$$\|u^{\mathsf{M}}; H^1(\Pi^\varepsilon)\|^2 \leq C\mathcal{B}^\varepsilon(u^{\mathsf{M}}, u^{\mathsf{M}})$$

for all $u^{\mathsf{M}} \in H^1(\Pi^\varepsilon)^3$.

As a consequence of this, the operator $\mathcal{M}^\varepsilon : H^1(\Pi^\varepsilon)^3 \to H^1(\Pi^\varepsilon)^3$, defined by

$$\mathcal{B}^\varepsilon(\mathcal{M}^\varepsilon u^{\mathsf{M}}, v^{\mathsf{M}}) = (\varrho u^{\mathsf{M}}, v^{\mathsf{M}})_{\Pi^\varepsilon} \quad \forall\, u^{\mathsf{M}}, v^{\mathsf{M}} \in H^1(\Pi^\varepsilon)^3 \tag{22}$$

is bounded, positive and self-adjoint. The problems (12) and thus (5)–(6) are equivalent to the abstract equation

$$\mathcal{M}^\varepsilon u^{\mathsf{M}} = \mu u^{\mathsf{M}}. \tag{23}$$

with the new spectral parameter $\mu = (1 + \lambda)^{-1}$. To be precise, by the spectrum (respectively, point, continuous or essential spectrum) of the original problem (5)–(6) we mean those numbers $\lambda = \mu^{-1} - 1$, where μ belongs to the spectrum (respectively, point, continuous or essential spectrum) of the well-defined spectral problem (23). The spectrum is a disjoint union of discrete and essential components. It is not known, if the essential and continuous spectra coincide, in other words, if there exist eigenvalues with infinite multiplicity. The essential spectrum is obviously not empty, due to the unboundedness of Π^ε.

1.4. Formulation of the main result and structure of the paper.

The spectral problem (23) concerns a bounded, positive, self-adjoint operator $\mathcal{M}^\varepsilon$ related to an unbounded periodic domain, hence, in view of the Floquet-Bloch-Gelfand-theory one expects that the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{M}^\varepsilon)$ of the operator $\mathcal{M}^\varepsilon$ and thus the essential spectrum σ_{ess} of the original problem (5)–(6) have band-gap structure,

$$\sigma_{\mathrm{ess}} = \bigcup_{p=1}^{\infty} \Upsilon_p^\varepsilon, \tag{24}$$

where Υ_p^ε are closed subintervals of $[0, +\infty)$. While this fact indeed holds true, it does not follow directly from the existing literature due to the complicated structure of $\mathcal{M}^\varepsilon$; we shall give a proof for (24) in Theorem 2.3 of Section 2.3. Here, we formulate our main results as follows.

Theorem 1.4. *There exist an unbounded non-negative monotone sequence $\{\Lambda_j\}_{j=1}^\infty$ and positive sequences $\{c_j\}_{j=1}^\infty$, $\{\beta_j\}_{j=1}^\infty$ and $\{\varepsilon_j\}_{j=1}^\infty$ such that, for all j,*

$$\Upsilon_j \subset [\Lambda_j - c_j\varepsilon^{\beta_j}, \Lambda_j + c_j\varepsilon^{\beta_j}] \quad \text{for } \varepsilon \in (0, \varepsilon_j].$$

Taking into account the band-gap structure (24) yields the following result.

Theorem 1.5. *If $\Lambda_j < \Lambda_{j+1}$ holds for some index $j \in \mathbb{N}$, then the essential spectrum σ_{ess} has a gap between the segments Υ_j^ε and $\Upsilon_{j+1}^\varepsilon$ for small enough ε. Consequently, given $N \in \mathbb{N}$, one can open at least N gaps in σ_{ess}, if ε is small enough.*

The sequence $\{\Lambda_j\}_{j=1}^\infty$ will just consist of the eigenvalues of the limit problem to be considered in Section 3.1. Theorem 1.4 follows from Lemmas 3.5 and 4.4 by defining $\beta_j = \min(\beta(j), \gamma(j))$ in the notation of the cited lemmas. To verify Theorem 1.5, we choose a given number N of indices $j \in \mathbb{N}$ such that eigenvalues Λ_j, Λ_{j-1} of the limit problem (78) are distinct. Then, for small enough ε and for all such j, the spectral bands $\Upsilon_1^{\varepsilon,\eta}, \ldots, \Upsilon_{j-1}^{\varepsilon,\eta}$ are contained in the interval $[0, \Lambda_{j-1} + C_j\varepsilon^{\beta_j}]$ and the bands $\Upsilon_j^{\varepsilon,\eta}$, $\Upsilon_{j+1}^{\varepsilon,\eta}, \ldots$ to the interval $[\Lambda_j - C_j\varepsilon^{\beta_j}, +\infty)$. Thus, there is a spectral gap between Λ_{j-1} and Λ_j, for all chosen j.

We explain the contents of the paper. Section 2 concentrates on the Floquet-Bloch-Gelfand-method, which turns the piezoelectricity problem on the waveguide Π^ε into a parameter dependent problem on the bounded domain ϖ^ε. The self-adjoint reduction is described in Section 2.1, and the operator theoretic formulation of the spectral problem in Section 2.2. The connection of the spectra of the original and model problems is clarified in Section 2.3. The limit model problem corresponding to the value $\varepsilon = 0$ is dealt with in Section 3.1; in particular, the reference eigenvalues Λ_p are defined there. The rest of the section is concentrated on the proof Lemma 3.5, which contains the upper bound for the spectral bands, cf. above. We apply suitable test functions to the max-min-principle and thus prove accurate enough upper estimates for the eigenvalues $\Lambda_p^{\varepsilon,\eta}$. To this end, we derive weighted Sobolev estimates for the eigenfunctions of the limit problem (Section 3.4) and apply these to estimates of the non-local operators T and $T^{\varepsilon,\eta}$ (Section 3.5).

Section 4 contains the proof for the lower bound of the spectral bands, see Lemma 4.4. The contents of this section is analogous to Section 3, save a number of inevitable changes.

1.5. On the physical background.

The absence of the spectral parameter λ on the last, "electric" line of the system (5) is caused by the fact that the electric potential u^{E} cannot affect the kinetic energy at low and middle frequencies, which are accepted by the oscillations u^{M}, see [32, 30] for details.

The two above mentioned peculiarities, namely, the non-self-adjointness of L and the absence of u^{E} on the right of (5), compensate each other in the mathematical formulation so that a vector eigenfunction $u^\circ \in H^1(\Pi^\varepsilon)^4$ of the piezoelectricity problem (5)–(6) always corresponds to a real eigenvalue $\lambda^\circ \geq 0$. To see this, one takes the complex conjugation of the electric lines in (5)–(6), integrates by parts to obtain

$$\begin{aligned}&(A^{\mathsf{MM}} D^{\mathsf{M}} u^{\circ\mathsf{M}}, D^{\mathsf{M}} u^{\circ\mathsf{M}})_{\Pi^\varepsilon} + (A^{\mathsf{ME}} D^{\mathsf{E}} u^{\circ\mathsf{E}}, D^{\mathsf{M}} u^{\circ\mathsf{M}})_{\Pi^\varepsilon}\\ &-(A^{\mathsf{EM}} D^{\mathsf{M}} \overline{u^{\circ\mathsf{M}}}, D^{\mathsf{M}} \overline{u^{\circ\mathsf{M}}})_{\Pi^\varepsilon} + (A^{\mathsf{MM}} D^{\mathsf{M}} \overline{u^{\circ\mathsf{M}}}, D^{\mathsf{M}} \overline{u^{\circ\mathsf{M}}})_{\Pi^\varepsilon} = \lambda^\circ (\varrho u^{\circ\mathsf{M}}, u^{\circ\mathsf{M}})_{\Pi^\varepsilon},\end{aligned}$$

and observes that the second and third scalar products on the left hand side cancel each other.

Also, the changes

$$u = (u^{\mathsf{M}}, u^{\mathsf{E}}) \mapsto u^\bullet = (u^{\mathsf{M}}, -u^{\mathsf{E}}) \quad , \quad A \mapsto A^\bullet = \begin{pmatrix} A^{\mathsf{MM}} & A^{\mathsf{ME}} \\ A^{\mathsf{EM}} & -A^{\mathsf{EE}} \end{pmatrix} \tag{25}$$

easily turn the problem (5)–(6) into a formally self-adjoint one with the variational formulation

$$\mathcal{A}^\bullet(u^\bullet, v^\bullet) = (A^\bullet D u^\bullet, D v^\bullet)_{\Pi^\varepsilon} = \lambda (\varrho {u^\bullet}^{\mathsf{M}}, {v^\bullet}^{\mathsf{M}})_{\Pi^\varepsilon}. \tag{26}$$

The sesquilinear form on the left hand side is closely related to the electric enthalpy, cf. [32, 31]. However, it is obviously not positive, due to the minus sign on A^{EE} in (25), and therefore it cannot keep the "good" properties of the spectrum of (26). Indeed, since the problem has for any $\lambda \in \mathbb{C}$ the constant solution

$$u^{\bullet\circ} = (0,0,0,1)\top, \tag{27}$$

one can observe by repeating the calculation in Section 2.3 that the operator of the problem (26) is not a Fredholm operator from $H^1(\Pi^\varepsilon)^4$ into $\big(H^1(\Pi^\varepsilon)^4\big)^*$. In other words, the (essential) spectrum covers the whole complex plane $\mathbb{C}$. We emphasize that (27) is a parasite solution, because any potential in mathematical physics is determined up to an additive constant, and indeed, the reduction procedure in Section 1.3 excludes the subspace of constant electric potentials and produces a self-adjoint positive operator $\mathcal{M}^\varepsilon$ with a nice spectrum contained in $\overline{\mathbb{R}^+} = [0, +\infty) \subset \mathbb{C}$.

The Neumann boundary conditions (6) correspond to a waveguide with a traction-free surface in contact with an absolute electric isolator. If the surrounding medium is an absolute electric conductor, then the piezoelectricity system (5) has to be supplied with the mixed boundary conditions

$$EB(x, \nabla) u(x) = 0 \ , \ u^{\mathsf{E}}(x) = 0 \ \text{ for almost every } x \in \partial\Pi^\varepsilon. \tag{28}$$

Since the auxiliary Dirichlet problem (14), posed in the subspace $H_0^1(\Pi^\varepsilon) = \{u^{\mathsf{E}} \in H^1(\Pi^\varepsilon) : u^{\mathsf{E}} = 0 \text{ on } \partial\Pi^\varepsilon\}$, is uniquely solvable, the variational problems (12) and (26) are properly stated in $H^1(\Pi^\varepsilon)^3 \times H_0^1(\Pi^\varepsilon)$, and in principle the reduction scheme can be avoided. For this reason we do not treat the problem (5), (28), although all of our results can easily be adapted to this case, too.

2. Essential spectrum of the piezoelectricity waveguide.

2.1. Floquet-Bloch-Gelfand-transform and model problem in the cell.

The natural approach to the piezoelectricity problem on the periodic waveguide is to apply the Bloch-Floquet-Gelfand (FBG) -transform, which leads into a parameter dependent problem on the bounded periodicity cell.

Given the unknown function u, its FBG-transform is defined by (see [12] and, e.g. [19, 14, 27, 15]):

$$\mathcal{G} : u(y,z) \mapsto U(y,z;\eta) = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \exp(-i\eta(z+m))u(y,z+m),$$

where $(y,z) \in \Pi^\varepsilon$ on the left, $(y,z) \in \varpi^\varepsilon$ on the right, and $\eta \in [-\pi,\pi)$ is the dual variable (Floquet parameter). The inverse transform is given by

$$u(y,z) = (2\pi)^{-1/2} \int\limits_{-\pi}^{\pi} e^{i\eta z}\hat{u}(y, z-[z];\eta)d\eta,$$

where $[z]$ denotes the integer part, $[z] = \max\{\zeta \in \mathbb{Z} : \zeta \le z\}$. The Parseval identity takes the form

$$(u,v)_{\Pi^\varepsilon} = \int\limits_{-\pi}^{\pi} (\hat{u},\hat{v})_{\varpi^\varepsilon} d\eta.$$

The FBG-transform is an isomorphism

$$\mathcal{G} : L^2(\Pi^\varepsilon) \to L^2(-\pi,\pi; L^2(\varpi^\varepsilon)) \quad \text{and}$$
$$\mathcal{G} : H^1(\Pi^\varepsilon) \to L^2(-\pi,\pi; H^1_{\rm per}(\varpi^\varepsilon)),$$

where the spaces are complex valued and $L^2(-\pi,\pi;L^2(\varpi^\varepsilon))$ consists of $L^2(\varpi^\varepsilon)$–valued L^2–functions on $[-\pi,\pi]$, the space $L^2(-\pi,\pi;H^1_{\rm per}(\varpi^\varepsilon))$ is defined analogously, and $H^1_{\rm per}(\varpi^\varepsilon)$ is the space of Sobolev–functions 1–periodic with respect to z. In the case of $L^2(\Pi^\varepsilon)$ the mapping $\mathcal{G}$ is even an isometry (see e.g. [27, § 3.4] and [25, Cor. 3.4.3]).

Applying $\mathcal{G}$, the problem (5) converts into the following model spectral problem in the periodicity cell (4):

$$\overline{D(-\nabla - i\eta e_3)}^\top A(x)D(\nabla + i\eta e_3)U(x;\eta) = \Lambda^{\eta,\varepsilon}\varrho(x)U(x;\eta), \qquad x \in \varpi^\varepsilon, \tag{29}$$

$$\overline{D(n(x))}^\top A(x)D(\nabla_y, \partial_z + i\eta)U(x;\eta) = 0, \quad x \in \upsilon^\varepsilon := \partial\varpi^\varepsilon \setminus \overline{(\omega^{\varepsilon,+}\cup\omega^{\varepsilon,-})}, \tag{30}$$

$$U(y,\tfrac12;\eta) = U(y,-\tfrac12;\eta), \quad \partial_z U(y,\tfrac12;\eta) = \partial_z U(y,-\tfrac12;\eta), \ y \in \omega^\varepsilon, \tag{31}$$

where $U(x;\eta)$ also depends on ε. Notice that the periodicity conditions are imposed only on the small cross–sections $\omega^\pm = \omega^\varepsilon \times \{\pm 1/2\}$ of the needle. The weak formulation of the problem (29) for the unknown $U \in H^1_{\rm per}(\varpi^\varepsilon)^4$, $U \neq 0$, and $\Lambda^{\varepsilon,\eta}$ reads as

$$\big(AD(\nabla + i\eta e_3)U, D(\nabla + i\eta e_3)V\big)_{\varpi^\varepsilon} = \Lambda^{\varepsilon,\eta}(\varrho U^{\sf M}, V^{\sf M})_{\varpi^\varepsilon}, \tag{32}$$

for all $V \in H^1_{\rm per}(\varpi^\varepsilon)^4$.

As in the case of the original problem (12), it is necessary to make it self-adjoint using the reduction scheme; the procedure is the same, although the boundedness of ϖ^ε allows for a crucial simplification. So we again write (32) componentwise as

$$\begin{aligned} & \big(A^{\sf MM}D^{\sf M}(\nabla + i\eta e_3)U^{\sf M}, D^{\sf M}(\nabla + i\eta e_3)V^{\sf M}\big)_{\varpi^\varepsilon} \\ + \ & \big(A^{\sf ME}D^{\sf E}(\nabla + i\eta e_3)U^{\sf E}, D^{\sf M}(\nabla + i\eta e_3)V^{\sf M}\big)_{\varpi^\varepsilon} \\ = \ & \Lambda^{\varepsilon,\eta}(\varrho U^{\sf M}, V^{\sf M})_{\varpi^\varepsilon} \quad \forall\, V^{\sf M} \in H^1_{\rm per}(\varpi^\varepsilon)^3, \end{aligned} \tag{33}$$

$$\begin{aligned} & \big(A^{\sf EE}D^{\sf E}(\nabla + i\eta e_3)U^{\sf E}, D^{\sf E}(\nabla + i\eta e_3)V^{\sf E}\big)_{\varpi^\varepsilon} \\ = \ & \big(A^{\sf EM}D^{\sf M}(\nabla + i\eta e_3)U^{\sf M}, D^{\sf E}(\nabla + i\eta e_3)V^{\sf E}\big)_{\varpi^\varepsilon} \quad \forall\, V^{\sf E} \in H^1_{\rm per}(\varpi^\varepsilon) \end{aligned} \tag{34}$$

with $U^{\mathsf{M}} \in H^1_{\rm per}(\varpi^\varepsilon)^3$ in the problem (34). For $\eta \neq 0$ the problem (34) is uniquely solvable in $H^1_{\rm per}(\varpi^\varepsilon)$, but for $\eta = 0$ the homogeneous Neumann problem has the constant solution. However, the right hand side of (34) with $\eta = 0$ degenerates for the constant test function V^{E} so that the Fredholm alternative provides a solution, which becomes unique by requiring the orthogonality condition

$$\int\limits_{\varpi^\varepsilon} U^{\mathsf{E}}(x;0)dx = 0,$$

which we assume from now on. In this way, the integral identity (34) defines a continuous mapping

$$T^{\varepsilon,\eta} : H^1_{\rm per}(\varpi^\varepsilon)^3 \to H^1_{\rm per}(\varpi^\varepsilon) \ , \ T^{\varepsilon,\eta}(U^{\mathsf{M}}) = U^{\mathsf{E}} \tag{35}$$

for all $\eta \in [-\pi, \pi)$.

With the help of (35) the problem (33) can be reformulated as

$$\begin{aligned} &\big(A^{\mathsf{MM}} D^{\mathsf{M}}(\nabla + i\eta e_3)U^{\mathsf{M}}, D^{\mathsf{M}}(\nabla + i\eta e_3)V^{\mathsf{M}}\big)_{\varpi^\varepsilon} + \mathcal{R}^{\varepsilon,\eta}(U^{\mathsf{M}}, V^{\mathsf{M}}) \\ = \ & \Lambda^{\varepsilon,\eta}(\varrho U^{\mathsf{M}}, V^{\mathsf{M}})_{\varpi^\varepsilon} \quad \forall\, V^{\mathsf{M}} \in H^1_{\rm per}(\varpi^\varepsilon)^3, \end{aligned} \tag{36}$$

by denoting

$$\mathcal{R}^{\varepsilon,\eta}(U^{\mathsf{M}}, V^{\mathsf{M}}) := (A^{\mathsf{ME}} D^{\mathsf{E}}(\nabla + i\eta e_3)T^{\varepsilon,\eta}U^{\mathsf{M}}, D^{\mathsf{M}}(\nabla + i\eta e_3)V^{\mathsf{M}})_{\varpi^\varepsilon}. \tag{37}$$

We emphasize that $\mathcal{R}^{\varepsilon,\eta}$ depends continuously on $\eta \in [-\pi, \pi)$ with respect to the operator norm, since the operator norms of $T^{\varepsilon,\eta}$ are bounded uniformly in ε and η; this fact will be proven in Lemma 2.2, below. Also, owing to the special structure of the right and left hand sides of (34), $\mathcal{R}^{\varepsilon,\eta}$ depends 2π-periodically on $\eta \in [-\pi, \pi)$. To see this, we have $U^{\mathsf{E}}(x; \eta + 2\pi) = e^{-2\pi i z} U^{\mathsf{E}}(x; \eta)$ and $U^{\mathsf{M}}(x; \eta + 2\pi) = e^{-2\pi i z} U^{\mathsf{M}}(x; \eta)$, hence,

$$T^{\varepsilon,\eta+2\pi} U^{\mathsf{E}}(\cdot; \eta + 2\pi) = e^{-2\pi i z} T^{\varepsilon,\eta} U^{\mathsf{E}}(\cdot; \eta).$$

Moreover,

$$\begin{aligned} &\big(A^{\mathsf{ME}} D^{\mathsf{E}}(\nabla + i(\eta + 2\pi)e_3)T^{\varepsilon,\eta+2\pi}U^{\mathsf{E}}(\cdot; \eta + 2\pi), D^{\mathsf{M}}(\nabla + i(\eta + 2\pi)e_3)V^{\mathsf{M}}\big)_{\varpi^\varepsilon} \\ = \ &\big(A^{\mathsf{ME}} D^{\mathsf{E}}(\nabla + i(\eta + 2\pi)e_3)e^{-2\pi i z}U^{\mathsf{E}}(\cdot; \eta), D^{\mathsf{M}}(\nabla + i(\eta + 2\pi)e_3)V^{\mathsf{M}}\big)_{\varpi^\varepsilon} \\ = \ &\big(e^{-2\pi i z} A^{\mathsf{ME}} D^{\mathsf{E}}(\nabla + i\eta e_3)U^{\mathsf{E}}(\cdot; \eta), D^{\mathsf{M}}(\nabla + i(\eta + 2\pi)e_3)V^{\mathsf{M}}\big)_{\varpi^\varepsilon} \\ = \ &\big(A^{\mathsf{ME}} D^{\mathsf{E}}(\nabla + i\eta e_3)T^{\varepsilon,\eta}U^{\mathsf{M}}(\cdot; \eta), e^{2\pi i z} D^{\mathsf{M}}(\nabla + i(\eta + 2\pi)e_3)V^{\mathsf{M}}\big)_{\varpi^\varepsilon} \\ = \ &\big(A^{\mathsf{ME}} D^{\mathsf{E}}(\nabla + i\eta e_3)T^{\varepsilon,\eta}U^{\mathsf{M}}(\cdot; \eta), D^{\mathsf{M}}(\nabla + i\eta e_3)(e^{2\pi i z}V^{\mathsf{M}})\big)_{\varpi^\varepsilon}. \end{aligned}$$

2.2. Spectrum of the model problem.

The proofs of Lemma 1.2 and Corollary 1.3 show that the sesquilinear form $\mathcal{R}^{\varepsilon,\eta}$ is Hermitian and positive in $H^1_{\rm per}(\varpi^\varepsilon)^3 \times H^1_{\rm per}(\varpi^\varepsilon)^3$ and that the sesquilinear form

$$\begin{aligned} \mathcal{B}^{\varepsilon,\eta}(U^{\mathsf{M}}, V^{\mathsf{M}}) \ &:= \ \big(A^{\mathsf{MM}} D^{\mathsf{M}}(\nabla + i\eta e_3)U^{\mathsf{M}}, D^{\mathsf{M}}(\nabla + i\eta e_3)V^{\mathsf{M}}\big)_{\varpi^\varepsilon} \\ &\quad + \ \mathcal{R}^{\varepsilon,\eta}(U^{\mathsf{M}}, V^{\mathsf{M}}) + (\varrho U^{\mathsf{M}}, V^{\mathsf{M}})_{\varpi^\varepsilon} \\ &=: \ \mathcal{A}^{\varepsilon,\eta}(U^{\mathsf{M}}, V^{\mathsf{M}}) + \mathcal{R}^{\varepsilon,\eta}(U^{\mathsf{M}}, V^{\mathsf{M}}) + (\varrho U^{\mathsf{M}}, V^{\mathsf{M}})_{\varpi^\varepsilon} \end{aligned} \tag{38}$$

defines an inner product in the space $H^1_{\rm per}(\varpi^\varepsilon)^3$. The operator $\mathcal{M}^{\varepsilon,\eta} : H^1_{\rm per}(\varpi^\varepsilon)^3 \to H^1_{\rm per}(\varpi^\varepsilon)^3$, defined by

$$\mathcal{B}^{\varepsilon,\eta}(\mathcal{M}^{\varepsilon,\eta}U^{\mathsf{M}}, V^{\mathsf{M}}) = (\varrho U^{\mathsf{M}}, V^{\mathsf{M}})_{\varpi^\varepsilon} \ \ \forall\, U^{\mathsf{M}}, V^{\mathsf{M}} \in H^1_{\rm per}(\varpi^\varepsilon)^3$$

is positive and self-adjoint. Since the domain ϖ^ε is bounded, the embedding $H^1_{\rm per}(\varpi^\varepsilon)^3 \hookrightarrow L^2(\varpi^\varepsilon)^3$ and thus also the operator $\mathcal{M}^{\varepsilon,\eta}$ are compact, and the spectrum of the η-dependent problem is discrete. The problems (32) and thus (29) are equivalent to the following abstract equation in $H^1_{\rm per}(\varpi^\varepsilon)^3$,

$$\mathcal{M}^{\varepsilon,\eta}U^{\mathsf{M}} = M^{\varepsilon,\eta}U^{\mathsf{M}},$$

where the spectral parameters are related by $M^{\varepsilon,\eta} = (1+\Lambda^{\varepsilon,\eta})^{-1}$. As a consequence, the eigenvalue sequence of (32) can be written, counting multiplicities, as

$$0 \le \Lambda_1^{\varepsilon,\eta} \le \Lambda_2^{\varepsilon,\eta} \le \ldots \le \Lambda_p^{\varepsilon,\eta} \le \ldots \to +\infty. \tag{39}$$

Due to the remarks below (36) and Lemma 2.2, the numbers $\Lambda_p^{\varepsilon,\eta}$ depend continuously and 2π-periodically on η. The corresponding eigenvectors are denoted by $U_{(p)}^{\varepsilon,\eta,\mathsf{M}} \in H^1_{\rm per}(\varpi^\varepsilon)^3$, and they are subject to the orthogonality and normalization conditions

$$(U_{(p)}^{\varepsilon,\eta,\mathsf{M}}, U_{(q)}^{\varepsilon,\eta,\mathsf{M}})_{\varpi^\varepsilon} = \delta_{p,q}\ ,\ p,q = 1,2,\ldots \tag{40}$$

At the end of this section we complete the study of the model problem by proving a technical necessity:

Lemma 2.1. *The operator norms of $T^{\varepsilon,\eta}$ are uniformly bounded, i.e. there exists a constant $C>0$, independent of ε and η, such that*

$$\|T^{\varepsilon,\eta}U^{\mathsf{M}}; H^1(\varpi^\varepsilon)\| \le C\|U^{\mathsf{M}}; H^1(\varpi^\varepsilon)\| \quad \forall\, U^{\mathsf{M}} \in H^1_{\rm per}(\varpi^\varepsilon)^3. \tag{41}$$

This is a consequence of the following Lemma 2.2 concerning the equation

$$\begin{aligned} &\big(A^{\mathsf{EE}}D^{\mathsf{E}}(\nabla + i\eta e_3)U^{\mathsf{E}}, D^{\mathsf{E}}(\nabla + i\eta e_3)V^{\mathsf{E}}\big)_{\varpi^\varepsilon} \\ =\ & F^{\mathsf{E}}(\eta; V^{\mathsf{E}}) \quad \forall\, V^{\mathsf{E}} \in H^1_{\rm per}(\varpi^\varepsilon), \end{aligned} \tag{42}$$

where $\eta \in [-\pi,\pi)$ and $F^{\mathsf{E}}(\eta;\cdot) \in H^1_{\rm per}(\varpi^\varepsilon)^*$. It is obvious that the linear form defined by the right hand side of (34) satisfies the assumptions on F^{E} of the lemma, since $D^{\mathsf{E}}1 = D^{\mathsf{E}}(\nabla)1 = 0$.

Lemma 2.2. *Assume that the functional $F^{\mathsf{E}}(\eta;\cdot)$ satisfies the bounds $\|F^{\mathsf{E}}(\eta;\cdot);\, H^1_{\rm per}(\varpi^\varepsilon)^*\| \le c$ and $|F^{\mathsf{E}}(\eta;1)| \le c|\eta|$ for all η. Then the problem (42) has for any $\eta \in [-\pi,\pi)$ a solution $U^{\mathsf{E}}(\cdot;\eta) \in H^1_{\rm per}(\varpi^\varepsilon)$, and*

$$\begin{aligned} &\|\nabla U^{\mathsf{E}}(\cdot;\eta); L^2(\varpi^\varepsilon)\| + |\eta|\|U^{\mathsf{E}}(\cdot;\eta); L^2(\varpi^\varepsilon)\| \\ \le\ & c_1\big(\|F^{\mathsf{E}}(\eta;\cdot); H^1_{\rm per}(\varpi^\varepsilon)^*\| + |\eta|^{-1}|F^{\mathsf{E}}(\eta;1)|\big) \le C, \end{aligned} \tag{43}$$

where c_1 and C do not depend on ε or η.

Proof. For $\eta \in [-\pi,\pi)$ with $|\eta| \ge \delta > 0$ the statement is obvious, because the problem (42) with $\delta \neq 0$ is uniquely solvable: it is self-adjoint and $D^{\mathsf{E}}(\nabla + i\eta e_3)U^{\mathsf{E}} = 0$ implies $U^{\mathsf{E}} = 0$ (due to the periodicity in z), and, moreover, the solution depends continuously on η in a compact interval.

It suffices to consider small η. Thus, for $|\eta| < \delta$ we search for a solution in the form

$$U^{\mathsf{E}}(x:\eta) = a(\eta)\eta^2(1 + i\eta\phi(x)) + \widetilde{U}^{\mathsf{E}}(x;\eta), \tag{44}$$

where $\phi \in H^1_{\rm per}(\varpi^\varepsilon)$ is a solution of the problem

$$(A^{\mathsf{EE}}D^{\mathsf{E}}\phi, D^{\mathsf{E}}V^{\mathsf{E}})_{\varpi^\varepsilon} + (A^{\mathsf{EE}}e_3, D^{\mathsf{E}}V^{\mathsf{E}})_{\varpi^\varepsilon} = 0 \quad \forall\, V^{\mathsf{E}} \in H^1_{\rm per}(\varpi^\varepsilon). \tag{45}$$

For the last term $\widetilde{U}^{\mathsf{E}}$ in (44) we obtain the problem

$$\begin{aligned} (A^{\mathsf{EE}}D^{\mathsf{E}}\widetilde{U}^{\mathsf{E}}, D^{\mathsf{E}}V^{\mathsf{E}})_{\varpi^\varepsilon} &= F^{\mathsf{E}}(\eta;V^{\mathsf{E}}) + \eta\widetilde{F}^{\mathsf{E}}(\eta;V^{\mathsf{E}}) \\ &+ a(\eta)F^{\mathsf{E}}_\phi(\eta;V^{\mathsf{E}}) \quad \forall\, V^{\mathsf{E}} \in H^1_{\rm per}(\varpi^\varepsilon), \end{aligned} \tag{46}$$

where

$$\begin{aligned} \widetilde{F}^{\mathsf{E}}(\eta;V^{\mathsf{E}}) &= i(A^{\mathsf{EE}}D^{\mathsf{E}}\widetilde{U}^{\mathsf{E}}, e_3V^{\mathsf{E}})_{\varpi^\varepsilon} - i(A^{\mathsf{EE}}e_3\widetilde{U}^{\mathsf{E}}, D^{\mathsf{E}}V^{\mathsf{E}})_{\varpi^\varepsilon} - \eta(A^{\mathsf{EE}}e_3\widetilde{U}^{\mathsf{E}}, e_3V^{\mathsf{E}})_{\varpi^\varepsilon}, \\ F^{\mathsf{E}}_\phi(\eta;V^{\mathsf{E}}) &= (A^{\mathsf{EE}}D^{\mathsf{E}}\phi, e_3V^{\mathsf{E}})_{\varpi^\varepsilon} - (A^{\mathsf{EE}}e_3\phi, D^{\mathsf{E}}V^{\mathsf{E}})_{\varpi^\varepsilon} \\ &+ (A^{\mathsf{EE}}e_3, e_3V^{\mathsf{E}})_{\varpi^\varepsilon} + i\eta(A^{\mathsf{EE}}e_3\phi, e_3V^{\mathsf{E}})_{\varpi^\varepsilon} \end{aligned}$$

We note that (45) implies

$$\begin{aligned} & (A^{\mathsf{EE}} D^{\mathsf{E}}\phi, e_3)_{\varpi^\varepsilon} - (A^{\mathsf{EE}} e_3\phi, D^{\mathsf{E}}1)_{\varpi^\varepsilon} + (A^{\mathsf{EE}} e_3, e_3)_{\varpi^\varepsilon} \\ = \; & (A^{\mathsf{EE}} e_3, e_3)_{\varpi^\varepsilon} + 2(A^{\mathsf{EE}} D^{\mathsf{E}}\phi, e_3)_{\varpi^\varepsilon} - (A^{\mathsf{EE}} e_3, D^{\mathsf{E}}\phi)_{\varpi^\varepsilon} \\ = \; & (A^{\mathsf{EE}} e_3, e_3)_{\varpi^\varepsilon} + 2(A^{\mathsf{EE}} D^{\mathsf{E}}\phi, e_3)_{\varpi^\varepsilon} + (A^{\mathsf{EE}} D^{\mathsf{E}}\phi, D^{\mathsf{E}}\phi)_{\varpi^\varepsilon} \\ = \; & (A^{\mathsf{EE}} (e_3 + D^{\mathsf{E}}\phi), e_3 + D^{\mathsf{E}}\phi)_{\varpi^\varepsilon} = \varphi > 0, \end{aligned} \tag{47}$$

since $e_3 + D^{\mathsf{E}}\phi \neq 0$ cannot vanish everywhere in ϖ^ε due to the periodicity of ϕ in z.

Although the right hand side of (46) depends on $\widetilde{U}^{\mathsf{E}}$, we regard it for a while as given and write the combatibility condition for the Neumann problem (46) as

$$F^{\mathsf{E}}(\eta;1) + \eta \widetilde{F}^{\mathsf{E}}(\eta;1) + a(\eta) F^{\mathsf{E}}(\eta;1) = 0,$$

which in view of (47) turns into

$$a(\eta) = -\big(\varphi + i\eta (A^{\mathsf{EE}} e_3\phi, e_3)_{\varpi^\varepsilon}\big)^{-1} \big(F^{\mathsf{E}}(\eta;1) + \eta \widetilde{F}^{\mathsf{E}}(\eta;1)\big), \tag{48}$$

while

$$|a(\eta)| \leq c\big(|F^{\mathsf{E}}(\eta;1)| + |\eta|\,|\widetilde{F}^{\mathsf{E}}(\eta;1)|\big). \tag{49}$$

Now we insert (48) into (46), and using a perturbation argument for small $|\eta|$, find a solution $\widetilde{U}^{\mathsf{E}} \in H^1_{\mathrm{per}}(\varpi^\varepsilon)$ with the estimate

$$\|\widetilde{U}^{\mathsf{E}}; H^1(\varpi^\varepsilon)\| \leq c\big(\|F^\varepsilon; H^1_{\mathrm{per}}(\varpi^\varepsilon)^*\| + |a(\eta)|\big). \tag{50}$$

The representation (46) and the estimates (49), (50) yield the desired inequality (43). ⊠

2.3. Essential spectrum of the piezoelectricity waveguide.

It is known for example in the case of the elasticity system that the essential spectrum of the problem on the periodic waveguide Π^ε coincides with the union of the parameter dependent spectra on the bounded periodicity cell ϖ^ε (see for example [25, Thm. 2.1]). However, the existing results do not cover the present piezoelectricity case due to the appearance of the non-local operator T^ε, and we thus have to reprove the theorem in this more general setting.

Theorem 2.3. *The essential spectrum of the operator $\mathcal{M}^\varepsilon$, (23), is the union*

$$\sigma_{\mathrm{ess}}(\mathcal{M}^\varepsilon) = \bigcup_{p\in\mathbb{N}} \upsilon^\varepsilon_p, \tag{51}$$

of the spectral bands

$$\upsilon^\varepsilon_p = \{\mu \,:\, \lambda = 1 - \mu^{-1} \in \Upsilon^\varepsilon_p\}, \tag{52}$$

where

$$\Upsilon^\varepsilon_p = \{\lambda \,:\, \lambda = \Lambda^{\varepsilon,\eta}_p \text{ for some } \eta \in [-\pi,\pi)\}. \tag{53}$$

The complement $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{M}^\varepsilon)$ of (51) is the resolvent set of the operator $\mathcal{M}^\varepsilon$.

Remark 2.4. The (bounded, closed, connected) intervals (53) must be regarded as spectral bands of the reduced piezoelectricity problem (19) of the original problem (12) modulo constant electric potentials.

Proof. I. Assume first that μ does not belong to any band (52) and consider the abstract equation

$$\mathcal{M}^\varepsilon u^{\mathsf{M}} - \mu u^{\mathsf{M}} = f^{\mathsf{M}} \in H^1(\Pi^\varepsilon)^3.$$

In view of (22), (21), this is equivalent to the integral identity

$$\begin{aligned} & \mathcal{A}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) + \mathcal{R}^\varepsilon(u^{\mathsf{M}}, v^{\mathsf{M}}) - \lambda(\varrho u^{\mathsf{M}}, v^{\mathsf{M}})_{\Pi^\varepsilon} \\ = \; & \langle f^{\mathsf{M}}, v^{\mathsf{M}}\rangle_{\Pi^\varepsilon} \;\; \forall v^{\mathsf{M}} \in H^1(\Pi^\varepsilon)^3, \end{aligned} \tag{54}$$

where we denote by $\langle\cdot,\cdot\rangle_\Omega$ the inner product of the Sobolev space $H^1(\Omega)^k$ and $\lambda = 1-\mu^{-1}$ does not belong to any band (53). Using the FBG-transform and Parseval identity we arrive at a family of problems depending on $\eta\in[-\pi,\pi)$,

$$\begin{aligned} &\mathcal{A}^{\varepsilon,\eta}(U^{\mathsf M},V^{\mathsf M})+\mathcal{R}^{\varepsilon,\eta}(U^{\mathsf M},V^{\mathsf M})-\lambda(\varrho U^{\mathsf M},V^{\mathsf M})_{\varpi^\varepsilon}\\ =\ &\langle F^{\mathsf M},V^{\mathsf M}\rangle_{\varpi^\varepsilon}\quad \forall V^{\mathsf M}\in H^1_{\rm per}(\varpi^\varepsilon)^3,\end{aligned}$$

where $U^{\mathsf M}$, $V^{\mathsf M}$, and $F^{\mathsf M}$ are the FBG-images of $u^{\mathsf M}$, $v^{\mathsf M}$, and $f^{\mathsf M}$. The FBG-transform applied to (18) and (14) also yield the relations (cf. (37), (34))

$$\begin{aligned} &\mathcal{R}^{\varepsilon,\eta}(U^{\mathsf M},V^{\mathsf M})=\big(A^{\mathsf{ME}}D^{\mathsf E}(\nabla+i\eta e_3)U^{\mathsf E},D^{\mathsf E}(\nabla+i\eta e_3)V^{\mathsf M}\big)_{\varpi^\varepsilon}\\ &\big(A^{\mathsf{EE}}D^{\mathsf E}(\nabla+i\eta e_3)U^{\mathsf E},D^{\mathsf E}(\nabla+i\eta e_3)V^{\mathsf E}\big)_{\varpi^\varepsilon}\\ =\ &\big(A^{\mathsf{EM}}D^{\mathsf E}(\nabla+i\eta e_3)U^{\mathsf M},D^{\mathsf E}(\nabla+i\eta e_3)V^{\mathsf E}\big)_{\varpi^\varepsilon}. \end{aligned}\tag{55}$$

Due to our assumption on $\mu=(1+\lambda)^{-1}$, the problem (55) has a unique solution $U^{\mathsf M}\in H^1(\varpi^\varepsilon)$ for all $\eta\in[-\pi,\pi)$. Hence, the estimate

$$\int_{-\pi}^{\pi}\|U^{\mathsf M}(\cdot;\eta);H^1(\varpi^\varepsilon)\|^2d\eta\le C_\lambda\int_{-\pi}^{\pi}\|F^{\mathsf M}(\eta;\cdot);H^1(\varpi^\varepsilon)\|^2d\eta$$

is valid, and the Parseval identity turns this into

$$\|u^{\mathsf M}(\cdot;\eta);H^1(\Pi^\varepsilon)\|^2\le C_\lambda\|f^{\mathsf M}(\cdot;\eta);H^1(\Pi^\varepsilon)\|^2.$$

Thus, $\mathbb{C}\setminus\bigcup_p \upsilon_p^\varepsilon$ is contained in the resolvent set.

II. We shall prove that for all $p\in\mathbb{N}$, $\eta\in[-\pi,\pi)$, every number $\mu\in\upsilon_p^\varepsilon$ belongs to the essential spectrum of $\mathcal{M}^\varepsilon$. To this end we construct a singular Weyl sequence $\{u^{\mathsf M}_{(N)}\}_{N\in\mathbb{N}}$, which by definition must have the following properties:
1°. $\|u^{\mathsf M}_{(N)};H^1(\Pi^\varepsilon)\|=1$,
2°. $u^{\mathsf M}_{(N)}\to 0$ weakly in $H^1(\Pi^\varepsilon)$,
3°. $\|(\mathcal{M}^\varepsilon-\mu)u^{\mathsf M}_{(N)};H^1(\Pi^\varepsilon)\|\to 0$ as $N\to+\infty$.

By definitions, $\mu=(1+\lambda)^{-1}$, where $\lambda=\Lambda_p^{\varepsilon,\eta}$ for some p and η. Let $U^{\varepsilon,\eta,\mathsf M}_{(p)}$ be the corresponding normalised eigenfunction (40) of the problem (36), so that $U^{\varepsilon,\eta}_{(p)}=(U^{\varepsilon,\eta,\mathsf M}_{(p)},U^{\varepsilon,\eta,\mathsf E}_{(p)}):=(U^{\varepsilon,\eta,\mathsf M}_{(p)},T^\varepsilon U^{\varepsilon,\eta,\mathsf M}_{(p)})$ is the vector eigenfunction of the problem (29)–(31). Then, the Floquet wave

$$w(y,z)=e^{i\eta z}U^{\varepsilon,\eta}_{(p)}(y,z;\eta)\tag{56}$$

satisfies the homogeneous problem (5)–(6), which means that there holds the integral identity

$$\begin{aligned} &\big(A^{\mathsf{MM}}D^{\mathsf M}w^{\mathsf M},D^{\mathsf M}(\chi_N v^{\mathsf M})\big)_{\Pi^\varepsilon}+\big(A^{\mathsf{ME}}D^{\mathsf E}w^{\mathsf E},D^{\mathsf M}(\chi_N v^{\mathsf M})\big)_{\Pi^\varepsilon}\\ -\ &\big(A^{\mathsf{EM}}D^{\mathsf M}w^{\mathsf M},D^{\mathsf E}(\chi_N v^{\mathsf E})\big)_{\Pi^\varepsilon}+\big(A^{\mathsf{EE}}D^{\mathsf E}w^{\mathsf E},D^{\mathsf E}(\chi_N v^{\mathsf E})\big)_{\Pi^\varepsilon}\\ =\ &\lambda(\varrho w^{\mathsf M},\chi_N v^{\mathsf M})_{\Pi^\varepsilon}\quad\forall v=(v^{\mathsf E},v^{\mathsf E})\in H^1(\Pi^\varepsilon)^4, \end{aligned}\tag{57}$$

where for all $N\in\mathbb{N}$ we denote by $X_N:\mathbb{R}\to[0,1]$ a smooth cut-off-function $X_N(z)=\chi(z-2^N+\frac12)\chi(2^{N+1}+\frac12-z)$, where $\chi:\mathbb{R}\to[0,1]$ is a C^∞-function with $\chi(z)=0$ for $z\le 0$ and $\chi(z)=1$ for $z\ge 1$. So, X_N has compact support and satisfies

$$X_N(z)=\begin{cases}1\,, & z\in[2^N+1/2,2^{N+1}-1/2,]\\ 0\,, & z\notin[2^N-1/2,2^{N+1}+1/2]\end{cases}$$

and also $|\partial_z X_N(z)|\le C$ for all z; see Fig. 3.

We set

$$u^{\mathsf M}_{(N)}=\|X_N w^{\mathsf M};L^2(\Pi^\varepsilon)\|^{-1}X_N w^{\mathsf M}$$

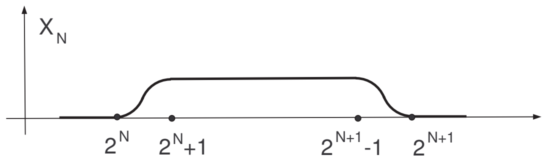

FIGURE 3. Cut-off-function.

and observe that

$$\|X_N w^{\mathsf{M}}; L^2(\Pi^\varepsilon)\|^2 \geq (2^{N+1} - 2^N - 2)\|U_p^{\varepsilon\mathsf{M}}; L^2(\varpi^\varepsilon)\|^2 \tag{58}$$

for $N \geq 2$. Thus, the properties 1° and 2° hold true, at least for a subsequence of these functions, because $\operatorname{supp} u_{(N)}^{\mathsf{M}} \cap \operatorname{supp} u_{(Q)}^{\mathsf{M}} = \varnothing$ for $N \neq Q$.

We have, with the notation of (54),

$$\begin{aligned} & \|(\mathcal{M}^\varepsilon - \mu)u_{(N)}^{\mathsf{M}}; H^1(\Pi^\varepsilon)\| = \sup \left|\langle(\mathcal{M}^\varepsilon - \mu)u_{(N)}^{\mathsf{M}}, v^{\mathsf{M}}\rangle_{\Pi^\varepsilon}\right. \\ = \; & \|X_N w^{\mathsf{M}}; L^2(\Pi^\varepsilon)\|^{-1}\mu^{-1} \sup \left|\mathcal{A}^\varepsilon(X_N w^{\mathsf{M}}, v^{\mathsf{M}}) + \mathcal{R}^\varepsilon(X_N w^{\mathsf{M}}, v^{\mathsf{M}})\right. \\ - \; & \left.\lambda(\varrho X_N w^{\mathsf{M}}, v^{\mathsf{M}})_{\Pi^\varepsilon}\right|, \end{aligned} \tag{59}$$

where the supremum is computed over all $v^{\mathsf{M}} \in H^1(\Pi^\varepsilon)^3$ such that $\|v^{\mathsf{M}}; H^1(\Pi^\varepsilon)\| = 1$. Let us process the expression within the supremum on the right of (59). To this end, we commute in (57) the cut-off function X_N and the operators D^{M}, D^{E} several times, and a simple but lengthy calculation yields

$$\begin{aligned} & \left(A^{\mathsf{MM}} D^{\mathsf{M}}(X_N w^{\mathsf{M}}), D^{\mathsf{M}} v^{\mathsf{M}}\right)_{\Pi^\varepsilon} + \left(A^{\mathsf{ME}} D^{\mathsf{E}}(X_N w^{\mathsf{E}}), D^{\mathsf{M}} v^{\mathsf{M}}\right)_{\Pi^\varepsilon} \\ = \; & F_N^{\mathsf{M}0}(v^{\mathsf{M}}) + F_N^{\mathsf{M}1}(v^{\mathsf{M}}), \end{aligned} \tag{60}$$

$$\begin{aligned} & \left(A^{\mathsf{EE}} D^{\mathsf{E}}(X_N w^{\mathsf{E}}), D^{\mathsf{E}} v^{\mathsf{E}}\right)_{\Pi^\varepsilon} - \left(A^{\mathsf{EM}} D^{\mathsf{M}}(X_N w^{\mathsf{M}}), D^{\mathsf{E}} v^{\mathsf{E}}\right)_{\Pi^\varepsilon} \\ = \; & F_N^{\mathsf{E}0}(v^{\mathsf{E}}) + F_N^{\mathsf{E}1}(v^{\mathsf{E}}), \end{aligned} \tag{61}$$

where

$$\begin{aligned} F_N^{\mathsf{M}0}(v^{\mathsf{M}}) &= -\left(A^{\mathsf{MM}} D^{\mathsf{M}} w^{\mathsf{M}} + A^{\mathsf{ME}} D^{\mathsf{E}} w^{\mathsf{E}}, [D^{\mathsf{M}}, X_N] v^{\mathsf{M}}\right)_{\Pi^\varepsilon}, \\ F_N^{\mathsf{M}1}(v^{\mathsf{M}}) &= \left(A^{\mathsf{MM}}[D^{\mathsf{M}}, X_N] w^{\mathsf{M}} + A^{\mathsf{ME}}[D^{\mathsf{E}}, X_N] w^{\mathsf{E}}, D^{\mathsf{M}} v^{\mathsf{M}}\right)_{\Pi^\varepsilon}, \end{aligned}$$

and

$$\begin{aligned} F_N^{\mathsf{E}0}(v^{\mathsf{E}}) &= -\left(A^{\mathsf{EE}} D^{\mathsf{E}} w^{\mathsf{E}} - A^{\mathsf{EM}} D^{\mathsf{M}} w^{\mathsf{M}}, [D^{\mathsf{E}}, X_N] v^{\mathsf{E}}\right)_{\Pi^\varepsilon}, \\ F_N^{\mathsf{E}1}(v^{\mathsf{M}}) &= \left(A^{\mathsf{EE}}[D^{\mathsf{E}}, X_N] w^{\mathsf{E}} - A^{\mathsf{EM}}[D^{\mathsf{M}}, X_N] w^{\mathsf{M}}, D^{\mathsf{E}} v^{\mathsf{E}}\right)_{\Pi^\varepsilon}. \end{aligned} \tag{62}$$

Since the support of the matrix function $[D^{\mathsf{M}}, X_N] = D^{\mathsf{M}}(\nabla) X_N$ is contained in the union of the closures of the two cells $\varpi_k^\varepsilon = \{x \in \Pi^\varepsilon : z \in (k - 1/2, k + 1/2)\}$ with $k = 2^N$ and $k = 2^{N+1} - 1$, we have

$$\left|F_N^{\mathsf{M}0}(v^{\mathsf{M}})\right| + \left|F_N^{\mathsf{M}1}(v^{\mathsf{M}})\right| \leq C\|v^{\mathsf{M}}; H^1(\Pi^\varepsilon)\|, \tag{63}$$

where C is independent of N and v^{M}. Furthermore, (14), (17) and (61) yield

$$X_N w^{\mathsf{E}} = T^\varepsilon(X_N w^{\mathsf{E}}) - W^{\mathsf{E}0} - W^{\mathsf{E}1}, \tag{64}$$

where $W^{\mathsf{E}q} \in H^{\mathsf{E}}(\Pi^\varepsilon)$ (cf. (11)) is a solution of the Neumann problem

$$\left(A^{\mathsf{EE}} D^{\mathsf{E}} W^{\mathsf{E}q}, D^{\mathsf{E}} v^{\mathsf{E}}\right)_{\Pi^\varepsilon} = F^{\mathsf{E}q}(v^{\mathsf{E}}) \ \ , \ v^{\mathsf{E}} \in H^{\mathsf{E}}(\Pi^\varepsilon).$$

The right hand side of (62) involves $D^{\mathsf{E}}v^{\mathsf{E}}$ only and therefore we obtain similarly to (63)

$$|F^{\mathsf{E}1}(v^{\mathsf{E}})| \leq C\|w^{\mathsf{M}}; L^2(\varpi^\varepsilon_{2^N} \cup \varpi^\varepsilon_{2^{N+1}-1})\| \, \|\nabla v^{\mathsf{E}}; L^2(\Pi^\varepsilon)\| \leq C\|v^{\mathsf{E}}; H^{\mathsf{E}}(\Pi^\varepsilon)\|, \tag{65}$$

$$\|D^{\mathsf{E}}W^{\mathsf{E}1}; L^2(\Pi^\varepsilon)\| \leq C. \tag{66}$$

Since the norm of the space $H^{\mathsf{E}}(\Pi^\varepsilon)$ contains a weight, the estimate (65) does not hold for $F^{\mathsf{E}0}(v^{\mathsf{E}})$. However, using the Poincaré inequality in $\varpi^\varepsilon_{2^N}$ and $\varpi^\varepsilon_{2^{N+1}-1}$ we obtain

$$\begin{aligned} & F^{\mathsf{E}0}_N(v^{\mathsf{E}}) = F^{\mathsf{E}\bullet}_{2^N}(v^{\mathsf{E}}) - F^{\mathsf{E}\bullet}_{2^{N+1}-1/2}(v^{\mathsf{E}}) \quad , \quad F^{\mathsf{E}0}_{2^N}(v^{\mathsf{E}}) = \\ & F^{\mathsf{E}\bullet}_{2^N}(v^{\mathsf{E}}) = \big([D^{\mathsf{E}}, \chi(z-2^N-1/2)](A^{\mathsf{EE}}w^{\mathsf{E}} - A^{\mathsf{EM}}D^{\mathsf{M}}w^{\mathsf{M}}), v^{\mathsf{E}}\big)_{\varpi^\varepsilon_{2^n}}, \\ & \big|F^{\mathsf{E}\bullet}_{2^N}(v^{\mathsf{E}} - v^{\mathsf{E}\bullet}_{2^N})\big| + \big|F^{\mathsf{E}\bullet}_{2^{N+1}-1}(v^{\mathsf{E}} - v^{\mathsf{E}\bullet}_{2^{N+1}-1})\big| \\ \leq \; & C\|\nabla v^{\mathsf{E}}; L^2(\Pi^\varepsilon)\| \leq C\|v^{\mathsf{E}}; H^{\mathsf{E}}(\Pi^\varepsilon)\|. \end{aligned}$$

Moreover, the estimate

$$\|D^{\mathsf{E}}W^{\mathsf{E}0}; L^2(\Pi^\varepsilon)\| \leq C \tag{67}$$

is a consequence of the relation

$$\begin{aligned} F^{\mathsf{E}\bullet}_{2^N}(v^{\mathsf{E}\bullet}_{2^N}) \;&=\; \overline{v^{\mathsf{E}\bullet}_{2^N}} \int\limits_{\varpi^\varepsilon_{2^N}} [D^{\mathsf{E}}, \chi_N]^\top (A^{\mathsf{EE}}D^{\mathsf{E}}w^{\mathsf{E}} - \\ &-\; A^{\mathsf{EM}}D^{\mathsf{M}}w^{\mathsf{M}})dx = 0, \end{aligned} \tag{68}$$

which in turn follows from Lemma 2.5, below; we denote $\chi_N(z) = \chi(z - 2^N - \frac{1}{2})$ here. Now we can conclude with the property 3°: we insert (64) into (60) and use the estimates (63), (65) and (66), (67) to obtain

$$\big|\mathcal{A}^\varepsilon(X_N w^{\mathsf{M}}, v^{\mathsf{M}}) + \mathcal{R}^\varepsilon(X_N w^{\mathsf{M}}, v^{\mathsf{M}}) - \lambda(\varrho X_N w^{\mathsf{M}}, v^{\mathsf{M}})_{\Pi^\varepsilon}\big| \leq C.$$

The property 3° follows from this, (59) and (58). ⊠

We are left with the proof of the following

Lemma 2.5. *Let w be the Floquet wave* (56) *corresponding to the eigenvalue $\Lambda^{\varepsilon,\eta}_p$ and the eigenfunction $U^{\varepsilon,\eta,\mathsf{M}}_{(p)}$ of the problem* (36)*. Then w satisfies the formula*

$$\int\limits_{\Pi^\varepsilon} [D^{\mathsf{E}}, \chi]^\top (A^{\mathsf{EE}}D^{\mathsf{E}}w^{\mathsf{E}} - A^{\mathsf{EM}}D^{\mathsf{M}}w^{\mathsf{M}})dx = 0, \tag{69}$$

where $\chi \in C^\infty(\mathbb{R})$ is any function which equals one near $+\infty$ and null near $-\infty$, for example, $\chi(z) = 1$ for $z > z_+$ and $\chi(z) = 0$ for $z < z_- < z_+$ for some numbers $z_- < z_+$.

Proof. This result can be obtained by different arguments, for example, by using the orthogonality and normalization conditions for Jordan chains of the quadratic pencil $\eta \mapsto \mathfrak{A}(\eta; \lambda)$ generated by the model problem on the periodicity cell, (29)–(31) (or (32) in the variational form). Accordingly, in [27, § 5.2–§ 5.4] it was verified that in the case of a formally self-adjoint elliptic problem with smooth coefficients (see Section 1.5 and the substitutions (25)), one can choose a basis $\{w^0, w^2, \ldots, w^m, \ldots\}$ in the linear span of Floquet waves such that

$$\begin{aligned} Q^\bullet(\chi w^j, \chi w^k) \;&=\; (L^\bullet \chi w^j, \chi w^k)_{\Pi^\varepsilon} + (B^\bullet \chi w^j, \chi w^k)_{\partial\Pi^\varepsilon} \\ &-\; (\chi w^j, L^\bullet \chi w^k)_{\Pi^\varepsilon} - (\chi w^j, B^\bullet \chi w^k)_{\partial\Pi^\varepsilon} = 0 \end{aligned}$$

as $j \neq k$. We observe that the symmetric Green formula implies

$$Q^\bullet((1-\chi)w^j, \chi w^k) = 0,$$

because $(1-\chi)w^j = 0$ for $z > z^+$ and $\chi w^k = 0$ for $z < z^-$. Thus,

$$\begin{aligned} & 0 = Q^\bullet(w^j, \chi w^k) \\ = \;& \big(w^j, D(\nabla)^\top A^\bullet D(\nabla)\chi w^k\big)_{\Pi^\varepsilon} - \big(w^j, D(\nabla)^\top A^\bullet D(\nabla)\chi w^j\big)_{\Pi^\varepsilon} \\ = \;& \big(w^j, D(\nabla)^\top A^\bullet [D(\nabla),\chi] w^k\big)_{\Pi^\varepsilon} + \big(w^j, [D(\nabla),\chi]^\top A^\bullet D(\nabla)\chi w^k\big)_{\Pi^\varepsilon} \\ - \;& \big(w^j, D(\nabla)^\top A^\bullet D(\nabla)\chi w^k\big)_{\Pi^\varepsilon} \\ = \;& \big(w^j, [D(\nabla),\chi]^\top A^\bullet D(\nabla) w^k\big)_{\Pi^\varepsilon} - \big(A^\bullet D(\nabla) w^j, [D(\nabla),\chi] w^k\big)_{\Pi^\varepsilon}. \end{aligned} \tag{70}$$

Notice that this orthogonality condition can be extended for non-smooth coefficients and boundaries by a completion argument.

The particular Floquet wave $w^0 = (0,0,0,1)$ exists for every λ and it differs from the Floquet waves (56) because of our reduction scheme, which excludes constant electric potentials. Hence, we derive from (70) the desired equality

$$0 = -\big(A^\bullet D(\nabla) w, [D(\nabla),\chi] w^0\big)_{\Pi^\varepsilon} = \big(A^{\mathsf{EE}} D^{\mathsf{E}} w^{\mathsf{E}} + A^{\mathsf{EM}} D^{\mathsf{M}} w^{\mathsf{E}}, [D^{\mathsf{E}},\chi]\big)_{\Pi^\varepsilon}.$$

Let us describe another way to conclude with (69) in the case of the same cut-off function $\chi_N(z)$ as in (68). We again assume that the coefficients are smooth and obtain

$$\begin{aligned} & \int_{\varpi^\varepsilon} [D^{\mathsf{E}}, \chi_N]\big(A^{\mathsf{EE}} D^{\mathsf{E}} w^{\mathsf{E}} - A^{\mathsf{EM}} D^{\mathsf{M}} w^{\mathsf{E}}\big) dx \\ = & \int_{\varpi^\varepsilon} \chi_N D^{\mathsf{E}}\big(A^{\mathsf{EE}} D^{\mathsf{E}} w^{\mathsf{E}} - A^{\mathsf{EM}} D^{\mathsf{M}} w^{\mathsf{E}}\big) dx + I(1), \end{aligned} \tag{71}$$

where

$$I(t) \;=\; \int_{G(t)} D^{\mathsf{E}}(e_3)^\top \big(A^{\mathsf{EE}} D^{\mathsf{E}} w^{\mathsf{E}} - A^{\mathsf{EM}} D^{\mathsf{M}} w^{\mathsf{E}}\big) dy, \tag{72}$$

and $G(t) := \{(y,z) \in \overline{\varpi^\varepsilon} : z = t\}$. The first integral on the right side of (71) vanishes, due to the "electricty" line in the system (5), and the integral (72) is independent of $t \in [0,1]$ by the Green formula. Moreover, we have

$$\begin{aligned} I(1) = e^{i\eta t}\Big|_{t=1} \int_{G(t)} D^{\mathsf{E}}(e_3)^\top \big(\quad & A^{\mathsf{EE}} D^{\mathsf{E}}(\nabla + i\eta e_3) U^{\varepsilon,\eta,\mathsf{E}}_{(p)} \\ - \quad & A^{\mathsf{EM}} D^{\mathsf{M}}(\nabla + i\eta e_3) U^{\varepsilon,\eta,\mathsf{E}}_{(p)}\big) dy = e^{i\eta} I(0). \end{aligned}$$

We thus see that for $\eta \in [-\pi,\pi) \setminus \{0\}$ the independence property is possible in the case $I(1) = 0$ only, that is, (69) is true. The case $\eta = 0$ follows by a continuity argument. ⊠

3. Upper bound for spectral bands.

In this section we prove an upper bound for the spectral bands. The proof of the lower bound is partially similar and it is given in Section 4. However, there are some differences, and in particular the proof of Lemma 4.1 in Section 4 uses in an essential way the upper bound obtained in this section.

We start in Section 3.1 with a review of the spectrum of the limit problem in the isolated cell. Moreover, we shall need a number of preparatory results. In [28], standard local elliptic estimates were used to derive pointwise estimates for the eigenfunctions of the limit problem near points corresponding to $\mathcal{O}^\pm$ of this paper, and similar estimates will also be used here, see Lemma 3.1. However, the piezoelectricity case contains more terms to be estimated, in particular those related with the non-local operators, and following the scheme of [28] would also require pointwise estimates of the eigenfunctions of the η-dependent problem. These are not available, since the boundary of the domain ϖ^ε is not smooth around the points $\mathcal{O}^{\pm}$, but that problem can be circumvented by using the weighted Sobolev estimates of Lemma 3.2 and 4.1.

3.1. Spectrum of the limit model problem.

We next consider the problem (29)–(31) at $\varepsilon = 0$ (fig.2, b)), in which case the ligaments of Π^{ε} vanish. Indeed, the periodicity conditions (31) cannot be stated any more, and the remaining problem for the unknown function U on ϖ consist of the equation (29) on ϖ and the condition (30) on the entire boundary $\partial\varpi$. We observe that for any η this problem has the same eigenvalues as the case $\eta = 0$: if u is a solution to the problem (73), below, with the eigenvalue Λ, then $U = \exp(-i\eta z)u$ solves (29) on ϖ with the same eigenvalue Λ. We thus led to consider the standard piezoelectricity problem for the isolated bounded body ϖ corresponding to the case $\eta = 0$ only (see fig. 3, c):

$$\begin{aligned} &\overline{D(-\nabla)}^{\top} A(x)D(\nabla)u = \varrho\Lambda u , \qquad x \in \varpi, \\ &\overline{D(n(x))}^{\top} A(x)D(\nabla)u = 0 , \qquad x \in \partial\varpi. \end{aligned} \tag{73}$$

By the remark above, we can now proceed similarly to Section 2.1 by setting $\eta = 0$, $\varepsilon = 0$. The weak formulation is written as

$$(AD(\nabla)u, D(\nabla)v)_{\varpi} = \Lambda\big(\varrho u^{\mathsf{M}}, v^{\mathsf{M}}\big)_{\varpi} \quad \forall v \in H^1(\varpi)^4. \tag{74}$$

The reduction scheme once more yields a self-adjoint problem with variational formulation similar to (12). We only need to fix some notation here. First, the operator $T : H^1(\varpi)^3 \to H^1(\varpi)$, $Tu^{\mathsf{M}} = u^{\mathsf{E}}$ is defined by the equation

$$\big(A^{\mathsf{EE}}D^{\mathsf{E}}Tu^{\mathsf{M}}, D^{\mathsf{E}}v^{\mathsf{E}}\big)_{\varpi^{\varepsilon}} = \big(A^{\mathsf{EM}}D^{\mathsf{M}}u^{\mathsf{M}}, D^{\mathsf{E}}v^{\mathsf{E}}\big)_{\varpi^{\varepsilon}} \quad \forall\, v^{\mathsf{E}} \in H^1(\varpi^{\varepsilon}).$$

The compatibility condition is treated in the same way as in (35), and the operator T is thus uniquely defined up to an additive constant. This we determine by requiring

$$\int\limits_{\varpi} Tu^{\mathsf{M}}dx = \int\limits_{\varpi} u^{\mathsf{E}}dx = 0.$$

The sesquilinear form $\mathcal{R} : H^1(\varpi)^3 \times H^1(\varpi)^3 \to \mathbb{C}$ is defined as the form $\mathcal{R}^{\varepsilon}$ in (18) with the help of the operator T:

$$\mathcal{R}(u^{\mathsf{M}}, v^{\mathsf{M}}) := (A^{\mathsf{ME}}D^{\mathsf{E}}Tu^{\mathsf{M}}, D^{\mathsf{M}}v^{\mathsf{M}})_{\varpi}. \tag{75}$$

Let us also set for u^{M}, $v^{\mathsf{M}} \in H^1(\varpi)^3$,

$$\begin{aligned} \mathcal{A}(u^{\mathsf{M}}, v^{\mathsf{M}}) &:= \big(A^{\mathsf{MM}}D^{\mathsf{M}}u^{\mathsf{M}}, D^{\mathsf{M}}v^{\mathsf{M}}\big)_{\varpi} , \\ \mathcal{B}(u^{\mathsf{M}}, v^{\mathsf{M}}) &:= \mathcal{A}(u^{\mathsf{M}}, v^{\mathsf{M}}) + \mathcal{R}(u^{\mathsf{M}}, v^{\mathsf{M}}) + (\varrho u^{\mathsf{M}}, v^{\mathsf{M}})_{\varpi} \end{aligned} \tag{76}$$

The proofs of Lemma 1.2 and Corollary 1.3 again imply that the form $\mathcal{R}$ is Hermitian, positive in $H^1(\varpi)$; moreover, it is coercive:

$$\|u^{\mathsf{M}}; H^1(\varpi)\|^2 \le C\mathcal{B}(u^{\mathsf{M}}, u^{\mathsf{M}}) \tag{77}$$

for a constant $C > 0$, for all $u^{\mathsf{M}} \in H^1(\varpi)^3$.

The equation

$$\mathcal{B}(\mathcal{M}u^{\mathsf{M}}, v^{\mathsf{M}}) = (\varrho u^{\mathsf{M}}, v^{\mathsf{M}})_{\varpi} \quad \forall\, u^{\mathsf{M}}, v^{\mathsf{M}} \in H^1(\varpi)^3$$

defines a positive, self-adjoint, compact operator $\mathcal{M} : H^1(\varpi)^3 \to H^1(\varpi)^3$. The problem (73) is equivalent to the abstract equation

$$\mathcal{M}u^{\mathsf{M}} = \nu u^{\mathsf{M}},$$

and the spectral concepts are defined via the connection $\nu = (1+\Lambda)^{-1}$ in the same way as above. The operator $\mathcal{M}$ has a decreasing sequence of positive eigenvalues converging to 0, and the problem (73) has the increasing eigenvalue sequence (counting multiplicities)

$$0 = \Lambda_1 = \ldots = \Lambda_6 < \Lambda_7 \le \Lambda_8 \le \ldots \le \Lambda_p \le \ldots \to +\infty . \tag{78}$$

For every p, let $u^{\mathsf{M}}_{(p)} \in H^1(\varpi)^3$ be the eigenfunction corresponding to Λ_p, normalized and orthogonal in $L^2(\varpi)^3$:

$$(u^{\mathsf{M}}_{(p)}, u^{\mathsf{M}}_{(q)})_{\varpi} = \delta_{p,q}\ ,\ \ p,q = 1,2,\dots. \tag{79}$$

We recall that by the max–min principle (see, e.g., [4, Th.10.2.2])

$$\Lambda_j = \max_{\mathcal{H}_j} \inf_{\mathcal{U}\in\mathcal{H}_j\setminus\{0\}} \frac{\mathcal{B}(\mathcal{U},\mathcal{U};\varpi)}{\|\mathcal{U}; L^2(\varpi)\|^2}, \tag{80}$$

where $\mathcal{H}_j$ stands for any subspace in $H^1(\varpi)^3$ of codimension $j-1$, in particular, $\mathcal{H}_1 = H^1(\varpi)^3$.

3.2. Some technical tools.

We need to define some cut-off functions. First, fix the geometric parameters $0 < r_1 < r_2 < r_3$ such that the surfaces $\partial\varpi \cap B(\mathcal{O}^{\pm}, r_j)$ are C^3 (see Section 1.1), and we denote

$$\varpi_{\pm}(r^j) := \varpi \cap B(\mathcal{O}^{\pm}, r_j)\ ,\ \ j = 1,2,3. \tag{81}$$

It follows from the basic geometric setting that, for a constant $\rho \ge 1$,

$$\varpi^{\varepsilon} \setminus \bigcup_{\pm} B(\mathcal{O}^{\pm}, \rho\varepsilon/4) \subset \varpi$$

for all $\varepsilon > 0$. In the following we always consider ε small enough: $\rho\varepsilon < r_1/3$. Let χ_ε be a C^∞ cut-off function which vanishes in the sets $B(\mathcal{O}^{\pm}, \rho\varepsilon)$ and equals 1 outside the sets $B(\mathcal{O}^{\pm}, 2\rho\varepsilon)$, and satisfies $|\nabla\chi_\varepsilon| \le C\varepsilon^{-1}$. As a consequence, functions defined on ϖ can be extended to ϖ^ε by multiplying them with χ_ε. We also define X_ε to be a C^∞ cut-off function with a slightly bigger support: $X_\varepsilon = 1$ in $B(\mathcal{O}^{\pm}, \rho\varepsilon)$ and $X_\varepsilon = 0$ in $B(\mathcal{O}^{\pm}, \rho\varepsilon/2)$ with $\nabla X_\varepsilon \le C\varepsilon^{-1}$.

For $x \in \mathbb{R}^3$, let us define the distance function

$$\delta(x) = \min(|x - \mathcal{O}^+|, |x - \mathcal{O}^-|)$$

and, for all θ, $0 < \theta < r_1$, the truncated distance function

$$\delta_\theta(x) = \begin{cases} \theta\,, & \text{if } \delta(x) < \theta \\ \delta(x)\,, & \text{if } \delta(x) \ge \theta. \end{cases} \tag{82}$$

The following Hardy-type inequality is well-known:

$$\left\|\delta_\theta^{-1-\beta} f; L^2(\varpi')\right\| \le C\left\|\delta_\theta^{-\beta}\nabla f; L^2(\varpi')\right\|\ ,\quad f \in H^1(\varpi'), \tag{83}$$

where ϖ' equals ϖ or ϖ^ε, and the constant $C > 0$ can be chosen independently of ε, θ and β, when $0 \le \beta < 1/2 - c$ for some constant $0 < c < 1$. We sketch the proof for the convenience of the reader, replacing the domain by $\mathbb{R}^3$ and the function $\delta(x)$ by $|x|$ in (82), and assuming $\beta > 0$ (the case $\beta = 0$ is easier). Indeed, let (ρ, σ) be the spherical coordinates of $\mathbb{R}^3$ with $\rho \in [0,\infty)$, $\sigma \in [-\pi,\pi] \times [0,\pi]$. For every fixed σ, the classical Hardy inequality for $g(\cdot,\sigma) \in C_0^\infty[0,\infty)$ reads as

$$\int_a^\infty |g(r,\sigma)|^2 dr \le 4 \int_a^\infty r^2 |g'(r,\sigma)|^2 dr,$$

where originally $a = 0$, but obviously any $a > 0$ can also be used. Changing the integration variable by $r = \varphi(\rho) := \rho^{1-2\beta}$ and choosing a properly lead to the following inequality for the function $f = g \circ \varphi$,

$$\int_\theta^\infty \rho^{-2\beta} |f(\rho,\sigma)|^2 d\rho \le C \int_\theta^\infty \rho^{2-2\beta} |f'(\rho,\sigma)|^2 d\rho \tag{84}$$

where we take $C := \sup 4(1-2\beta)^{-2} < \infty$ (see the choice of β above). Assuming $f \in C_0^\infty(\mathbb{R}^3)$ and integrating over the sphere $\partial B(0,\rho)$ with respect to the surface measure ds_ρ, using the Fubini theorem and taking into account $dx = C\rho^2 d\rho ds_\rho$, (84) implies (with usual abuse of notation in the variables)

$$\int\limits_{\mathbb{R}^3\setminus B(0,\theta)} |x|^{-2-2\beta}|f(x)|^2 dx \le C \int\limits_{\mathbb{R}^3\setminus B(0,\theta)} |x|^{-2\beta}|\nabla f(x)|^2 dx. \tag{85}$$

On the other hand,

$$\int\limits_{B(0,\theta)} \theta^{-2}|f(x)|^2 dx \le \int\limits_{\mathbb{R}^3} |x|^{-2}|f(x)|^2 dx \le C\int\limits_{\mathbb{R}^3} |\nabla f(x)|^2 dx. \tag{86}$$

where the second step is just a standard Hardy inequality, see for example [17], formula (1.3.3) with $p=2$, $s=2$, $m=3$. Multiplying (86) by $\theta^{-2\beta}$ and combining with (85) yields

$$\int\limits_{\mathbb{R}^3} \big(\max(|x|,\theta)\big)^{-2-2\beta}|f(x)|^2 dx \le C\int\limits_{\mathbb{R}^3} \big(\max(|x|,\theta)\big)^{-2\beta}|\nabla f(x)|^2 dx.$$

Passing to the Lipschitz domain ϖ' can be done by rectifying the boundary.

Next we derive pointwise estimates for the eigenfunctions $u_{(p)}$. Here, it is essential that the boundaries of $\partial\varpi_\pm(r^j)$ are sufficiently smooth (see (81)).

Lemma 3.1. *The pointwise estimates*

$$|u_{(p)}(x)| \le c_\varpi(1+\Lambda_p)\ , \quad |\nabla u_{(p)}(x)| \le c_\varpi(1+\Lambda_p)^{3/2} \tag{87}$$

hold for all $x \in \varpi_\pm(r^1)$.

Proof. We apply the local estimates [1, Ch. 10] to the problem (73) with $v := u_{(p)}$ and $f := \Lambda_p u_{(p)}$. We thus find for every $l = 0,1,2$ a constant $C = C_{\varpi,l,j}$ such that

$$\begin{aligned} &\|u_{(p)}; H^{l+1}(\varpi_\pm(r^j))\| \\ \le\ & C\Big(\Lambda_p\|u_{(p)}; H^{l-1}(\varpi_\pm(r^{j+1}))\| + \|u_{(p)}; L^2(\varpi_\pm(r^{j+1}))\|\Big) \end{aligned} \tag{88}$$

for $j=1,2$. Moreover, $\|u_{(p)}; L^2(\varpi_\pm(r^{j+1}))\| \le 1$, by the normalization (79), and applying (77) yields

$$\begin{aligned} &\|u_{(p)}; H^1(\varpi_\pm(r^j))\|^2 \le \|u_{(p)}; H^1(\varpi)\|^2 \\ \le\ & C\mathcal{B}(u_{(p)},u_{(p)}) = C\Lambda_p(u_{(p)},u_{(p)})_\varpi = C\Lambda_p. \end{aligned}$$

Taking $l=1$, $j=2$ in (88) we obtain

$$\begin{aligned} &\|u_{(p)}; H^2(\varpi_\pm(r^2))\| \\ \le\ & C_l\Big(\Lambda_p\|u_{(p)}; L^2(\varpi_\pm(r^3))\| + \|u_{(p)}; L^2(\varpi_\pm(r^3))\|\Big) \le C_1(\Lambda_p+1) \end{aligned} \tag{89}$$

and for $l=2$, $j=1$,

$$\begin{aligned} &\|u_{(p)}; H^3(\varpi_\pm(r^1))\| \le C_l\Big(\Lambda_p\|u_{(p)}; H^1(\varpi_\pm(r^2))\| + \|u_{(p)}; L^2(\varpi_\pm(r^2))\|\Big) \\ \le\ & C_2\big(\Lambda_p^2 C_1(\Lambda_p+1)^2+1\big) \end{aligned} \tag{90}$$

In view of the standard embeddings $H^2(\varpi_\pm(r^2)) \subset C_B(\varpi_\pm(r^1))$ and $H^3(\varpi_\pm(r^1)) \subset C_B^1(\varpi_\pm(r^1))$, the estimates (87) follow from (89) and (90). ⊠

3.3. Weighted Sobolev estimates.

The following weighted Sobolev estimates will be used for treating the difficulties caused by the non-local operators T and $T^{\varepsilon,\eta}$.

Lemma 3.2. *For every $j \in \mathbb{N}$ there exist numbers $\beta = \beta(j) > 0$ and C_j such that for all $\varepsilon > 0$*

$$\left\| \delta^{-\beta} \nabla u^{\mathsf{M}}_{(j)}; L^2(\varpi) \right\| + \left\| \delta^{-1-\beta} u^{\mathsf{M}}_{(j)}; L^2(\varpi) \right\| \le C_j, \tag{91}$$

$$\left\| \delta^{-\beta} \nabla (\chi_\varepsilon u^{\mathsf{M}}_{(j)}); L^2(\varpi) \right\| + \left\| \delta^{-1-\beta} \chi_\varepsilon u^{\mathsf{M}}_{(j)}; L^2(\varpi) \right\| \le C_j, \tag{92}$$

$$\left\| \delta^{-\beta} \nabla T(\chi_\varepsilon u^{\mathsf{M}}_{(j)}); L^2(\varpi) \right\| + \left\| \delta^{-1-\beta} T(\chi_\varepsilon u^{\mathsf{M}}_{(j)}); L^2(\varpi) \right\| \le C_j. \tag{93}$$

Lemma 3.2 implies the following estimates.

Corollary 3.3. *We have, with $\beta = \beta(j)$ as in Lemma 3.2,*

$$\|(1-\chi_\varepsilon) u^{\mathsf{M}}_{(j)}; H^1(\varpi)\| \le C\varepsilon^\beta, \tag{94}$$

$$\|(1-X_\varepsilon) T(\chi_\varepsilon u^{\mathsf{M}}_{(j)}); H^1(\varpi)\| \le C\varepsilon^\beta, \tag{95}$$

$$\|(\nabla X_\varepsilon) T(\chi_\varepsilon u^{\mathsf{M}}_{(j)}); L^2(\varpi)\| \le C\varepsilon^\beta. \tag{96}$$

Proof. By the results of Section 3.2 we note that $|\nabla \chi_\varepsilon| \le C\varepsilon^{-1}$ and that in the support of $1-\chi_\varepsilon$ there holds $\delta(x) \le 2\rho\varepsilon$, hence, (94) follows from (91):

$$\begin{aligned} \|(1-\chi_\varepsilon) u^{\mathsf{M}}_{(j)}; H^1(\varpi)\|^2 &\le \int\limits_{B(\mathcal{O}^\pm, 2\rho\varepsilon)} \big((1 + |\nabla \chi_\varepsilon|^2) |u^{\mathsf{M}}_{(j)}|^2 + |\nabla u^{\mathsf{M}}_{(j)}|^2 \big) dx \\ &\le C\varepsilon^{2\beta} \int\limits_{B(\mathcal{O}^\pm, 2\rho\varepsilon)} \big(\delta^{-2-2\beta} \delta^2 \varepsilon^{-2} |u^{\mathsf{M}}_{(j)}|^2 + \delta^{-2\beta} |\nabla u^{\mathsf{M}}_{(j)}|^2 \big) dx \\ &\le C\varepsilon^{2\beta} \int\limits_{B(\mathcal{O}^\pm, 2\rho\varepsilon)} \big(\delta^{-2-2\beta} |u^{\mathsf{M}}_{(j)}|^2 + \delta^{-2\beta} |\nabla u^{\mathsf{M}}_{(j)}|^2 \big) dx \le C' \varepsilon^{2\beta}. \end{aligned}$$

The inequality (95) is proven in the same way with the help of (93).

The proof of (96) is similar: we use the position of the support of ∇X_ε, the estimate $|\nabla X_\varepsilon| \le C\varepsilon^{-1}$ and the weights in the inequality (93). ⊠

Proof of Lemma 3.2. 1° To verify (91) we start by writing (cf. (74))

$$\begin{aligned} \big(AD(\nabla)u, D(\nabla)v \big)_\varpi &= \big(A^{\mathsf{MM}} D^{\mathsf{M}} u^{\mathsf{M}}, D^{\mathsf{M}} v^{\mathsf{M}} \big)_\varpi + \big(A^{\mathsf{ME}} D^{\mathsf{E}} u^{\mathsf{E}}, D^{\mathsf{M}} v^{\mathsf{M}} \big)_\varpi \\ &- \big(A^{\mathsf{EM}} D^{\mathsf{M}} u^{\mathsf{M}}, D^{\mathsf{E}} v^{\mathsf{E}} \big)_\varpi + \big(A^{\mathsf{EE}} D^{\mathsf{E}} u^{\mathsf{E}}, D^{\mathsf{E}} v^{\mathsf{E}} \big)_\varpi, \end{aligned} \tag{97}$$

where $u = (u^{\mathsf{M}}, u^{\mathsf{E}})$, $v = (v^{\mathsf{M}}, v^{\mathsf{E}}) \in H^1(\varpi)^4$. If $v = u$, then, since $A^{\mathsf{ME}} = (A^{\mathsf{EM}})^\top$ (see the lines after (7)), the sum of the terms with A^{ME} and A^{EM} of (97) is imaginary, while the terms with A^{MM} and A^{EE} are real (the matrices A^{MM} and A^{EE} are symmetric positive definite, see (8)). We obtain

$$\mathrm{Re}\big(AD(\nabla)u, D(\nabla)u \big)_\varpi \ge \big(A^{\mathsf{MM}} D^{\mathsf{M}} u^{\mathsf{M}}, D^{\mathsf{M}} u^{\mathsf{M}} \big)_\varpi \tag{98}$$

for all $u = (u^{\mathsf{M}}, u^{\mathsf{E}}) \in H^1(\varpi)^4$

Let δ_θ be as in (82) and let us define the operator $Q_\theta f = \delta_\theta^{-\beta} f$, where the number $\beta > 0$ is to be fixed later. Writing, as before, $D = D(\nabla)$ and also $\tilde{u} := Q_\theta u^{\mathsf{M}}_{(j)}$ we notice that the commutator $[D, Q_\theta]$ is the same as the multiplication with the matrix function $D\delta_\varepsilon^{-\beta}$, which has the estimate

$$|D\delta_\theta^{-\beta}| \le C\beta \delta_\theta^{-\beta-1}. \tag{99}$$

Hence, we have

$$\begin{aligned} & \big(ADu^{\mathsf M}_{(j)}, DQ_\theta^2 u^{\mathsf M}_{(j)}\big)_\varpi \\ = \; & \big(Q_\theta ADu^{\mathsf M}_{(j)}, D\tilde u\big)_\varpi + \big(Q_\theta ADu^{\mathsf M}_{(j)}, Q_\theta^{-1}[D,Q_\theta]\tilde u\big)_\varpi \\ = \; & \big(AD\tilde u, D\tilde u\big)_\varpi - \big(Q_\theta^{-1}A[D,Q_\theta]\tilde u, D\tilde u\big)_\varpi \\ + \; & \big(AD\tilde u, Q_\theta^{-1}[D,Q_\theta]\tilde u\big)_\varpi - \big(AQ_\theta^{-1}[D,Q_\theta]\tilde u, Q_\theta^{-1}[D,Q_\theta]\tilde u\big)_\varpi \\ = \; & \big(AD\tilde u, D\tilde u\big)_\varpi + I(\tilde u,\tilde u), \end{aligned} \tag{100}$$

where (99) yields the estimate

$$\big|\mathrm{Re} I(\tilde u,\tilde u)\big| \le c\beta \|\delta_\theta^{-1}\tilde u; L^2(\varpi)\| \Big(\|\nabla \tilde u; L^2(\varpi)\| + \beta\|\delta_\theta^{-1}\tilde u; L^2(\varpi)\|\Big) \tag{101}$$

By (98) and the Korn inequality (10),

$$\begin{aligned} & (\tilde u,\tilde u)_\varpi + \mathrm{Re}(AD\tilde u, D\tilde u) \\ \ge \; & (\tilde u,\tilde u)_\varpi + (A^{\mathsf{MM}}D^{\mathsf M}\tilde u, D^{\mathsf M}\tilde u)_\varpi \ge c\|\nabla\tilde u; L^2(\varpi)\|. \end{aligned} \tag{102}$$

Choosing $V = Q_\theta^2 u_{(j)}$ in (100), the integral identity (74) can be written as

$$A(D\tilde u, D\tilde u)_\varpi + I(\tilde u,\tilde u) = \Lambda_j\big(\varrho\delta_\theta^{-\beta}u^{\mathsf M}_{(j)}, \delta_\theta^{-\beta}u^{\mathsf M}_{(j)}\big)_\varpi = \Lambda_j(\varrho\tilde u,\tilde u)_\varpi. \tag{103}$$

The right hand side is real and positive, and due to the Hardy inequality (83), it is bounded by a constant C_j, since $\|u^{\mathsf M}_{(j)}; H^1(\varpi)\|$ is bounded by a constant depending on j only. This, (101), and (102) imply

$$\begin{aligned} & \|\tilde u; L^2(\varpi)\|^2 + \|\nabla\tilde u; L^2(\varpi)\|^2 \le C\mathrm{Re}A(D\tilde u, D\tilde u) + C(\tilde u,\tilde u)_\varpi \\ \le \; & C_j'\big(1 + |\mathrm{Re} I(\tilde u,\tilde u)|\big) \\ \le \; & C_j' + c_j\beta\|\delta_\theta^{-1}\tilde u; L^2(\varpi)\|\Big(\|\nabla\tilde u; L^2(\varpi)\| + \beta\|\delta_\theta^{-1}\tilde u; L^2(\varpi)\|\Big). \end{aligned}$$

On the right hand side we use (83) to bound $\|\delta_\theta^{-1}\tilde u; L^2(\varpi)\|$ by $C\|\tilde u; H^1(\varpi)\|$. Choosing a small enough β and moving terms from the right to the left yield

$$\|\tilde u; L^2(\varpi)\|^2 + \|\nabla\tilde u; L^2(\varpi)\|^2 \le C.$$

By this and (83),

$$\|\delta_\theta^{-1-\beta}u^{\mathsf M}_{(j)}; L^2(\varpi)\| = \|\delta_\theta^{-1}\tilde u; L^2(\varpi)\| \le \|\nabla\tilde u; L^2(\varpi)\| \le C, \tag{104}$$

hence, (99) and $\delta_\theta^{-\beta}\nabla u^{\mathsf M}_{(j)} = -[D,Q_\theta]u^{\mathsf M}_{(j)} + \nabla\tilde u$ give us

$$\|\delta_\theta^{-\beta}\nabla u^{\mathsf M}_{(j)}; L^2(\varpi)\| \le C\big(\|\delta_\theta^{-1-\beta}u; L^2(\varpi)\| + \|\nabla\tilde u; L^2(\varpi)\big)\| \le C', \tag{105}$$

Passing to the limit $\theta \to 0$, (104) and (105) imply (91).

2°. We compare (92) with (91) and note that since $|\chi_\varepsilon| \le 1$, the only new term requiring an estimate is $\|\delta^{-\beta}(\nabla\chi_\varepsilon)u^{\mathsf M}_{(j)}; L^2(\varpi)\|$. But we have $|\nabla\chi_\varepsilon| \le C\varepsilon^{-1} \le C\delta^{-1}$ for x belonging to the support of χ_ε, hence, this bound also follows from (91).

3°. Proof of (93). We write $\mathcal V := \chi_\varepsilon u^{\mathsf M}_{(j)}$ and let the operator Q_θ be as above. First, by the Hardy inequality (83), we have

$$\Big\|\delta_\theta^{-1-\beta}T\mathcal V; L^2(\varpi)\Big\| \le C\Big\|\delta_\theta^{-\beta}\nabla T\mathcal V; L^2(\varpi)\Big\| \tag{106}$$

The definition of T gives for the test function $v^{\mathsf E} := Q_\theta^2 T\mathcal V \in H^1(\varpi)$,

$$(A^{\mathsf{EE}}D^{\mathsf E}T\mathcal V, D^{\mathsf E}Q_\theta^2 T\mathcal V)_\varpi = (A^{\mathsf{EM}}D^{\mathsf M}\mathcal V, D^{\mathsf E}Q_\theta^2 T\mathcal V)_\varpi. \tag{107}$$

The left hand side equals

$$(A^{\mathsf{EE}}Q_\theta D^{\mathsf E}T\mathcal V, Q_\theta D^{\mathsf E}T\mathcal V)_\varpi + (A^{\mathsf{EE}}Q_\theta D^{\mathsf E}T\mathcal V, Q_\theta^{-1}[D^{\mathsf E}, Q_\theta^2]T\mathcal V)_\varpi,$$

and the right hand side

$$(A^{\mathsf{EM}}Q_\theta D^{\mathsf{M}}\mathcal{V}, Q_\theta D^{\mathsf{E}}T\mathcal{V})_\varpi + (A^{\mathsf{EM}}Q_\theta D^{\mathsf{M}}\mathcal{V}, Q_\theta^{-1}[D^{\mathsf{E}}, Q_\theta^2]T\mathcal{V})_\varpi,$$

while the commutator $[D^{\mathsf{E}}, Q_\theta^2]$ is multiplication with a matrix function satisfying

$$|[D^{\mathsf{E}}, Q_\theta^2]| \leq C\beta\delta_\theta^{-1-2\beta}. \tag{108}$$

Combining (107)–(108) and using the positive definiteness of A^{EE} and the Cauchy-Bunyakovski-Schwartz (=CBS)-inequality yield

$$\begin{aligned}
&\|\delta_\theta^{-\beta}\nabla T\mathcal{V}; L^2(\varpi)\|^2 = \|Q_\theta D^{\mathsf{E}}T\mathcal{V}; L^2(\varpi)\|^2 \\
\leq\ & C(A^{\mathsf{EE}}Q_\theta D^{\mathsf{E}}T\mathcal{V}, Q_\theta D^{\mathsf{E}}T\mathcal{V})_\varpi \\
\leq\ & \big|(A^{\mathsf{EE}}Q_\theta D^{\mathsf{E}}T\mathcal{V}, Q_\theta^{-1}[D^{\mathsf{E}}, Q_\theta^2]T\mathcal{V})_\varpi\big| \\
+\ & \big|(A^{\mathsf{EM}}Q_\theta D^{\mathsf{M}}\mathcal{V}, Q_\theta D^{\mathsf{E}}T\mathcal{V})_\varpi\big| + \big|(A^{\mathsf{EM}}Q_\theta D^{\mathsf{M}}\mathcal{V}, Q_\theta^{-1}[D^{\mathsf{E}}, Q_\theta^2]T\mathcal{V})_\varpi\big| \\
\leq\ & C\beta\|\delta_\theta^{-\beta}\nabla T\mathcal{V}; L^2(\varpi)\|\|\delta_\theta^{-\beta-1}T\mathcal{V}; L^2(\varpi)\| \\
+\ & \|\delta_\theta^{-\beta}\nabla\mathcal{V}; L^2(\varpi)\|\Big(\|\delta_\theta^{-\beta}\nabla T\mathcal{V}; L^2(\varpi)\| + \|\delta_\theta^{-\beta-1}T\mathcal{V}; L^2(\varpi)\|\Big) \\
\leq\ & C\beta\|\delta_\theta^{-\beta}\nabla T\mathcal{V}; L^2(\varpi)\|^2 + C'\|\delta_\theta^{-\beta}\nabla T\mathcal{V}; L^2(\varpi)\|,
\end{aligned} \tag{109}$$

where we used (106) for the last inequality and also (92) to evaluate the norm $\|\delta_\theta^{-\beta}\nabla\mathcal{V}; L^2(\varpi)\|$. Dividing by $\|\delta_\theta^{-\beta}\nabla T\mathcal{V}; L^2(\varpi)\|$ and choosing a small enough $\beta > 0$ we obtain $\|\delta_\theta^{-\beta}\nabla T\mathcal{V}; L^2(\varpi)\| \leq C$ and (93) follows by combining this with (106) and taking the limit $\theta \to 0$. ⊠

3.4. Estimates for the non-local operators.

We next prove a lemma which allows to control the expression of the non-local operator $T^{\varepsilon,0}$ when it acts on "wrong" eigenfunctions. This will be needed in the use of the max-min principle, (128), when eigenfunctions of the limit problem have to be used as approximate eigenfunctions for the η-dependent problem.

Lemma 3.4. *For all $j \in \mathbb{N}$ there exist numbers $\beta := \beta(j) > 0$ and C_j such that*

$$\Big\|X_\varepsilon T(\chi_\varepsilon u_{(j)}^{\mathsf{M}}) - Tu_{(j)}^{\mathsf{M}}; H^1(\varpi)\Big\| < C\varepsilon^\beta \quad \textit{and} \tag{110}$$

$$\Big\|D^{\mathsf{E}}\big(T^{\varepsilon,0}(\chi_\varepsilon u_{(j)}^{\mathsf{M}}) - X_\varepsilon T(\chi_\varepsilon u_{(j)}^{\mathsf{M}})\big); L^2(\varpi^\varepsilon)\Big\| < C\varepsilon^\beta. \tag{111}$$

Here the function $\chi_\varepsilon u_{(j)}^{\mathsf{M}}$ is extended to ϖ^ε as null, see Section 3.2.

Proof. Let us denote $u = u_{(j)}^{\mathsf{M}}$, and $\mathcal{V} := \chi_\varepsilon u_{(j)}^{\mathsf{M}}$.

1°. Since the operator T is bounded in the norm of $H^1(\varpi)$, (110) follows from (94) and (95):

$$\begin{aligned}
&\|X_\varepsilon T\mathcal{V} - Tu; H^1(\varpi)\| \\
\leq\ & \|(X_\varepsilon - 1)T\mathcal{V}; H^1(\varpi)\| + \|T(\mathcal{V} - u); H^1(\varpi)\| \leq C\varepsilon^\beta.
\end{aligned}$$

2°. Let us prove (111). We denote by $\varphi \in H^1(\varpi^\varepsilon)$ a test function with $\|\varphi; H^1(\varpi^\varepsilon)\| \leq 1$. We observe by (96) that

$$\|D^{\mathsf{E}}X_\varepsilon T\mathcal{V} - X_\varepsilon D^{\mathsf{E}}T\mathcal{V}, L^2(\varpi^\varepsilon)\| = \|(\nabla X_\varepsilon)T\mathcal{V}; L^2(\varpi)\| \leq C\varepsilon^\beta. \tag{112}$$

Next, since the support of $1 - X_\varepsilon$ is contained in $B(\mathcal{O}^\pm, \rho\varepsilon)$, we have

$$\begin{aligned}
&\big|\big(A^{\mathsf{EE}}X_\varepsilon D^{\mathsf{E}}R\mathcal{V}, D^{\mathsf{E}}\varphi\big)_\varpi - \big(A^{\mathsf{EE}}D^{\mathsf{E}}T\mathcal{V}, D^{\mathsf{E}}\varphi\big)_\varpi\big| \\
\leq & \int\limits_{B(\mathcal{O}^\pm,\rho\varepsilon)} \big|A^{\mathsf{EE}}D^{\mathsf{E}}R\mathcal{V}\big|\,\big|\nabla\varphi\big|dx \leq C\Big(\int\limits_{B(\mathcal{O}^\pm,\rho\varepsilon)} \big|\nabla R\mathcal{V}\big|^2\Big)^{1/2}\|\varphi; H^1(\omega)\|^2 \leq C'\varepsilon^\beta,
\end{aligned} \tag{113}$$

where, as before, the last integral was estimated using the information on the integration domain and the inequality (93) of Lemma 4.1.

We now combine (113) and the definitions of the operators T and $T^{\varepsilon,0}$ to argue as follows (here, it is important to observe the supports of the given functions)

$$\begin{aligned} & \left|\big(A^{\mathsf{EE}}(X_\varepsilon D^{\mathsf{E}}T\mathcal{V} - D^{\mathsf{E}}T^{\varepsilon,0}\mathcal{V}), D^{\mathsf{E}}\varphi\big)_{\varpi^\varepsilon}\right| \\ = & \left|\big(A^{\mathsf{EE}}X_\varepsilon D^{\mathsf{E}}T\mathcal{V}, D^{\mathsf{E}}\varphi\big)_{\varpi} - \big(A^{\mathsf{EE}}D^{\mathsf{E}}T^{\varepsilon,0}\mathcal{V}, D^{\mathsf{E}}\varphi\big)_{\varpi^\varepsilon}\right| \\ = & \left|\big(A^{\mathsf{EE}}D^{\mathsf{E}}T\mathcal{V}, D^{\mathsf{E}}\varphi\big)_{\varpi} - \big(A^{\mathsf{EE}}D^{\mathsf{E}}T^{\varepsilon,0}\mathcal{V}, D^{\mathsf{E}}\varphi\big)_{\varpi^\varepsilon}\right| + O(\varepsilon^\beta) \\ = & \left|\big(A^{\mathsf{EM}}D^{\mathsf{M}}\mathcal{V}, D^{\mathsf{E}}\varphi\big)_{\varpi} - \big(A^{\mathsf{EM}}D^{\mathsf{M}}\mathcal{V}, D^{\mathsf{E}}\varphi\big)_{\varpi^\varepsilon}\right| + O(\varepsilon^\beta) \\ = & \left|\big(A^{\mathsf{EM}}D^{\mathsf{M}}\mathcal{V}, D^{\mathsf{E}}\varphi\big)_{\varpi} - \big(A^{\mathsf{EM}}D^{\mathsf{M}}\mathcal{V}, D^{\mathsf{E}}\varphi\big)_{\varpi}\right| + O(\varepsilon^\beta) = O(\varepsilon^\beta). \end{aligned} \tag{114}$$

We replace φ by $X_\varepsilon T\mathcal{V} - T^{\varepsilon,0}\mathcal{V}$. By Lemma 2.1, there exists a constant $C > 0$, independent of ε, such that $\|X_\varepsilon T\mathcal{V} - T^{\varepsilon,0}\mathcal{V}; H^1(\varpi^\varepsilon)\| \leq C$. This, (8), and (114) imply

$$\begin{aligned} & C\|D^{\mathsf{E}}(X_\varepsilon T\mathcal{V} - T^{\varepsilon,0}\mathcal{V}); L^2(\varpi)\|^2 \\ \leq & \left|\big(A^{\mathsf{EE}}D^{\mathsf{E}}(X_\varepsilon T\mathcal{V} - T^{\varepsilon,0}\mathcal{V}), D^{\mathsf{E}}(X_\varepsilon T\mathcal{V} - T^{\varepsilon,0}\mathcal{V})\big)_{\varpi}\right| \\ \leq & \left|\big(A^{\mathsf{EE}}(X_\varepsilon D^{\mathsf{E}}T\mathcal{V} - D^{\mathsf{E}}T^{\varepsilon,0}\mathcal{V}), D^{\mathsf{E}}(X_\varepsilon T\mathcal{V} - T^{\varepsilon,0}\mathcal{V})\big)_{\varpi}\right| + O(\varepsilon^\beta) = O(\varepsilon^\beta), \end{aligned}$$

where we also used (112) to commute the cut-off function and the derivative on the left hand side. ⊠

3.5. Proof for the upper estimate for the spectral bands.

In this section we produce upper estimates for the endpoints of spectral bands. More precisely, we show for small ε that for every j, the eigenvalues $\Lambda_j^{\varepsilon,\eta}$, $\eta \in [-\pi, \pi)$ are bounded from above by Λ_j plus a term of order at most $\varepsilon^{\beta(j)}$, see Lemma 3.5. A corresponding lower bound is produced in the next section, Lemma 4.4. These lead to the main results, Theorems 1.4 and 1.5.

Lemma 3.5. *There exists a decreasing sequence $0 < \varepsilon_j < \varepsilon_{j-1}$ and a constant $C = C(j, \varpi, A, \varrho)$, such that*

$$\Lambda_j^{\varepsilon,\eta} \leq \Lambda_j + C\varepsilon^{\beta(j)} \tag{115}$$

for all $j \in \mathbb{N}$, $\varepsilon \leq \varepsilon_j$ and $\eta \in [0, 2\pi)$.

Proof. Let us start by defining approximate eigenfunctions for the η-dependent problem with the help of the eigenfunctions $u_{(j)}^{\mathsf{M}}$ of the limit problem. For all j, η and ε, we write $\mathcal{V}_{(j)}^\varepsilon = \chi_\varepsilon u_{(j)}^{\mathsf{M}}$. The product can be extended to ϖ^ε with the help of the cut-off function χ_ε, see the beginning of Section 3.2. The above choices of parameters guarantee that the extension becomes smooth and that the following holds:

$$\begin{aligned} & \mathrm{supp}\mathcal{V}_{(j)}^\varepsilon \subset \varpi \ \text{ and} \\ & \mathrm{supp}\big(\mathcal{V}_{(j)}^\varepsilon\big|_{\varpi} - u_{(j)}^{\mathsf{M}}\big) \subset \varpi \cap \cup_\pm B(\mathcal{O}^\pm, 2\rho\varepsilon) \subset \cup_\pm \varpi_\pm(r^1). \end{aligned} \tag{116}$$

Using (87) we can thus bound

$$\|u_{(j)}^{\mathsf{M}} - \mathcal{V}_{(j)}^\varepsilon; L^2(\varpi)\|^2 \leq C \int_{\varpi\cap\cup_\pm B(\mathcal{O}^\pm, 2\rho\varepsilon)} (1+\Lambda_j)^2 dx \leq C'\varepsilon^3(1+\Lambda_j)^2, \tag{117}$$

$$\begin{aligned} \|\nabla u_{(j)}^{\mathsf{M}} - \nabla\mathcal{V}_{(j)}^\varepsilon; L^2(\varpi)\|^2 & \leq \int_{\varpi\cap\cup_\pm B(\mathcal{O}^\pm, 2\rho\varepsilon)} \big(|\nabla u_{(j)}^{\mathsf{M}}|^2 + |\nabla\chi_\varepsilon|^2 |u_{(j)}^{\mathsf{M}}|^2\big) dx \\ & \leq C\Big(\varepsilon^3(1+\Lambda_j)^3 + \varepsilon(1+\Lambda_j)^2\Big) . \end{aligned} \tag{118}$$

since the volume of the set $\varpi \cap \cup_\pm B(\mathcal{O}^\pm, 2\rho\varepsilon)$ is of order ε^3.

We remark that the above defined approximate eigenfunctions preserve a linear independence property: one can find for every $j \in \mathbb{N}$ a number $\varepsilon_j > 0$ such that the functions $\mathcal{V}^\varepsilon_{(1)}, \ldots, \mathcal{V}^\varepsilon_{(j)}$ are linearly independent in the space $L^2(\varpi)^3$ (hence also in $L^2(\varpi^\varepsilon)^3$), if $\varepsilon \le \varepsilon_j$. This follows for example from general results in functional analysis: a small enough perturbation of a Schauder basis of a Banach space is again a Schauder basis, hence, any finite collection of the perturbed basis elements must be linearly independent, cf. [16, Prop 1.a.9]. A detailed proof of this statement can also be found e.g. in [28], Section 4.

Fixing j, let $(b_p)_{p=1}^j$ be any sequence of numbers normalized so that $\sum_{p=1}^j |b_p|^2 = 1$, and let

$$\mathcal{W}^\varepsilon_{(j)} = \sum_{p=1}^j b_p \mathcal{V}^\varepsilon_{(p)} \quad , \quad W_j := \sum_{p=1}^j b_p u^{\mathsf{M}}_{(p)}, \tag{119}$$

hence, by (79),

$$\|W_j; L^2(\varpi)\| = 1. \tag{120}$$

Due to the eigenvector property, cf. (76)–(79), and (120) there holds

$$|\mathcal{B}(W_j, W_j)| \le \Lambda_j. \tag{121}$$

Moreover, we denote

$$\mathbf{W}^{\varepsilon,\eta}_{(j)}(x, \eta) = e^{-i\eta z} \mathcal{W}^\varepsilon_{(j)}(x). \tag{122}$$

Since the supports of these functions are contained in ϖ (see (116)), the first estimate (117) and (119), (120) imply

$$\|\mathbf{W}^{\varepsilon,\eta}_{(j)}, L^2(\varpi^\varepsilon)\| = \|\mathcal{W}^\varepsilon_{(j)}, L^2(\varpi)\| \ge 1 - c\varepsilon^{3/2}. \tag{123}$$

We shall also soon check that the following relation holds true:

$$\Big| \mathcal{B}^{\varepsilon,\eta}(\mathbf{W}^{\varepsilon,\eta}_{(j)}, \mathbf{W}^{\varepsilon,\eta}_{(j)}) - \mathcal{B}(W_j, W_j) \Big| \le C\varepsilon^\beta \ . \tag{124}$$

This and the max-min principle for the sesquilinear form $\mathcal{B}^{\varepsilon,\eta}$, (38), will complete the proof. Indeed, by [4, Th.10.2.2],

$$\Lambda^{\varepsilon,\eta}_j = \max_{\mathcal{H}_j} \inf_{\mathcal{V} \in \mathcal{H}_j \setminus \{0\}} \frac{\mathcal{B}^{\varepsilon,\eta}(\mathcal{U}, \mathcal{U})}{\|\mathcal{U}; L^2(\varpi^\varepsilon)\|^2},$$

where $\mathcal{H}_j$ stands for any subspace in $H^1_{\rm per}(\varpi^\varepsilon)$ of codimension $j-1$. Since the sequence of functions $(\mathcal{V}^\varepsilon_{(p)})_{p=1}^j$ is linearly independent we can find from any $\mathcal{H}_j$ an element $\mathbf{W}^{\varepsilon,\eta}_{(j)}$ of the form (122). This, (121), (123), and (124), imply

$$\Lambda^{\varepsilon,\eta}_j \quad \le \quad \frac{\mathcal{B}^{\varepsilon,\eta}(\mathbf{W}^{\varepsilon,\eta}_{(j)}, \mathbf{W}^{\varepsilon,\eta}_{(j)})}{\|\mathbf{W}^{\varepsilon,\eta}_{(j)}; L^2(\varpi^\varepsilon)\|^2} \le \frac{\mathcal{B}(W_j, W_j) + C\varepsilon^\beta}{1 - c\varepsilon^{3/2}} \le \Lambda_j + C'\varepsilon^\beta.$$

So, there only remains to prove (124). Since the support of $\mathbf{W}^{\varepsilon,\eta}_{(j)}$ is contained in ϖ (see (116)), we have

$$\big(\varrho \mathbf{W}^{\varepsilon,\eta}_{(j)}), \mathbf{W}^{\varepsilon,\eta}_{(j)}\big)_{\varpi^\varepsilon} = \big(\varrho \mathcal{W}^\varepsilon_{(j)}, \mathcal{W}^\varepsilon_{(j)}\big)_{\varpi},$$

hence,

$$\Big| \big(\varrho \mathbf{W}^{\varepsilon,\eta}_{(j)}, \mathbf{W}^{\varepsilon,\eta}_{(j)}\big)_{\varpi^\varepsilon} - \big(\varrho W_j, W_j\big)_{\varpi} \Big| \le C\varepsilon^{3/2} \tag{125}$$

follows from (117), (119), (120) and the CBS-inequality. Also, by direct differentiation and taking into account the support and (38),

$$\mathcal{A}^{\varepsilon,\eta}(\mathbf{W}^{\varepsilon,\eta}_{(j)}, \mathbf{W}^{\varepsilon,\eta}_{(j)}) = \mathcal{A}^{\varepsilon,0}(\mathcal{W}^\varepsilon_{(j)}, \mathcal{W}^\varepsilon_{(j)}) = (A^{\mathsf{MM}} D^{\mathsf{M}} \mathcal{W}^\varepsilon_{(j)}, D^{\mathsf{M}} \mathcal{W}^\varepsilon_{(j)})_{\varpi},$$

hence,

$$\Big|\mathcal{A}^{\varepsilon,\eta}(\mathbf{W}^{\varepsilon,\eta}_{(j)},\mathbf{W}^{\varepsilon,\eta}_{(j)})-\mathcal{A}(W_j,W_j;\varpi)\Big|\le C\varepsilon^{1/2}, \tag{126}$$

by (76), (118), and a similar argument as above. Finally,

$$\mathcal{R}^{\varepsilon,\eta}(\mathbf{W}^{\varepsilon,\eta}_{(j)},\mathbf{W}^{\varepsilon,\eta}_{(j)})=\mathcal{R}^{\varepsilon,0}(\mathcal{W}^{\varepsilon}_{(j)},\mathcal{W}^{\varepsilon}_{(j)})$$

and this consists of terms like

$$\mathcal{R}^{\varepsilon,0}(\mathcal{V}^{\varepsilon}_{(j)},\mathcal{V}^{\varepsilon}_{(k)})=\big(A^{\mathsf{ME}}D^{\mathsf{E}}T^{\varepsilon,0}\mathcal{V}^{\varepsilon}_{(j)},D^{\mathsf{M}}\mathcal{V}^{\varepsilon}_{(k)}\big)_{\varpi^\varepsilon}$$

By (117) and (118) we have

$$\|\mathcal{V}^{\varepsilon}_{(j)}-u^{\mathsf{M}}_{(j)};H^1(\varpi)\|\le C\varepsilon^{\beta}, \tag{127}$$

hence, $\|D^{\mathsf{M}}\mathcal{V}^{\varepsilon}_{(j)};L^2(\varpi)\|=\|D^{\mathsf{M}}\mathcal{V}^{\varepsilon}_{(j)};L^2(\varpi^\varepsilon)\|$ is bounded by a constant depending on j only, and Lemma 3.4 and (127) thus imply (recall $\mathcal{V}^{\varepsilon}_{(j)}=\chi_\varepsilon u^{\mathsf{M}}_{(j)}$)

$$\begin{aligned}
&\big(A^{\mathsf{ME}}D^{\mathsf{E}}T^{\varepsilon,0}\mathcal{V}^{\varepsilon}_{(j)},D^{\mathsf{M}}\mathcal{V}^{\varepsilon}_{(k)}\big)_{\varpi^\varepsilon}\\
=\;&\big(A^{\mathsf{ME}}D^{\mathsf{E}}X_\varepsilon T\mathcal{V}^{\varepsilon}_{(j)},D^{\mathsf{M}}\mathcal{V}^{\varepsilon}_{(k)}\big)_{\varpi^\varepsilon}+O(\varepsilon^\beta)\\
=\;&\big(A^{\mathsf{ME}}D^{\mathsf{E}}X_\varepsilon T\mathcal{V}^{\varepsilon}_{(j)},D^{\mathsf{M}}\mathcal{V}^{\varepsilon}_{(k)}\big)_{\varpi}+O(\varepsilon^\beta)\\
=\;&\big(A^{\mathsf{ME}}D^{\mathsf{E}}Tu^{\mathsf{M}}_{(j)},D^{\mathsf{M}}\mathcal{V}^{\varepsilon}_{(k)}\big)_{\varpi}+O(\varepsilon^\beta)\\
=\;&\big(A^{\mathsf{ME}}D^{\mathsf{E}}Tu^{\mathsf{M}}_{(j)},D^{\mathsf{M}}u^{\mathsf{M}}_{(k)}\big)_{\varpi}+O(\varepsilon^\beta)\\
=\;&\mathcal{R}(u^{\mathsf{M}}_{(j)},u^{\mathsf{M}}_{(k)})+O(\varepsilon^\beta),
\end{aligned} \tag{128}$$

where also the supports were taken into account. We obtained

$$\big|\mathcal{R}^{\varepsilon,0}(\mathcal{V}^{\varepsilon}_{(j)},\mathcal{V}^{\varepsilon}_{(k)})-\mathcal{R}(u^{\mathsf{M}}_{(j)},u^{\mathsf{M}}_{(k)})\big|\le C\varepsilon^\beta$$

which yields

$$\big|\mathcal{R}^{\varepsilon,0}(\mathbf{W}^{\varepsilon,\eta}_{(j)},\mathbf{W}^{\varepsilon,\eta}_{(k)})-\mathcal{R}(W_j,W_j)\big|\le C\varepsilon^\beta$$

This, (125) and (126) imply (124). ⊠

4. Lower estimate for the spectral bands.

To prove the lower estimate one can proceed as with the upper estimate, exchanging the roles of the eigenvalues of the limit and η-dependent problems. However, there are some complications: one of them is that the boundary of the periodicity cell ϖ^ε cannot be assumed smooth near the points $\mathcal{O}^\pm$, hence, the local elliptic estimates cannot be used to prove L^∞-bounds like Lemma 3.1 for the eigenfunctions $U^{\varepsilon,\eta}_{(j)}$. Moreover, the weight exponents $\beta(j)$ in Section 3 depended on the eigenvalues Λ_j, so, applying the same proof here would yield exponents depending on ε and η, which might be useless. In order to obtain weighted Sobolev estimates with weight exponents independent of ε and η we shall use the already proven upper bound (115) for the eigenvalues $\Lambda^{\varepsilon,\eta}_j$. In particular, we can fix for every j the number $L_j>0$ such that

$$\sup_{\varepsilon,\eta}\Lambda^{\varepsilon,\eta}_j\le L_j. \tag{129}$$

We keep the notation introduced in Section 3.2 throughout this section.

4.1. Weighted Sobolev estimates.

For the lower bound we shall need the following versions of the weighted Sobolev estimates.

Lemma 4.1. *For every $j \in \mathbb{N}$, there exist constants $\gamma = \gamma(j) > 0$ and $C_j > 0$ such that for all $\varepsilon > 0$*

$$\left\|\delta^{-\gamma}\nabla U^{\varepsilon,\eta,\mathsf{M}}_{(j)}; L^2(\varpi^\varepsilon)\right\| + \left\|\delta^{-1-\gamma}U^{\varepsilon,\eta,\mathsf{M}}_{(j)}; L^2(\varpi^\varepsilon)\right\| \leq C_j, \tag{130}$$

$$\left\|\delta^{-\gamma}\nabla \chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}; L^2(\varpi^\varepsilon)\right\| + \left\|\delta^{-1-\gamma}\chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}; L^2(\varpi^\varepsilon)\right\| \leq C_j, \tag{131}$$

$$\begin{aligned}&\left\|\delta^{-\gamma}\nabla T^{\varepsilon,\eta}\big(\chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}\big); L^2(\varpi^\varepsilon)\right\| \\ &\quad + \left\|\delta^{-1-\gamma}T^{\varepsilon,\eta}\big(\chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}\big); L^2(\varpi^\varepsilon)\right\| \leq C_j\end{aligned} \tag{132}$$

$$\left\|\delta^{-\gamma}\nabla T\big(\chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}\big); L^2(\varpi)\right\| + \left\|\delta^{-1-\gamma}T\big(\chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}\big); L^2(\varpi)\right\| \leq C_j \tag{133}$$

We emphasize that the last inequality is special in the sense that it contains the limit case operator T acting on eigenfunctions of a "wrong", ε-dependent operator. This estimate will be crucial for the proof of the lower bound in Section 4; it is used to prove the most difficult step, interchanging the operators T and $T^{\varepsilon,0}$ which happens in the inequality (142).

The following inequalities (134)–(137) are proven in the same way as in Corollary 3.3 with the help of (130), (132), and (133).

Corollary 4.2. *We have*

$$\|(1-\chi_\varepsilon)U^{\varepsilon,\eta,\mathsf{M}}_{(j)}; H^1(\varpi^\varepsilon)\| \leq C\varepsilon^\gamma, \tag{134}$$

$$\|(1-X_\varepsilon)T^{\varepsilon,0}(\chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}); H^1(\varpi^\varepsilon)\| \leq C\varepsilon^\gamma, \tag{135}$$

$$\|(\nabla X_\varepsilon)T^{\varepsilon,0}(\chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}); L^2(\varpi)\| \leq C\varepsilon^\gamma \tag{136}$$

$$\|(\nabla X_\varepsilon)T(\chi_\varepsilon U^{\varepsilon,\eta,\mathsf{M}}_{(j)}); L^2(\varpi)\| \leq C\varepsilon^\gamma. \tag{137}$$

Proof of Lemma 4.1. 1°. We prove (130). Let us fix the numbers j, ε and η, and denote in this proof $U := (U^\mathsf{M}, U^\mathsf{E}) := \big(U^{\varepsilon,\eta,\mathsf{M}}_{(j)}, T^{\varepsilon,\eta}U^{\varepsilon,\eta,\mathsf{M}}_{(j)}\big)$. In addition to the normalization (40), we also need a bound for $\|U; H^1(\varpi^\varepsilon)\|$. For this we write the eigenfunction property (32), or (33)–(36) as

$$\begin{aligned}
&\big(AD^\mathsf{M}(\nabla + i\eta e_3)U, D(\nabla + i\eta e_3)V\big)_{\varpi^\varepsilon} \\
=\;& \big(A^{\mathsf{MM}}D^\mathsf{M}(\nabla + i\eta e_3)U^\mathsf{M}, D^\mathsf{M}(\nabla + i\eta e_3)V^\mathsf{M}\big)_{\varpi^\varepsilon} \\
+\;& \big(A^{\mathsf{ME}}D^\mathsf{E}(\nabla + i\eta e_3)U^\mathsf{E}, D^\mathsf{M}(\nabla + i\eta e_3)V^\mathsf{M}\big)_{\varpi^\varepsilon} \\
-\;& \big(A^{\mathsf{EM}}D^\mathsf{M}(\nabla + i\eta e_3)U^\mathsf{M}, D^\mathsf{E}(\nabla + i\eta e_3)V^\mathsf{E}\big)_{\varpi^\varepsilon} \\
+\;& \big(A^{\mathsf{EE}}D^\mathsf{E}(\nabla + i\eta e_3)U^\mathsf{E}, D^\mathsf{E}(\nabla + i\eta e_3)V^\mathsf{E}\big)_{\varpi^\varepsilon} \\
=\;& \Lambda^{\varepsilon,\eta}_j(\varrho U^\mathsf{M}, V^\mathsf{M})_{\varpi^\varepsilon} \quad \forall\, V = (V^\mathsf{M}, V^\mathsf{E}) \in H^1_{\rm per}(\varpi^\varepsilon)^4.
\end{aligned} \tag{138}$$

We proceed as before choosing $V := U$. The identity $A^{\mathsf{ME}} = (A^{\mathsf{EM}})^\top$ again imply that the sum of the second and third lines of (138) is imaginary, while the first, fourth, and fifth lines are real. As a consequence of (40) we obtain the estimate

$$\begin{aligned}
&\big(A^{\mathsf{MM}}D^\mathsf{M}(\nabla + i\eta e_3)U^\mathsf{M}, D^\mathsf{M}(\nabla + i\eta e_3)U^\mathsf{M}\big)_{\varpi^\varepsilon} \\
+\;& \big(A^{\mathsf{EE}}D^\mathsf{E}(\nabla + i\eta e_3)U^\mathsf{E}, D^\mathsf{E}(\nabla + i\eta e_3)U^\mathsf{E}\big)_{\varpi^\varepsilon} \\
=\;& \Lambda^{\varepsilon,\eta}_j(\varrho U^\mathsf{M}, U^\mathsf{M})_{\varpi^\varepsilon} \leq C\Lambda^{\varepsilon,\eta}_j,
\end{aligned}$$

thus, using (9), (10), $D^{\mathsf{M}} = D^{\mathsf{M}}(\nabla)$ and $\|U^{\mathsf{M}}; L^2(\varpi^\varepsilon)\| = 1$,

$$\begin{aligned} & C\Lambda_j^{\varepsilon,\eta} \geq \big(A^{\mathsf{MM}} D^{\mathsf{M}}(\nabla + i\eta e_3)U^{\mathsf{M}}, D^{\mathsf{M}}(\nabla + i\eta e_3)U^{\mathsf{M}}\big)_{\varpi^\varepsilon} \\ = \;& \big(A^{\mathsf{MM}} D^{\mathsf{M}} U^{\mathsf{M}}, D^{\mathsf{M}} U^{\mathsf{M}}\big)_{\varpi^\varepsilon} + \big(A^{\mathsf{MM}} D^{\mathsf{M}} U^{\mathsf{M}}, D^{\mathsf{M}}(i\eta e_3)U^{\mathsf{M}}\big)_{\varpi^\varepsilon} \\ + \;& \big(A^{\mathsf{MM}} D^{\mathsf{M}}(i\eta e_3) U^{\mathsf{M}}, D^{\mathsf{M}} U^{\mathsf{M}}\big)_{\varpi^\varepsilon} + \big(A^{\mathsf{MM}} D^{\mathsf{M}}(i\eta e_3) U^{\mathsf{M}}, D^{\mathsf{M}}(i\eta e_3)U^{\mathsf{M}}\big)_{\varpi^\varepsilon} \\ \geq \;& C\|U^{\mathsf{M}}; H^1(\varpi^\varepsilon)\|^2 \\ - \;& C'\Big(\|\nabla U^{\mathsf{M}}; L^2(\varpi^\varepsilon)\|\|U^{\mathsf{M}}; L^2(\varpi^\varepsilon)\| + \|U^{\mathsf{M}}; L^2(\varpi^\varepsilon)\|^2\Big) \\ = \;& C\|U^{\mathsf{M}}; H^1(\varpi^\varepsilon)\|^2 - C'\big(\|\nabla U^{\mathsf{M}}; L^2(\varpi^\varepsilon)\| + 1\big). \end{aligned}$$

Dividing this by $\|U^{\mathsf{M}}; H^1(\varpi^\varepsilon)\|$ yields

$$\|U^{\mathsf{M}}; H^1(\varpi^\varepsilon)\| \leq C\Lambda_j^{\varepsilon,\eta} \tag{139}$$

Notice that by the remark below (10), the constant C can be chosen here independently of ε.

From now on we follow the considerations (99)–(104): we recall the definition of δ_θ in (82) with $Q_\theta f = \delta_\theta^{-\gamma} f$, and write $D := D(\nabla + i\eta e_3)$ and replace $u_{(j)}$ by $U = (U_{(j)}^{\varepsilon,\eta}, T^{\varepsilon,\eta} U_{(j)}^{\varepsilon,\eta})$, ϖ by ϖ^ε, Λ_j by $\Lambda_j^{\varepsilon,\eta}$. The number $\gamma = \gamma(j) > 0$ is fixed as in (99)–(104), where the bound (139) is needed after (103). These arguments again yield

$$\|\tilde U; L^2(\varpi^\varepsilon)\|^2 + \|\nabla \tilde U; L^2(\varpi^\varepsilon)\|^2 \leq C$$

for $\tilde U = \delta_\theta^{-\gamma} U_{(j)}^{\varepsilon,\eta}$. The only essential difference to the argument in Section 3.3 is that in (103), the eigenvalue $\Lambda_j^{\varepsilon,\eta}$ appears, but in order to get here a bound independent of ε, η, we have to use the bound (129). Finally, (130) follows by passing to the limit in the same way as after (104).

2°. The bound (131) follows from (130) in the same way as (92) follows from (91) in Section 3.3.

3°. The proof of (132) and (133) is similar to that of (93) in Section 3.3, where we take $\mathcal{V} := \chi_\varepsilon U_{(j)}^{\varepsilon,\eta,\mathsf{M}}$. Moreover, in the case (132), we consider the operator $T^{\varepsilon,\eta}$ and the domain ϖ^ε instead of T and ϖ, respectively. In the case (133) we consider T and ϖ. In (109) we use (131) instead of (92). ⊠

4.2. Estimates for nonlocal operators.

We state the following result, which is analogous to Lemma 3.4. Let us denote $\mathcal{U} := \chi_\varepsilon U_{(p)}^{\varepsilon,\eta,\mathsf{M}} \in H^1(\varpi^\varepsilon)^3$ for this section.

Lemma 4.3. *We have*

$$\mathcal{R}(\mathcal{U}, \mathcal{U}) = (A^{\mathsf{ME}} D^{\mathsf{E}} X_\varepsilon T\mathcal{U}, D^{\mathsf{M}}\mathcal{U})_\varpi, \tag{140}$$

$$\big\|X_\varepsilon T^{\varepsilon,0}\mathcal{U} - T^{\varepsilon,0} U_{(p)}^{\varepsilon,\eta,\mathsf{M}}; H^1(\varpi^\varepsilon)\big\| \leq C\varepsilon^\gamma \;, \tag{141}$$

$$\begin{aligned} &\big\|D^{\mathsf{E}}(X_\varepsilon T\mathcal{U} - X_\varepsilon T^{\varepsilon,0}\mathcal{U}); L^2(\varpi)\big\| \\ &\quad = \big\|D^{\mathsf{E}}(X_\varepsilon T\mathcal{U} - X_\varepsilon T^{\varepsilon,0}\mathcal{U}); L^2(\varpi^\varepsilon)\big\| \leq C\varepsilon^\gamma \end{aligned} \tag{142}$$

Proof. 1°. The statement (140) follows from the observation that the cut-off function X_ε equals 1 in the support of $D^{\mathsf{M}}\mathcal{U}$, see the definition (75),

2°. The proof of (141) is the same as that of (111), once one replaces ϖ by ϖ^ε, $H^1(\varpi)$ by $H^1_{\rm per}(\varpi^\varepsilon)$, T by $T^{\varepsilon,0}$, $\mathcal{V}$ by $\mathcal{U}$, and u by $U_{(p)}^{\varepsilon,\eta}$. In addition one has to take into account that by (41), the operator norms of $T^{\varepsilon,0} : H^1_{\rm per}(\varpi^\varepsilon)^3 \to H^1_{\rm per}(\varpi^\varepsilon)$ are uniformly bounded .

3°. The proof of (142) is slightly different from (110), so let us give the details. In the following the test function φ is assumed to be equal to $X_\varepsilon\tilde\varphi$ for some $\tilde\varphi \in H^1(\varpi)$, hence, $\varphi \in H^1(\varpi^\varepsilon)$. Also we require $\|\varphi; H^1(\varpi)\| \leq 1$. Moreover, all inner products will be well defined both in $L^2(\varpi)$ and $L^2(\varpi^\varepsilon)$.

The estimate (136) implies

$$\|D^{\mathsf{E}} X_\varepsilon T^{\varepsilon,0}\mathcal{U} - X_\varepsilon D^{\mathsf{E}} T^{\varepsilon,0}\mathcal{U}, L^2(\varpi)\| = \|(\nabla X_\varepsilon) T^{\varepsilon,0}\mathcal{U}; L^2(\varpi)\| \leq C\varepsilon^\gamma \tag{143}$$

In the same way, using (137) instead of (136), we find that

$$\|D^{\mathsf{E}} X_\varepsilon T\mathcal{U} - X_\varepsilon D^{\mathsf{E}} T\mathcal{U}, L^2(\varpi)\| \le C\varepsilon^\gamma$$

Combining this with (143) we obtain

$$\|D^{\mathsf{E}} X_\varepsilon (T\mathcal{U} - T^{\varepsilon,0}\mathcal{U}) - X_\varepsilon D^{\mathsf{E}}(T\mathcal{U} - T^{\varepsilon,0}\mathcal{U}), L^2(\varpi)\| \le C\varepsilon^\gamma \tag{144}$$

Since the support of $1 - X_\varepsilon$ is contained in $B(\mathcal{O}^\pm, \rho\varepsilon)$, we also have by (132) and by the same reasons as in (113),

$$\begin{aligned} &\big|\big(A^{\mathsf{EE}} X_\varepsilon D^{\mathsf{E}} T^{\varepsilon,0}\mathcal{U}, D^{\mathsf{E}}\varphi\big)_\varpi - \big(A^{\mathsf{EE}} D^{\mathsf{E}} T^{\varepsilon,0}\mathcal{U}, D^{\mathsf{E}}\varphi\big)_\varpi\big| \\ \le\; & \int\limits_{B(\mathcal{O}^\pm,\rho\varepsilon)} \big|A^{\mathsf{EE}} D^{\mathsf{E}} T^{\varepsilon,0}\mathcal{U}\big|\,\big|\nabla\varphi\big| dx \\ \le\; & C\Big(\int\limits_{B(\mathcal{O}^\pm,\rho\varepsilon)} \big|\nabla T^{\varepsilon,0}\mathcal{U}\big|^2\Big)^{1/2} \|\varphi; H^1(\omega)\|^2 \le C'\varepsilon^\gamma. \end{aligned} \tag{145}$$

In the same way, using (133) instead of (132), we prove that

$$\big|\big(A^{\mathsf{EE}} X_\varepsilon D^{\mathsf{E}} R\mathcal{U}, D^{\mathsf{E}}\varphi\big)_\varpi - \big(A^{\mathsf{EE}} D^{\mathsf{E}} T\mathcal{U}, D^{\mathsf{E}}\varphi\big)_\varpi\big| \le C\varepsilon^\gamma, \tag{146}$$

Keeping in mind the choice of the support of $\varphi \in H^1(\varpi^\varepsilon)$ in the beginning of 3°, we use (146), (145) and argue as in (114), to get

$$\begin{aligned} &\Big|\big(A^{\mathsf{EE}} X_\varepsilon D^{\mathsf{E}}(T\mathcal{U} - T^{\varepsilon,0}\mathcal{U}), D^{\mathsf{E}}\varphi\big)_\varpi\Big| \\ =\; & \Big|\big(A^{\mathsf{EE}} X_\varepsilon D^{\mathsf{E}} T\mathcal{U}, D^{\mathsf{E}}\varphi\big)_\varpi - \big(A^{\mathsf{EE}} X_\varepsilon D^{\mathsf{E}} T^{\varepsilon,0}\mathcal{U}, D^{\mathsf{E}}\varphi\big)_\varpi\Big| \\ =\; & \Big|\big(A^{\mathsf{EE}} D^{\mathsf{E}} T\mathcal{U}, D^{\mathsf{E}}\varphi\big)_\varpi - \big(A^{\mathsf{EE}} D^{\mathsf{E}} T^{\varepsilon,0}\mathcal{U}, D^{\mathsf{E}}\varphi\big)_{\varpi^\varepsilon}\Big| + O(\varepsilon^\gamma) \\ =\; & \Big|\big(A^{\mathsf{EM}} D^{\mathsf{M}}\mathcal{U}, D^{\mathsf{E}}\varphi\big)_\varpi - \big(A^{\mathsf{EM}} D^{\mathsf{M}}\mathcal{U}, D^{\mathsf{E}}\varphi\big)_{\varpi^\varepsilon}\Big| + O(\varepsilon^\gamma) = O(\varepsilon^\gamma). \end{aligned} \tag{147}$$

At the end we used the choice of the support of φ a second time to conclude that the difference on the last row is null. As a consequence of (8) and (147) with $\varphi = X_\varepsilon(T\mathcal{U} - T^{\varepsilon,0}\mathcal{U})$,

$$\begin{aligned} & C\|X_\varepsilon(T\mathcal{U} - T^{\varepsilon,0}\mathcal{U}); H^1(\varpi)\|^2 \\ \le\; & \Big|\big(A^{\mathsf{EE}} D^{\mathsf{E}} X_\varepsilon(T\mathcal{U} - T^{\varepsilon,0}\mathcal{U}), D^{\mathsf{E}} X_\varepsilon(T\mathcal{U} - T^{\varepsilon,0}\mathcal{U})\big)_\varpi\Big| \\ =\; & \Big|\big(A^{\mathsf{EE}} X_\varepsilon D^{\mathsf{E}}(T\mathcal{U} - T^{\varepsilon,0}\mathcal{U}), D^{\mathsf{E}} X_\varepsilon(T\mathcal{U} - T^{\varepsilon,0}\mathcal{U})\big)_\varpi\Big| + O(\varepsilon^\gamma) = O(\varepsilon^\gamma), \end{aligned}$$

where we also used (144) to commute the derivative and the cut-off function in the left side factor. So, the result (142) follows. ⊠

4.3. Proof of the lower estimate for spectral bands.

We prove the following estimate.

Lemma 4.4. *For every j, there exist a constant $C_j > 0$ and a number $\tilde\varepsilon_j > 0$, $\tilde\varepsilon_j \le \varepsilon_j$, such that for all $0 < \varepsilon < \tilde\varepsilon_j$ we have*

$$\Lambda_j^{\varepsilon,\eta} \ge \Lambda_j - C\varepsilon^\gamma,$$

where the number $\gamma = \gamma(j)$ is as in Lemma 4.1.

Proof. Given j, ε and η, we denote $\mathcal{U}_{(j)}^{\varepsilon,\eta} = \chi_\varepsilon U_{(j)}^{\varepsilon,\eta} \in H^1(\varpi^\varepsilon)^3 \subset H^1(\varpi)^3$. It should be clear (cf. the remarks in the proof of Lemma 3.5) that we can pick up for every j a small enough number $\tilde\varepsilon_j > 0$, $\tilde\varepsilon_j \le \varepsilon_j$, such that $\tilde\varepsilon_j < \tilde\varepsilon_{j-1}$ and such that the functions $\mathcal{U}_{(1)}^{\varepsilon,\eta}, \dots, \mathcal{U}_{(j)}^{\varepsilon,\eta}$ still remain linearly independent in $L^2(\varpi)^3$ for all $\varepsilon \le \tilde\varepsilon_j$ and η; we require

$$\|U_{(j)}^{\varepsilon,\eta} - \mathcal{U}_{(j)}^{\varepsilon,\eta}; L^2(\varpi)\| \le 2^{-j-3}.$$

We recall that by (115) in Lemma 3.5, every $\Lambda_j^{\varepsilon,\eta}$ can be bounded by a positive number L_j depending only on j. We now fix j, $\eta \in [-\pi,\pi]$ and $\varepsilon \le \tilde\varepsilon_j$. Also the sequence $(e^{i\eta z}\mathcal{U}_{(p)}^{\varepsilon,\eta})_{p=1}^j$ is linearly independent in $L^2(\varpi)^3$, hence, any subspace $\mathcal{H}_j \subset L^2(\varpi)^3$ of codimension $j-1$ contains a linear combination

$$W = \sum_{p=1}^j b_p e^{i\eta z} U_{(p)}^{\varepsilon,\eta} \quad , \quad \text{such that} \quad \sum_{p=1}^j |b_p|^2 = 1; \tag{148}$$

clearly, $\|W; L^2(\varpi)\| = 1$. We denote

$$\mathcal{W} = \sum_{p=1}^j b_p \mathcal{U}_{(p)}^{\varepsilon,\eta}. \tag{149}$$

We make the following observations on the form $\mathcal{B}^{\varepsilon,\eta}$: First, since $U_{(p)}^{\varepsilon,\eta}$ are the eigenvectors of the operator $\mathcal{M}^{\varepsilon,\eta}$ and (39) holds, we have, also by (40), (149), and $\mathcal{B}^{\varepsilon,\eta}(U_{(p)}^{\varepsilon,\eta}, U_{(p)}^{\varepsilon,\eta}) = \Lambda_p^{\varepsilon,\eta}$, that

$$|\mathcal{B}^{\varepsilon,\eta}(W,W)| \le \Lambda_j^{\varepsilon,\eta}. \tag{150}$$

Second, $|\mathcal{B}^{\varepsilon,0}(e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W})| = |\mathcal{B}^{\varepsilon,\eta}(\mathcal{W},\mathcal{W})|$. We next show that

$$\mathcal{B}(e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W}) \le \mathcal{B}^{\varepsilon,\eta}(e^{i\eta z}W, e^{i\eta z}W) + C\varepsilon^{\gamma(j)}. \tag{151}$$

To this end we recall from (76) and (149) that

$$\begin{aligned}\mathcal{B}(e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W}) \quad &= \quad \mathcal{A}(e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W}) + (\varrho e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W})_\varpi \\ &+ \quad \sum_{p,q=1}^j b_p \bar b_q \mathcal{R}(e^{i\eta z}\mathcal{U}_{(p)}^{\varepsilon,\eta}, e^{i\eta z}\mathcal{U}_{(q)}^{\varepsilon,\eta}).\end{aligned} \tag{152}$$

Here,

$$\begin{aligned} &\mathcal{A}(e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W}) \le (A^{\mathsf{MM}} D^{\mathsf{M}} e^{i\eta z}\mathcal{W}, D^{\mathsf{M}} e^{i\eta z}\mathcal{W})_{\varpi^\varepsilon} = \mathcal{A}^{\varepsilon,0}(e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W}) \\ = \quad &\mathcal{A}^{\varepsilon,0}(e^{i\eta z}W, e^{i\eta z}W) + O(\varepsilon^\gamma), \\ &(\varrho e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W})_\varpi \le (\varrho e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W})_{\varpi^\varepsilon} = (\varrho e^{i\eta z}W, e^{i\eta z}W)_{\varpi^\varepsilon} + O(\varepsilon^\gamma) \end{aligned} \tag{153}$$

by the positivity of the matrix A^{MM} in the definitions of $\mathcal{A}$ and $\mathcal{A}^{\varepsilon,0}$, cf. (76), (9), (38), and by (148), (149), (134). Thus it remains to consider the $\mathcal{R}$-term. Let us fix a $p = 1,\dots,j$ and denote $U = U_{(p)}^{\varepsilon,\eta,\mathsf{M}}$, $\mathcal{U} = \chi_\varepsilon U$ as in Lemma 4.3. Since $X_\varepsilon = 1$ in the $\operatorname{supp}(\chi_\varepsilon)$ and $\operatorname{supp}(\chi_\varepsilon) \subset \varpi$, we have

$$\mathcal{R}(\mathcal{U},\mathcal{U}) = (A^{\mathsf{ME}} D^{\mathsf{E}} T\mathcal{U}, D^{\mathsf{E}}\mathcal{U})_\varpi = (A^{\mathsf{ME}} D^{\mathsf{E}} X_\varepsilon T\mathcal{U}, D^{\mathsf{E}}\mathcal{U})_{\varpi^\varepsilon}.$$

Hence, Lemma 4.3 and (134) imply

$$\begin{aligned} &\mathcal{R}(\mathcal{U},\mathcal{U}) = (A^{\mathsf{ME}} D^{\mathsf{E}} X_\varepsilon T\mathcal{U}, D^{\mathsf{E}}\mathcal{U})_{\varpi^\varepsilon} \\ = \quad &(A^{\mathsf{ME}} D^{\mathsf{E}} X_\varepsilon T^{\varepsilon,0}\mathcal{U}, D^{\mathsf{E}}\mathcal{U})_{\varpi^\varepsilon} + O(\varepsilon^\gamma) = (A^{\mathsf{ME}} D^{\mathsf{E}} T^{\varepsilon,0} U, D^{\mathsf{E}}\mathcal{U})_{\varpi^\varepsilon} + O(\varepsilon^\gamma) \\ = \quad &(A^{\mathsf{ME}} D^{\mathsf{E}} T^{\varepsilon,0} U, D^{\mathsf{E}} U)_{\varpi^\varepsilon} + O(\varepsilon^\gamma) = \mathcal{R}^{\varepsilon,0}(U,U) + O(\varepsilon^\gamma). \end{aligned}$$

Applying this to all terms of the last sum of (152) and using (153) yields (151), in view of (148) and (149).

By (134), (148), (149), we can also estimate

$$\begin{aligned} &\|e^{i\eta z}\mathcal{W}; L^2(\varpi)\| = \|\mathcal{W}; L^2(\varpi^\varepsilon)\| \\ \ge \quad &\|W; L^2(\varpi^\varepsilon)\| - \sum_{p=1}^j b_p \|(1-\chi_\varepsilon) U_{(p)}^{\varepsilon,\eta}; L^2(\varpi^\varepsilon)\| \ge 1 - C_j \varepsilon^{\gamma(j)}. \end{aligned} \tag{154}$$

Putting together (150)–(154) yields

$$\frac{\mathcal{B}(e^{i\eta z}\mathcal{W}, e^{i\eta z}\mathcal{W})}{\|e^{i\eta z}\mathcal{W}; L^2(\varpi)\|^2} \leq \mathcal{B}^{\varepsilon,0}(e^{i\eta z}W, e^{i\eta z}W) + C\varepsilon^{\gamma(j)}$$
$$= \mathcal{B}^{\varepsilon,\eta}(W,W) + C\varepsilon^{\gamma(j)} \leq \Lambda_j^{\varepsilon,\eta} + C\varepsilon^{\gamma(j)}.$$

Since $\mathcal{H}_j$ was an arbitrary $(j-1)$–codimensional subspace of $L^2(\varpi)$, we get from the max–min–principle (80) that $\Lambda_j \leq \Lambda_j^{\varepsilon,\eta} + C\varepsilon^{\gamma(j)}$. ⊠

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Sergei A. Nazarov
Saint-Petersburg State University, Mathematics and Mechanics Faculty, Universitetsky pr., 28, Peterhof, St. Petersburg, 198504, Russia
and Saint-Petersburg State Polytechnical University, Polytechnicheskaya ul., 29, St. Petersburg, 195251, Russia
and Institute of Problems of Mechanical Engineering RAS, V.O., Bolshoj pr., 61, St. Petersburg, 199178, Russia

Jari Taskinen
University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FI-00014 Helsinki, Finland.