Contents

9 Symmetries and cosmology

9.1 Symmetries of manifolds

9.1.1 Killing vectors

We have already mentioned the concepts **stationary** and **static** when discussing spacetimes. We have said that a spacetime is stationary if there exists a coordinate system where $\partial_0 g_{\alpha\beta} = 0$, and static if there exists a coordinate system where also $g_{0i} = 0$. This is like defining flat spacetime by saying that there exists a coordinate system where $g_{\alpha\beta}$ is everywhere constant. For flatness, the coordinate-independent criterion is the vanishing of the Riemann tensor. We similarly want a criterion for stationary and static, and also other symmetries of the manifold, that can be checked in any coordinates.

Consider the condition $\partial_0 g_{\alpha\beta} = 0$ that defines a stationary spacetime. In order to make it coordinate-independent, we should upgrade both the direction and the derivative to objects that live on the manifold and not just in a coordinate patch. We already know how to describe time direction in a coordinate-independent manner: just change the x^0 direction to the direction given by a timelike vector field $\underline{V}$. What about the derivative? We cannot simply shift from the partial to the covariant derivative, because it gives zero when applied to the metric. Instead, we use the **Lie derivative**. The Lie derivative of the metric in the direction of the vector field $\underline{V}$

is defined as

$$\begin{aligned} \mathcal{L}_{\underline{V}} g_{\alpha\beta} &\equiv V^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha V^\gamma g_{\gamma\beta} + \partial_\beta V^\gamma g_{\alpha\gamma} \\ &= V^\gamma \nabla_\gamma g_{\alpha\beta} + \nabla_\alpha V^\gamma g_{\gamma\beta} + \nabla_\beta V^\gamma g_{\alpha\gamma} \\ &= 2\nabla_{(\alpha} V_{\beta)} \ , \end{aligned} \tag{9.1}$$

where on the second line we have used the covariant derivative instead of the partial derivative, and on the third line chosen the Levi–Civita connection, for which $\nabla_\gamma g_{\alpha\beta} = 0$. The connection coefficients on the second line cancel, so all three forms are equivalent. Note that this is the change of the metric under the small coordinate change $x^\alpha \to x^\alpha + V^\alpha(x)$, and evaluated at the new coordinate position, which we discussed in section 6.1.4 when deriving the continuity equation $\nabla_\alpha T^{\alpha\beta} = 0$.

If the Lie derivative (9.1) is zero, $\underline{V}$ is called a **Killing vector**[1], and defines a symmetry of the manifold. Correspondingly the equation that it satisfies,

$$\mathcal{L}_{\underline{V}} g_{\alpha\beta} = 2\nabla_{(\alpha} V_{\beta)} = 0 \tag{9.2}$$

is called the **Killing equation**.

The Killing equation gives a conserved quantity along geodesics. Consider a geodesic with tangent vector field $\underline{A}$. The change of the dot product $\underline{A} \cdot \underline{V}$ along the geodesic is given by

$$A^\beta \nabla_\beta (A^\alpha V_\alpha) = A^\beta A^\alpha \nabla_{(\beta} V_{\alpha)} = 0 \ , \tag{9.3}$$

where in the first equality we have used the geodesic equation and in the second equality we have used the Killing equation. Thus, $\underline{A} \cdot \underline{V}$ is conserved along the geodesic. In chapter 5, we used a particular case of this result when discussing orbits of the Schwarzschild solution. There we obtained conserved quantities corresponding to time translation symmetry (energy) and rotation symmetry (angular momentum). If the Killing vector is timelike, (9.3) applied to a null geodesic says that photon energy is conserved along the null geodesic.

A spacetime is defined to be stationary when there is a timelike Killing vector field $\underline{V}$. If furthermore $\nabla_{[\beta} V_{\alpha]}$ projected orthogonally to V^α is zero, the spacetime is defined to be static. This second condition can be stated as $\epsilon^{\alpha\beta\gamma\delta} V_\beta \nabla_{[\gamma} V_{\delta]} = 0$. Physically, this means that $\underline{V}$ is **irrotational**. According to **Frobenius' theorem**, this condition is equivalent to the existence of hypersurfaces orthogonal to $\underline{V}$ that foliate the spacetime.

Let us see how these coordinate-independent conditions are equivalent to the coordinate-dependent ones we have used earlier. We choose the proper time along the worldline tangent to the Killing vector as the time coordinate, i.e. we adopt **comoving coordinates**. Then $V^\alpha = \delta^{\alpha 0}$. It immediately follows from the first line of (9.1) that $\mathcal{L}_{\underline{V}} g_{\alpha\beta} = 0$ is equivalent to $\partial_0 g_{\alpha\beta} = 0$. The spaces of constant x^0 are then orthogonal to V^α precisely when $g_{0i} = 0$. Looking at the definition in terms of $\nabla_{[\beta} V_{\alpha]}$, we have

$$0 = \epsilon^{\alpha\beta\gamma\delta} V_\beta \nabla_{[\gamma} V_{\delta]} = \epsilon^{\alpha\beta\gamma\delta} V_\beta \partial_{[\gamma} V_{\delta]} \ , \tag{9.4}$$

[1] The name refers to the mathematician Wilhelm Killing, not to extinguishing life.

where we have used the fact that the connection is symmetric. Given that $V_\alpha = g_{\alpha\beta}V^\beta = g_{\alpha 0}$, we see that $g_{0i} = 0$ is a sufficient condition for (9.4) to hold. It is more involved to show that it is also a necessary condition, so we skip the proof.

If there is more than one Killing vector, in general it is not possible to choose coordinates so that the components of all Killing vectors are constant. Therefore all of the symmetries they encode will not be transparent in the metric. In other words, as the derivatives of V^α also contribute in (9.1), the derivative of the metric in the direction of V^α is not zero. Note the analogy to the covariant derivative, where the connection coefficients correct for the coordinate dependence of the partial derivative.

As the Killing equation is linear, the sum of two Killing vectors is a Killing vector. As the Killing equation $\nabla_{(\beta}V_{\alpha)} = 0$ has 10 independent components, it follows that there are at most 10 independent Killing vectors. (This is in 4 dimensions. in d dimensions, the maximum number is $\frac{1}{2}d(d+1)$.) Also, the commutator of two Killing vectors is a Killing vector. (**Exercise.** Show this.) Therefore the Killing vectors form a **Lie algebra** corresponding to the symmetry group of the manifold, and we have

$$[\underline{V}_A, \underline{V}_B] = \sum_D C^D{}_{AB}\underline{V}_D \ , \tag{9.5}$$

where the indices A, B, D label Killing vectors, and the numbers $C^D{}_{AB}$ are the **structure constants** of the symmetry group. The relation (9.5) is purely algebraic, but vectors correspond to differential operators. In this way (9.5) relates the differential properties of the manifold to the algebraic properties of symmetry groups. For example, to rigorously derive a spherically symmetric metric, we would start from the set $C^D{}_{AB}$ that corresponds to the symmetry group $SO(2)$, find the corresponding differential operators, and solve the metric using the Killing equation.

The Killing equation can be read in two directions. On the one hand, the first line of (9.1) is an equation for the metric, given a vector field $\underline{V}$. We can thus construct a metric that corresponds to a given symmetry. On the other hand, the second line gives an equation for a vector field given a metric (and the corresponding connection). We can illustrate these two aspects with a second order equation for V^α. We start from the definition of the Riemann tensor, applied to the Killing vector V_α:

$$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} \ = \ R^\delta{}_{\alpha\beta\gamma}V_\delta \ . \tag{9.6}$$

Permuting the indices and adding the resulting equations, and using the Killing vector condition $V_{\alpha;\beta} = -V_{\beta;\alpha}$ (note the similarity to the derivation of the Levi–Civita connection in terms of the metric), we get

$$\begin{array}{rcl} V_{\alpha;\beta\gamma} + \cancel{V_{\gamma;\alpha\beta}} &=& R^\delta{}_{\alpha\beta\gamma}V_\delta \\ \cancel{V_{\beta;\gamma\alpha}} + V_{\alpha;\beta\gamma} &=& R^\delta{}_{\beta\gamma\alpha}V_\delta \\ \cancel{-V_{\gamma;\alpha\beta}} \cancel{-V_{\beta;\gamma\alpha}} &=& -R^\delta{}_{\gamma\alpha\beta}V_\delta \\ \hline \multicolumn{3}{c}{2V_{\alpha;\beta\gamma} = (R^\delta{}_{\alpha\beta\gamma} + R^\delta{}_{\beta\gamma\alpha} - R^\delta{}_{\gamma\alpha\beta})V_\delta} \end{array}$$

Using the first Bianchi identity $R^{\delta}{}_{[\alpha\beta\gamma]} = 0$, we can simplify the result into

$$V_{\alpha;\beta\gamma} = -R^{\delta}{}_{\gamma\alpha\beta} V_{\delta} \ . \tag{9.7}$$

This is a second order equation for the Killing vector V^{α}. It has 10 initial conditions: 4 initial values of V^{α} and 6 initial values of $V_{\alpha;\beta} = V_{[\alpha;\beta]}$. This is another way to see that there are at most 10 linearly independent Killing vectors in 4 dimensions. The equation (9.7) links the properties of the Killing vectors to the Riemann tensor.

For example, consider Minkowski space, where $R^{\delta}{}_{\gamma\alpha\beta} = 0$. We choose coordinates such that the connection vanishes, so (9.7) reduces to

$$V_{\alpha,\beta\gamma} = 0 \ , \tag{9.8}$$

which has the solution

$$V_{\alpha} = a_{\alpha} + b_{\alpha\beta} x^{\beta} \tag{9.9}$$

where a_{α} and $b_{\alpha\beta} = b_{[\alpha\beta]}$ are constants. From the first line of (9.1), it is transparent that the constants a_{α} correspond to translations; it can be shown that $b_{\alpha\beta}$ correspond to Lorentz transformations. We could also run this reasoning backwards: starting from the Lie algebra of the Poincaré group, we can solve the components of the Killing vectors from (9.5) (which does not involve the metric) to get the solution (9.9), and then it insert into (9.7) to find $R^{\delta}{}_{\gamma\alpha\beta} = 0$.

9.1.2 Maximally symmetric manifolds

A manifold is **maximally symmetric** if it has the maximum number of Killing vectors, i.e. in 4 dimensions 10. We will construct the metric for maximally symmetric manifolds in a simpler way than using the Killing equation. If a manifold is maximally symmetric, it has no preferred directions or positions. This means that the Riemann tensor has to be built from the metric and the Levi–Civita tensor alone. As $R^{\delta}{}_{[\alpha\beta\gamma]} = 0$, the Levi–Civita tensor cannot be involved, so the Riemann tensor is

$$R_{\alpha\beta\gamma\delta} = K(x)(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \ , \tag{9.10}$$

where K is a scalar function. The Ricci tensor is (in d dimensions)

$$R_{\alpha\beta} = (d-1) K(x) g_{\alpha\beta} \ , \tag{9.11}$$

and the Ricci scalar is

$$R = d(d-1) K(x) \ , \tag{9.12}$$

giving the Einstein tensor

$$G_{\alpha\beta} = \left(1 - \frac{d}{2}\right)(d-1) K(x) g_{\alpha\beta} \ . \tag{9.13}$$

The Einstein tensor for a maximally symmetric manifold in 2 dimensions is zero. (In fact, this is true for any two-dimensional manifold.) For $d > 2$, the contracted second Bianchi identity $\nabla_\alpha G^{\alpha\beta} = 0$ gives $\partial_\alpha K = 0$, i.e. K is constant. This is also true for $d = 2$.

We can now classify maximally symmetric manifolds according to two criteria: the sign of the curvature constant K (positive, negative or zero) and the sign of the determinant of the metric (positive or negative). (Even though we consider a four-dimensional Lorentzian manifold, we can look at submanifolds with no time directions.) The different possibilities are listed in table 1.

Sign of curvature	$\det(g_{\alpha\beta}) > 0$	$\det(g_{\alpha\beta}) < 0$
$K = 0$	Euclidean space $\mathbb{R}^d$	Minkowski space
$K > 0$	Hypersphere S^d	de Sitter space dS^d
$K < 0$	Hyperbolic space $\mathbb{H}^d$	anti-de Sitter space AdS^d

Table 1: Classification of maximally symmetric spacetimes with $d \geq 2$ with at most one time direction.

We assume that the manifolds have the simplest possible topology. We could construct more complicated manifolds by identifying some points with each other. For example, if we take a rectangle (a piece of $\mathbb{R}^2$) and identify each opposite side, we get the torus $\mathbb{T}^2$. We can tile $\mathbb{R}^2$ with rectangles and identify all of them like this, reducing the whole space to a finite torus. Similar identifications can be made in higher-dimensional spaces and for other values of the spatial curvature. Non-trivial spatial topology has been looked for in cosmological observations. No sign of it has been found, so if the topology of the real universe is non-trivial, the related scale is larger than the radius of the observable universe today, about 50 billion light years.

If $\det(g_{\alpha\beta}) < 0$, we have the three possible maximally symmetric spacetimes. If the curvature is zero, we have Minkowski space. If the curvature is positive, we have de Sitter space dS^d, and if it is negative, we have anti-de Sitter space AdS^d. If $\det(g_{\alpha\beta}) > 0$, we have the three i.e. maximally symmetric spaces (not spacetimes). If the curvature is zero, we have the Euclidean space $\mathbb{R}^n$. If the curvature is positive, we have the d-sphere S^d (hypersphere, for $d > 2$). If the curvature is negative, we have the d-dimensional hyperbolic space $\mathbb{H}^d$ (hyper-hyperboloid?).

We are familiar with viewing a two-sphere as a surface embedded in three-dimensional Euclidean space. We can do the same with the d-sphere. The Euclidean metric in $d + 1$ dimensions is, in Cartesian coordinates,

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ , \tag{9.14}$$

where i and j run from 1 to d. A sphere with radius α is the hypersurface in this space where the coordinates satisfy

$$u^2 + \delta_{ij}x^i x^j = \alpha^2 \ . \tag{9.15}$$

The $d + 1$-dimensional space is just a mathematical artifact that makes it easier to treat the hypersphere. We can obtain the metric on the hypersphere in two ways. We can solve u from (9.15) and insert it into the Euclidean metric (9.14). Or we can parametrise the coordinates in such a way that (9.15) is satisfied, and insert this parametrisation into (9.14). We take the second approach, and for $d = 3$ define the coordinates (χ, θ, φ) on the hypersphere as (χ and θ range from 0 to π, and φ ranges from 0 to 2π)

$$\begin{aligned} u &= \alpha \cos\chi \\ x &= \alpha \sin\chi \cos\theta \\ y &= \alpha \sin\chi \sin\theta \cos\varphi \\ z &= \alpha \sin\chi \sin\theta \sin\varphi \ . \end{aligned} \tag{9.16}$$

Inserting this into (9.14) gives the metric of the three-dimensional hypersphere:

$$\mathrm{d}s^2 = \alpha^2 \mathrm{d}\chi^2 + \alpha^2 \sin^2\chi (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) \ . \tag{9.17}$$

The generalisation to $d > 3$ is straightforward.

We can do the same for the hyperbolic space. Instead of Euclidean space, we have $d + 1$-dimensional Minkowski space as the embedding space, with the metric

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j \ . \tag{9.18}$$

Hyperbolic space is the hypersurface in this space where the coordinates satisfy

$$-u^2 + \delta_{ij} x^i x^j = -\alpha^2 \ . \tag{9.19}$$

In analogy with the hyperspherical case (9.16), for $d = 3$ we introduce the coordinates

$$\begin{aligned} u &= \alpha \cosh\chi \\ x &= \alpha \sinh\chi \cos\theta \\ y &= \alpha \sinh\chi \sin\theta \cos\varphi \\ z &= \alpha \sinh\chi \sin\theta \sin\varphi \ . \end{aligned} \tag{9.20}$$

Inserting this into (9.18), we get the metric of three-dimensional hyperbolic space,

$$\mathrm{d}s^2 = \alpha^2 \mathrm{d}\chi^2 + \alpha^2 \sinh^2\chi \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \right) \ . \tag{9.21}$$

Again, the generalisation to $d > 3$ is straightforward.

Let us now consider spacetimes where only the spatial sections are maximally symmetric.

9.2 Friedmann–Lemaître–Robertson–Walker universe

9.2.1 Metric

The 4d spacetime with three-dimensional maximally symmetric spatial hypersurfaces is called the **Friedmann–Lemaître–Robertson–Walker model** (FLRW model). We have earlier discussed the spatially flat case, and will now consider the general case. The FLRW model is one of the most useful exact solutions in GR, and has turned out to be a good description of the average properties of the universe.

Using it to model the real universe is often motivated with either the **Copernican principle** or the **cosmological principle**. The Copernican principle states that our position in space is not special, in particular we are not at a center of symmetry. The observed cosmic microwave background (CMB) shows that the universe on large scales is isotropic around us to about one part in 10^5. If our location is not special, this should be also true for other locations. If a three-dimensional space is exactly isotropic around three or more points (two is not sufficient), it is also exactly homogeneous. Although there are spacetimes where all observers see an exactly isotropic CMB even though space is not exactly isotropic, they are special cases and do not describe the real universe. However, from the fact that the CMB looks almost isotropic at all points it does not follow that the spacetime is almost homogeneous. So the Copernican principle does not get us to the FLRW model.

The cosmological principle states that on sufficiently large scales, the universe has no special positions or directions, i.e. it is homogeneous and isotropic. The FLRW model was first proposed by Aleksander Friedmann in 1922, before there were reliable cosmological observations. (The existence of galaxies other than the Milky Way was only shown in 1924.) Today this is not so much a principle as an observational fact[2]: on scales larger than about 500 million light years, all regions of the universe look statistically equivalent, up to small correlations. This is now predicted by the early universe scenario called cosmic inflation. But statistical homogeneity and isotropy are not the same as exact homogeneity and isotropy, and a space that is only statistically homogeneous and isotropic does not in general behave on average like a FLRW universe, even when averaged over large scales. Nevertheless, the FLRW model has proven to be a good approximation to the real universe, justified first by simplicity, second by success in explaining and predicting observations, and now there is also increasing theoretical understanding of why it is successful.

The metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\Sigma_3^2 \ , \tag{9.22}$$

where $\mathrm{d}\Sigma_3^2$ is the metric of one of the three maximally symmetric three-dimensional spaces discussed above. If we use spherical instead of angular coordinates, all three cases can be described simultaneously. One way to derive the metric for the three cases at the same time is to note that a maximally symmetric space is a subcase of an isotropic space. So, following what we did with the Schwarzschild metric in (5.4), we can write the metric as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2[e^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2] \ . \tag{9.23}$$

We know that the Riemann tensor of a slice of constant t has the form (9.10), so the spatial Ricci scalar is $6K/a(t)^2$. Deriving the spatial Ricci scalar from (9.23) and equating it to $6K/a(t)^2$ then allows to solve for $\beta(r)$, with the result (**Exercise:** Do this.)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\varphi^2 \right) \ . \tag{9.24}$$

[2] Modulo some caveats with regard to the interpretation of the observations.

In Cartesian coordinates, the metric reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{a(t)^2}{\left(1+\frac{K}{4}r^2\right)^2}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ , \tag{9.25}$$

where $r^2 \equiv \delta_{ij}x^i x^j$. In either form, this is called the **Robertson–Walker metric**, the **Friedmann–Robertson–Walker metric** (FRW metric) or the **Friedmann–Lemaître–Robertson–Walker metric** (FLRW metric).[3] Rotation symmetry is obvious in the metric, but not symmetry under spatial translations.

The spatial coordinates used in (9.24) and (9.25) are called **comoving coordinates**. Observers whose spatial coordinates are constant in these coordinate systems are called **comoving observers**. The time coordinate t is called the **cosmic time**. It is the proper time measured by comoving observers. Note that the proper distance between observers at constant coordinate distance grows or decreases proportionally to the **scale factor** $a(t)$. This is analogous to the oscillation of proper spatial distance between observers at constant coordinates in the case of gravitational waves described in the transverse gauge. The constant K is related to the curvature of space. The combination

$$R_{\mathrm{curv}}(t) \equiv a(t)/\sqrt{|K|} \tag{9.26}$$

is called the **curvature radius** of space. The FLRW metric also has a second length scale, which is related to the expansion, the **Hubble time**, $t_H \equiv H^{-1}$, where $H \equiv \dot{a}/a$ is the **Hubble parameter**, also called the **Hubble rate**.[4] The Hubble time multiplied by the speed of light $c = 1$ is the **Hubble length**, $\ell_H \equiv ct_H = H^{-1}$. In the case $K = 0$ the Hubble length is the only length scale.

The FLRW metric is invariant under a rescaling of the radial coordinate r and other quantities as

$$r \to \tilde{r} = \lambda r, \quad a \to \tilde{a} = \frac{1}{\lambda}a, \quad K \to \tilde{K} = \frac{1}{\lambda^2}K \ , \tag{9.27}$$

where $\lambda > 0$ is a constant. There are two common ways to use the rescaling to simplify the notation. If $K \neq 0$, we can rescale r to make K equal to ± 1. In this case the lowercase letter is usually used, so k instead of K. Then r is dimensionless, and $a(t)$ has the dimension of distance. Alternatively, we can set the scale factor today to unity , $a(t_0) \equiv a_0 = 1$.[5] (In cosmology, the subscript 0 usually denotes

3 The most common acronym used to be FRW, now the convention has shifted to FLRW. Some authors prefer to make a terminological distinction between the geometry (with the names Robertson and Walker attached) and the equations of motion (endowed with the name Friedmann and sometimes also Lemaître).

4 Georges Lemaître introduced this parameter and determined its value from observations in 1927. Two years later, Edwin Hubble introduced this parameter and determined its value from observations, so it became known as the Hubble parameter. In 2018, the International Astronomical Union decided to recommend calling an associated equation the Hubble–Lemaître law instead of the Hubble law, so perhaps H should correspondingly be called the Hubble–Lemaître parameter.

5 In some discussions of the early universe, it is more convenient to put a to unity at some other instead.

present day.) Unless otherwise noted, we use this latter convention. In this case $a(t)$ is dimensionless, and r and $K^{-1/2}$ have the dimension of distance.

If $K = 0$, the space is $\mathbb{R}^3$, and the proper radial distance is ar. It is often said that the "universe is flat" in this case, although if the universe is understood as the four-dimensional spacetime (as opposed to a spatial slice), "spatially flat" would be more correct.

If $K > 0$, the space is S^3. The spherical coordinates are singular at $r = r_K \equiv 1/\sqrt{K}$. With the coordinate transformation $r = r_K \sin\chi$ the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 K^{-1}\left[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right] . \tag{9.28}$$

The spatial part is conformal to the hypersphere metric (9.17). The angular coordinate χ has the range $0 < \chi < \pi$. These coordinates cover the whole space (apart from the usual problem at the poles and the seam line in the neighbourhood of $\varphi = 0$), whereas the original spherical coordinates cover only half of the space, i.e. one semihypersphere. Space is finite, with circumference $2\pi a r_K = 2\pi R_{\mathrm{curv}}$ and volume $2\pi^2 a^3 r_K^3 = 2\pi^2 R_{\mathrm{curv}}^3$. This positively curved universe is called **closed**.

If $K < 0$, the space is $\mathbb{H}^3$. There is no coordinate singularity in the metric (9.24), and r ranges from 0 to ∞. The substitution $r = |K|^{-1/2}\sinh\chi$ leads to the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2|K|^{-1}\left[\mathrm{d}\chi^2 + \sinh^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right] . \tag{9.29}$$

The spatial part is conformal to the metric (9.21). Space is infinite. In this negatively curved case the universe is called **open**. (The simplest topology for this space is same as for Euclidean space, so this terminology can be a bit misleading.)

If we want to keep the metric isotropic, we cannot do coordinate transformations that mix the time coordinate with the spatial coordinates. However, just as we redefined the radial coordinate as a function of itself, we can redefine the time coordinate alone. In comoving coordinates, the spatial part of the coordinate system stretches with the expansion of the universe. It is often practical to change the time coordinate so that the unit of time (i.e. the separation of time coordinate surfaces) grows at the same rate as the proper spatial distance. This is achieved by switching to the **conformal time** η defined by

$$\mathrm{d}\eta \equiv \frac{1}{a(t)}\mathrm{d}t \qquad \Rightarrow \qquad \eta = \int^t \frac{\mathrm{d}t'}{a(t')}. \tag{9.30}$$

If we choose the normalisation of $a(t)$ so that $K = \pm1, 0$, the above metrics read

$$\mathrm{d}s^2 \;=\; a(\eta)^2[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \left\{\begin{matrix}\sin^2\chi \\ \chi^2 \\ \sinh^2\chi\end{matrix}\right\}\mathrm{d}\Omega^2] \,, \tag{9.31}$$

where the cases in the curly brackets correspond, from top to bottom, to $K = +1, 0, -1$, and we have denoted $r \equiv \chi$ in the case $K = 0$. This form is particularly suited to studying light propagation. We can choose radial direction to be in the direction of propagation, so $\mathrm{d}\theta = \mathrm{d}\varphi = 0$, and the remaining part of the metric is conformal to the 1+1-dimensional Minkowski metric, with coordinates η and χ. The condition $\mathrm{d}s^2 = 0$ then gives simply $\mathrm{d}\eta = \pm\mathrm{d}\chi$, and light rays travel at 45° angles, which will be useful when we look at the causal properties of FLRW universes.

9.2.2 Friedmann equations

The function $a(t)$ is determined by the matter content (and the initial conditions) via the Einstein equation. The non-zero Levi–Civita connection coefficients for the FLRW metric in the spherical coordinates (9.24) are

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-Kr^2} \\
\Gamma^0_{22} &= a\dot{a}r^2 \\
\Gamma^0_{33} &= a\dot{a}r^2\sin^2\theta \\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a} \\
\Gamma^1_{11} &= \frac{kr}{1-Kr^2} \\
\Gamma^1_{22} &= -r(1-Kr^2) \\
\Gamma^1_{33} &= -r(1-Kr^2)\sin^2\theta \\
\Gamma^2_{12} &= \Gamma^3_{12} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \\
\Gamma^3_{23} &= \cot\theta\ .
\end{aligned} \tag{9.32}$$

The non-zero components of the Riemann tensor are

$$\begin{aligned}
R^{0i}{}_{0j} &= \frac{\ddot{a}}{a}\delta^i{}_j \\
R^{ij}{}_{kl} &= \left(\frac{\dot{a}^2}{a^2}+\frac{K}{a^2}\right)(\delta^i{}_k\delta^j{}_l-\delta^i{}_l\delta^j{}_k)\ .
\end{aligned} \tag{9.33}$$

The Weyl tensor is zero due to symmetry. The non-zero components of the Ricci tensor are

$$\begin{aligned}
R^0{}_0 &= 3\frac{\ddot{a}}{a} \\
R^i{}_j &= \left(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}+2\frac{K}{a^2}\right)\delta^i{}_j\ ,
\end{aligned} \tag{9.34}$$

so the Ricci scalar is

$$R = 6\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{K}{a^2}\right)\ , \tag{9.35}$$

and the non-zero components of the Einstein tensor are

$$\begin{aligned}
G^0{}_0 &= -3\frac{\dot{a}^2}{a^2}-3\frac{K}{a^2} \\
G^i{}_j &= -\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{K}{a^2}\right)\delta^i{}_j\ .
\end{aligned} \tag{9.36}$$

Let us now turn to the energy-momentum tensor. Because of symmetry, the energy flux and anisotropic stress are zero, and the non-zero components of the energy-momentum tensor are

$$\begin{aligned} T^0{}_0 &= -\rho(t) \\ T^i{}_j &= P(t)\delta^i{}_j \ . \end{aligned} \tag{9.37}$$

The Einstein equation (including the cosmological constant) now reduces to

$$\begin{aligned} 3\frac{\dot{a}^2}{a^2} + 3\frac{K}{a^2} &= 8\pi G_{\rm N}\rho + \Lambda \\ -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} &= 8\pi G_{\rm N} P - \Lambda \ . \end{aligned} \tag{9.38}$$

This pair of equations can be rearranged into a more convenient form, where first derivatives appear in only one equation:

$$3\frac{\dot{a}^2}{a^2} + 3\frac{K}{a^2} = 8\pi G_{\rm N}\rho + \Lambda \tag{9.39}$$

$$3\frac{\ddot{a}}{a} = -4\pi G_{\rm N}(\rho + 3P) + \Lambda \ . \tag{9.40}$$

These are the **Friedmann equations**. The expression **Friedmann equation** in the singular refers to (9.39). The second Bianchi identity relates these two equations together: it states that the equation (9.39) has to be the first integral of (9.40), which implies the relation

$$\dot{\rho} = -3(\rho + P)\frac{\dot{a}}{a} \ . \tag{9.41}$$

This is just the continuity equation $\nabla_\alpha T^\alpha{}_\beta = 0$ applied to the FLRW case. This equation, as we have noted, shows how energy is not conserved. We can rewrite (9.41) as

$$P = -\frac{1}{3H}\frac{1}{a^3}\frac{{\rm d}(a^3\rho)}{{\rm d}t} = -\frac{{\rm d}(a^3\rho)}{{\rm d}(a^3)} \equiv -\frac{{\rm d}E}{{\rm d}V} \ . \tag{9.42}$$

If pressure is zero, the energy E contained in a volume V (if we define it as the energy density integrated over the volume) remains constant as the universe expands or contracts. If pressure is positive, energy decreases with the expansion of the universe, and increases if the universe contracts. If pressure is negative, the opposite happens: energy increases with expansion, and decreases with contraction.

Out of the three equations (9.39), (9.40) and (9.41), one is redundant. If we drop (9.39), we lose information of the value of K, so usually the set (9.39) and (9.41) is the most convenient choice, and it involves only first derivative of a. We have 2 equations and 3 unknowns: a, ρ and P. In order to get a solution, we need to specify one function. What's missing physically is that we haven't stated what kind of matter we have. Often this is specified by giving the **equation of state** $w(t) \equiv P(t)/\rho(t)$.

Before looking at the solutions with a given equation of state in detail, let us make a few general observations about the Friedmann equations. First we can note that in general, space will expand or contract. More precisely, we can say that if $\Lambda \leq 0$, $\rho > 0$, $P \geq 0$, then $\dot{a} \neq 0$. If we have $\Lambda > 0$ and $\rho > 0$, we can have a static solution if $K > 0$. This requires balancing the energy density and the cosmological constant precisely. Such a solution in the case $P = 0$ is known as the **Einstein static universe** or the **Einstein universe**. In 1917 Einstein introduced the cosmological constant to GR to obtain this solution, convinced that space has to be static. However, this solution is unstable. (**Exercise:** Show this.) In 1927-29 Lemaître theoretically showed and Hubble observed that space does expand, and this quickly became the prevalent view (although Hubble remained sceptical until his death in 1953).

Let us now consider the spatially flat case (which is the most relevant for our universe) and put $\Lambda = 0$. (The cosmological constant can be reintroduced as a matter component with $w = -1$.) Assume that the equation of state w is constant. The continuity equation (9.41) reads

$$0 = \dot{\rho} + 3(1+w)H\rho \,, \tag{9.43}$$

which integrates to

$$\rho \propto a^{-3(1+w)} \,. \tag{9.44}$$

The Friedmann equation (9.39) now gives

$$\frac{\dot{a}^2}{a^2} \propto a^{-3(1+w)} \,, \tag{9.45}$$

which integrates to (assuming $w > -1$ and $\dot{a} > 0$)

$$a \propto (t - t_i)^{\frac{2}{3(1+\omega)}} \,. \tag{9.46}$$

At a finite time in the past, the scale factor becomes zero; without loss of generality, we choose the origin of the time coordinate to be there, $t_i = 0$. At this time the energy density is correspondingly infinite, and is the spacetime curvature. This singularity is called the **big bang**, and it is a general feature not only of FLRW models but of realistic cosmological models that include inhomogeneities. (Although there are exceptions.) So GR indicates that the age of the universe is finite (to the past). The time-reversed case is also a solution: such a universe has contracted forever, and will reach a **big crunch** singularity a finite time into the future.

The simplest possibility for matter is dust, $w = 0$. (Confusingly, in cosmology this form of matter is also called **matter**.) Physically, it corresponds to the situation when all non-gravitational interactions are negligible, such as a gas of non-interacting particles with masses much larger than their kinetic energies. The relation (9.46) shows that $\rho \propto a^{-3}$.

Another physically relevant case is $w = \frac{1}{3}$, called **radiation**. It describes a gas of particles with kinetic energies much larger than their masses. The relation (9.46) shows that $\rho \propto a^{-4}$. The energy density goes down one extra factor of a compared to dust because the energy of massless (or ultrarelativistic) particles redshifts with expansion.

The case $w = -1$ corresponds to **vacuum energy**. We see from (9.46) that $\rho =$ constant: every unit volume has the same constant amount of energy. Mathematically, this form of matter is equivalent to the cosmological constant, and the terms are often used interchangeably, although one is a modification of gravity and the other a form of matter. It leads to exponential expansion, $a \propto e^{Ht}$, where H is constant.

If we have more than one component, the energy densities add,

$$\rho = \rho_{\mathrm{r}0} a^{-4} + \rho_{\mathrm{m}0} a^{-3} + \rho_{\mathrm{vac}} \ . \tag{9.47}$$

The expansion of the universe is well described by the **ΛCDM model**, where the universe is spatially flat and contains dust, radiation, and vacuum energy. The letters CDM are an acronym for cold dark matter, which constitutes 84% of the dust, the rest being nuclei and electrons. This model is a good fit to the data if the Hubble constant (the current value of the Hubble parameter $H(t)$) is $H_0 = 67$ km/s/Mpc and the total energy density is 32% matter and 68% vacuum energy at present, $\Omega_{m0} \equiv \rho_{\mathrm{m}}(t_0)/\rho(t_0) = 0.32$ and $\Omega_{\Lambda 0} \equiv \rho_{\mathrm{vac}}(t_0)/\rho(t_0) = 0.68$.[6] (Radiation today is negligible, $\Omega_{\mathrm{r}0} = 0.5 \times 10^{-4}$.)

Exercise.

a) Find the age of the universe t_0. (Hint: Use the substitution $x^{3/2} = b \sinh\phi$ in the integral $\int \frac{x^{1/2}dx}{\sqrt{b^2+x^3}}$.)

b) At what time t_Λ were the matter and vacuum energy densities equal?

c) At present the expansion is accelerating, $\ddot{a} > 0$. When did the acceleration begin ($\ddot{a} = 0$), in redshift and in time?

9.2.3 Penrose diagram

Let us illustrate the causal structure of FLRW spacetimes with Penrose diagrams. Because of symmetry, null geodesics are straight lines in space, so we can always choose θ and φ so that light travels in the radial direction. When using conformal time and angular coordinates as in (9.31), light then travels at 45°, so we just need to figure out the range of the coordinates and compactify if necessary. We consider here only the spatially flat case, so $0 < \chi < \infty$.

Consider then the conformal time. We restrict to a constant equation of state w, although the case of time-varying w is similar if w does not cross $-\frac{1}{3}$. For constant $w > -1$, we have

$$\begin{aligned} \eta &= \int^t \frac{\mathrm{d}t'}{a(t')} \\ &\propto \int^t \mathrm{d}t' t'^{-\frac{2}{3(1+w)}} \\ &\propto (1+3w) t^{\frac{1+3w}{3(1+w)}} \ , \end{aligned} \tag{9.48}$$

[6] The precise value of the Hubble parameter is currently a point of contention, with different observations interpreted in the context of the ΛCDM model leading to incompatible results.

where on the second line we have used (9.46), the dropped constants of proportionality are positive, and we have assumed $w \neq -\frac{1}{3}$.[7] If $1+3w>0$, conformal time has the range $0<\eta<\infty$, where $\eta=0$ corresponds to the big bang. In the opposite case $1+3w<0$, we have $-\infty<\eta<0$, where $\eta=-\infty$ corresponds to the big bang. Let us assume that $1+3w>0$. The second Friedmann equation (9.40) shows that this corresponds to expansion that always decelerates.

We want to compactify both the space and the time coordinate. We do this the same way as in the Schwarzschild case, by rotating to null coordinates, bringing infinity to a finite range with arctan, and then rotating back to a space and time coordinate. We define the new dimensionless null variables

$$\begin{aligned} u &\equiv (\eta-\chi)/\eta_0 \\ v &\equiv (\eta+\chi)/\eta_0 \ , \end{aligned} \tag{9.49}$$

where $\eta_0>0$ is a constant with the dimension of time, whose value is not important. These coordinates have the range $-\infty<u<\infty$, $0<v<\infty$. The inverse transformation is

$$\begin{aligned} \eta &= \frac{1}{2}(v+u)\eta_0 \\ \chi &= \frac{1}{2}(v-u)\eta_0 \ . \end{aligned} \tag{9.50}$$

For constant angular coordinates (θ,φ), the FLRW metric (9.31) reduces to

$$\mathrm{d}s^2 = -a^2\eta_0^2\mathrm{d}u\mathrm{d}v \ . \tag{9.51}$$

We now make the range of coordinates finite with the transformation

$$\begin{aligned} U &\equiv \arctan u \\ V &\equiv \arctan v \ . \end{aligned} \tag{9.52}$$

The range of these new coordinates is $-\frac{\pi}{2}<U<\frac{\pi}{2}$, $0<V<\frac{\pi}{2}$. The inverse transformation is

$$\begin{aligned} u &= \tan U \\ v &= \tan V \ , \end{aligned} \tag{9.53}$$

so $\mathrm{d}u=\cos^{-2}U\mathrm{d}U$, $\mathrm{d}v=\cos^{-2}V\mathrm{d}V$, and we have

$$\mathrm{d}s^2 = -\frac{a^2\eta_0^2}{\cos^2 U\cos^2 V}\mathrm{d}U\mathrm{d}V \ . \tag{9.54}$$

When considering Penrose diagram and the causal properties of spacetime, the only important point about the conformal prefactor in the metric is whether it vanishes or diverges, as this restricts the range of validity of the coordinates. For the ranges

[7] If $w=-\frac{1}{3}$, then $a\propto t$, and spacetime is flat: it is just Minkowski space in expanding coordinates.

of U and V we have, $\cos U$ and $\cos V$ are always positive, so there are no constraints from the prefactor. Finally, let us rotate back to a time and a space coordinate,

$$\begin{aligned} T &\equiv V + U = \arctan\frac{\eta-\chi}{\eta_0} + \arctan\frac{\eta+\chi}{\eta_0} \\ R &\equiv V - U = \arctan\frac{\eta+\chi}{\eta_0} - \arctan\frac{\eta-\chi}{\eta_0}\ , \end{aligned} \tag{9.55}$$

with the inverse

$$\begin{aligned} U &= \frac{1}{2}(T-R) \\ V &= \frac{1}{2}(T+R)\ , \end{aligned} \tag{9.56}$$

where we have used (9.50) and (9.52). The metric is

$$\mathrm{d}s^2 = \frac{a^2\eta_0^2}{4\cos^2 U\cos^2 V}(-\mathrm{d}T^2 + \mathrm{d}R^2)\ , \tag{9.57}$$

with U and V given by (9.56). Let us see what is the range of coordinates on the RT-plane.

From (9.55) we see that the big bang singularity $\eta = 0$ corresponds to $T = 0$, $0 \leq R < \pi$. The infinite future $\eta = \infty$ is mapped to $T = \pi$, $R = 0$. In Minkowski space $\eta = t$ goes from $-\infty$ to ∞, and past infinity is mapped to $T = -\pi$, $R = 0$. The spatial origin $\chi = 0$ maps to $0 \leq T < \pi$, $R = 0$, and spatial infinity $\chi = \infty$ maps to $T = 0$, $R = \pi$. These ranges are the same in the FLRW and the Minkowski case. Finally, the condition $0 < V < \frac{\pi}{2}$ corresponds to $T + R < \pi$.

The Penrose diagram is shown in figure 1. The upper triangle is the FLRW universe. All timelike lines and null lines reach the big bang singularity when extended backwards. At a finite time after the big bang, every observer can have communicated to only a finite spatial range. For example, if you consider two timelike lines of constant r, then there is always a value of the time coordinate so small that their causal triangles do not overlap. The younger the universe, the less time the signals have had to travel. However, towards the future all observers can eventually send signals to each other. (The causal properties of the negatively curved FLRW model are identical, while the positively curved case is different due to finite size.) In the case of Minkowski space, the lower triangle is also present: there is no big bang singularity. All observers can always have sent signals to each other, as the universe is infinitely old. If the expansion starts to accelerate at some time, the causal structure in the future changes, but the causal structure in the past is unaffected. If the expansion has always accelerated, the causal structure is given by the lower triangle alone. Then the lower point of the triangle corresponds to the infinite past, and the line $T = 0$ to the infinite future. (**Exercise:** show this.) One example of accelerating expansion is provided by de Sitter space, to which we now turn.

9.3 de Sitter space

9.3.1 Hypersurface construction

Let us consider maximally symmetric spacetimes. The d-dimensional de Sitter space dS^d can be represented as a hypersurface in $d+1$-dimensional Minkowski space with

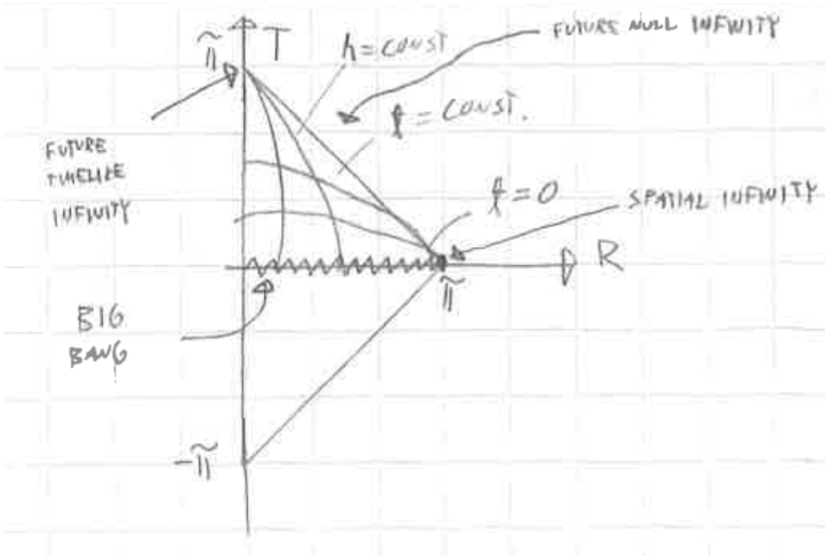

Figure 1: Penrose diagram for decelerating spatially flat FLRW universe and Minkowski space.

the metric (9.18) where the coordinates satisfy

$$-u^2 + \delta_{ij} x^i x^j = \alpha^2 \; . \tag{9.58}$$

The only difference between between $\mathbb{H}^d$ and dS^d is the sign of the α^2 term on the right-hand side of (9.19) and (9.58). For $d = 2$, we can draw both manifolds in the same picture, as shown in figure 2.

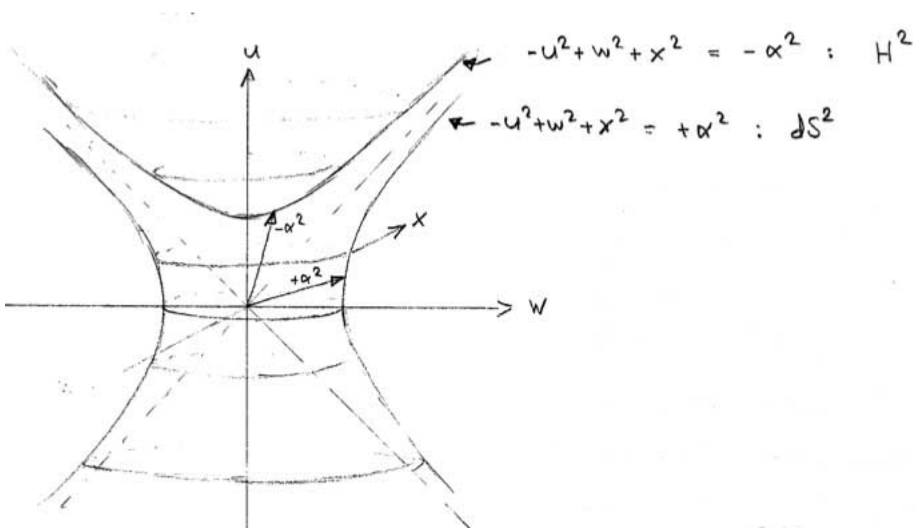

Figure 2: Relation between hyperbolic space and de Sitter space.

Let us now concentrate on de Sitter space, at first leaving two dimensions out. Two-dimensional de Sitter space is illustrated in figure 3. Each point in this figure corresponds to S^2 in the full four-dimensional de Sitter space.

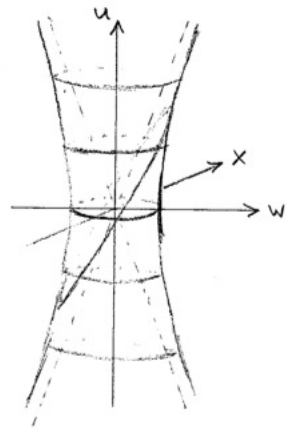

Figure 3: Two-dimensional de Sitter space.

Let us write the metric of the embedding space as

$$\mathrm{d}s^2_{2+1} = -\mathrm{d}u^2 + \mathrm{d}w^2 + \mathrm{d}x^2 \ , \tag{9.59}$$

where de Sitter space is the surface defined by

$$-u^2 + w^2 + x^2 = \alpha^2 \ . \tag{9.60}$$

We can parametrise this surface as (the parameter t has the range $-\infty < t < \infty$, while φ has the range $0 \leq \varphi < 2\pi$)

$$\begin{cases} u = \alpha \sinh \dfrac{t}{\alpha} \\ w = \alpha \cosh \dfrac{t}{\alpha} \cos\varphi \\ x = \alpha \cosh \dfrac{t}{\alpha} \sin\varphi \ , \end{cases} \tag{9.61}$$

so

$$\begin{cases} \mathrm{d}u = \cosh \dfrac{t}{\alpha} \mathrm{d}t \\ \mathrm{d}w = \sinh \dfrac{t}{\alpha} \cos\varphi \mathrm{d}t - \alpha \cosh \dfrac{t}{\alpha} \sin\varphi \mathrm{d}\varphi \\ \mathrm{d}x = \sinh \dfrac{t}{\alpha} \sin\varphi \mathrm{d}t + \alpha \cosh \dfrac{t}{\alpha} \cos\varphi \mathrm{d}\varphi \ . \end{cases} \tag{9.62}$$

Inserting (9.62) into the Minkowski metric (9.59) of the embedding space, we get the metric of two-dimensional de Sitter space:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \alpha^2 \cosh^2 \frac{t}{\alpha} \mathrm{d}\varphi^2 \ . \tag{9.63}$$

The metric (9.63) expresses algebraically what is depicted geometrically in figure 3. Two-dimensional de Sitter space is a stack of circles with radii ranging from α to infinity as $\alpha \cosh \frac{t}{\alpha}$. As the manifold is Lorentzian, there are spacelike, timelike and null directions, which the metric keeps track of, but the picture does not convey this information. But whether viewed in terms of the metric or the picture, the spacetime does not look maximally symmetric, as the throat seems special. However, this is an artifact of the embedding in three-dimensional Minkowski space. If we calculate the Riemann tensor, we see that it has the maximally symmetric form (9.10), with a constant K.

Adding the missing two dimensions, each point in figure 3 becomes a two-sphere, as noted. The embedding space is five-dimensional Minkowski space, with the metric

$$\mathrm{d}s^2_{4+1} = -\mathrm{d}u^2 + \mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ . \tag{9.64}$$

Now four-dimensional de Sitter space is the hypersurface defined by

$$-u^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 \ , \tag{9.65}$$

which we can parametrise with the coordinates (compare to the parametrisation (9.20) of three-dimensional hyperbolic space)

$$\begin{aligned} u &= \alpha \sinh \frac{t}{\alpha} \\ w &= \alpha \cosh \frac{t}{\alpha} \cos\chi \\ x &= \alpha \cosh \frac{t}{\alpha} \sin\chi \cos\theta \\ y &= \alpha \cosh \frac{t}{\alpha} \sin\chi \sin\theta \cos\varphi \\ z &= \alpha \cosh \frac{t}{\alpha} \sin\chi \sin\theta \sin\varphi \ . \end{aligned} \tag{9.66}$$

Taking the coordinate differentials and inserting them into (9.65), we get the metric of four-dimensional de Sitter space:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \alpha^2 \cosh^2 \frac{t}{\alpha} \underbrace{\Big[\mathrm{d}\chi^2 + \sin^2\chi \overbrace{(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)}^{\text{2-sphere } \mathrm{d}\Omega_2^2}\Big]}_{\text{3-sphere } \mathrm{d}\Omega_3^2} \ . \tag{9.67}$$

This metric represents a foliation of de Sitter space with three-spheres, i.e. with maximally symmetric spatial surfaces of constant positive curvature. We can also foliate de Sitter space with flat spacial slices.

9.3.2 Flat foliation

Let us again first look at two-dimensional de Sitter space before generalising to the four-dimensional case. We first write the condition (9.60) in terms of $(w+u, w-u)$ instead of (u, w):

$$(w+u)(w-u) = \alpha^2 - x^2 \quad \Rightarrow \quad w - u = \frac{\alpha^2 - x^2}{w+u} \tag{9.68}$$

Let us denote $\tilde{t} \equiv w + u$. At $\tilde{t} = 0$, the coordinate $w - u$ is not defined. The hypersurface $\tilde{t} = 0$ corresponds to two straight lines with $w = -u$ and $x = \pm\alpha$. The spacetime splits into two coordinate patches, with

1) $0 < \tilde{t} < \infty, \qquad -\infty < x < \infty$

2) $-\infty < \tilde{t} < 0, \qquad -\infty < x < \infty$,

neither of which covers the boundary $\tilde{t} = 0$. Solving for u and w in terms of $\tilde{t}$ and x, we have

$$\begin{aligned} w - u &= \frac{\alpha^2 - x^2}{\tilde{t}} \\ w + u &= \tilde{t} \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} w &= \frac{1}{2}\left(\tilde{t} + \frac{\alpha^2 - x^2}{\tilde{t}}\right) \\ u &= \frac{1}{2}\left(\tilde{t} - \frac{\alpha^2 - x^2}{\tilde{t}}\right) . \end{aligned} \tag{9.69}$$

Inserting (9.69) into the three-dimensional Minkowski metric (9.59), we get

$$ds^2 = -\frac{\alpha^2 - x^2}{\tilde{t}^2} d\tilde{t}^2 - \frac{2x}{\tilde{t}} d\tilde{t} dx + dx^2 . \tag{9.70}$$

We can get rid of the off-diagonal term and set $g_{00} = -1$ with the coordinate transformation (picking the patch with $\tilde{t} > 0$),

$$\hat{t} = \alpha \ln \frac{\tilde{t}}{\alpha} , \qquad \hat{x} = \frac{\alpha x}{\tilde{t}} , \tag{9.71}$$

which gives (**Exercise:** show this.)

$$ds^2 = -d\hat{t}^2 + e^{2H\hat{t}} d\hat{x}^2 . \tag{9.72}$$

where $H \equiv 1/\alpha$. The range of these coordinates, which cover only half of the spacetime, is $-\infty < \hat{t} < \infty$, $\quad -\infty < \hat{x} < \infty$, and the past coordinate infinity is the boundary to the other half of the spacetime.

The full four-dimensional case is very similar, we just add y and z and define

$$\hat{x} = \frac{\alpha x}{\tilde{t}} , \qquad \hat{y} = \frac{\alpha y}{\tilde{t}} , \qquad \hat{z} = \frac{\alpha z}{\tilde{t}} \tag{9.73}$$

to get the metric

$$ds^2 = -d\hat{t}^2 + a(\hat{t})^2 (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) , \tag{9.74}$$

where $a(\hat{t}) = e^{H\hat{t}}$. In these FLRW coordinates, the Friedmann equation (9.39) reads $3H^2 = \Lambda$, so $H = \sqrt{\Lambda/3}$, or $\alpha = \sqrt{3/\Lambda}$.

So half of de Sitter space looks like an FLRW universe that is eternal to the past and to the future and where the Hubble parameter is constant. If vacuum energy dominates the energy density of the universe towards the future, in the future the spacetime looks increasingly like part of de Sitter space.

Had we started with the FLRW coordinates (9.25) and looked for a maximally symmetric spacetime, we would have ended up with the metric (9.74), without it being apparent that it covers only half of the spacetime. As in the Schwarzschild case, we would have had to do coordinate transformations to get to the coordinates (9.67) that cover the whole spacetime.

In the coordinates (9.74) the six-dimensional symmetry under rotations and translations in $\mathbb{R}^3$ is evident, just as the hypersphere foliation (9.67) makes it clear that the spacetime has the (perhaps less familiar) six-parameter symmetry of S^3.

In neither of these representations is it obvious that there is a timelike Killing vector, because the metric depends on time. (Although in the coordinates (9.74) it is clear that the metric is invariant under a simultaneous time translation and constant dilation of the spatial coordinates.) However, already in chapter 5 we derived the metric for dS space in static coordinates where the time translation symmetry is evident.

9.3.3 Static coordinates

After deriving the Schwarzschild metric in chapter 5, we added the cosmological constant to get the Schwarzschild–de Sitter metric (5.27). If we put $r_\mathrm{s} = 0$, the black hole event horizon disappears, The remaining spacetime is maximally symmetric, and the Ricci tensor is $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$. The metric is

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda}{3} r^2\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{\Lambda}{3} r^2}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\big) \;, \tag{9.75}$$

where, as before, $\sqrt{\Lambda/3} = \alpha^{-1}$. Time translation invariance as well as symmetry under two-dimensional rotations is now evident, but not translation invariance. The radial coordinate has the range $0 < r < \sqrt{3/\Lambda}$. There is an event horizon at $r = \sqrt{3/\Lambda}$, for an observer sitting at $r = 0$. Unlike in the Schwarzschild case, the observer is enclosed by the event horizon.

Like the flat foliation, this coordinate system covers only part of the full de Sitter space. Rather than doing yet another coordinate transformation to show the relation between these coordinate systems, we will illustrate it with the Penrose diagram that will also make the causal structure of de Sitter space clear.

9.3.4 Penrose diagram

In the hypersphere foliation (9.67), the angular coordinates already have finite range. As in the Schwarzschild case, we can suppress two directions due to spherical symmetry: every point on the Penrose diagram corresponds to a two-sphere. The metric reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \alpha^2\cosh^2\frac{t}{\alpha}(\mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega^2) \;, \tag{9.76}$$

where $-\infty < t < \infty$, $0 < \chi < \pi$. So we just need to compactify the time coordinate and set $-g_{00} = g_{\chi\chi}$, so that light travels at 45° angles. These goals are achieved with the transformation

$$\cosh\frac{t}{\alpha} = \frac{1}{\cos t'} \;, \tag{9.77}$$

where the new coordinate t' has the range $-\pi/2 < t' < \pi/2$. Inserting (9.77) into (9.76), we obtain

$$\mathrm{d}s^2 = \frac{\alpha^2}{\cos^2 t'}(-\mathrm{d}t'^2 + \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega^2) \;. \tag{9.78}$$

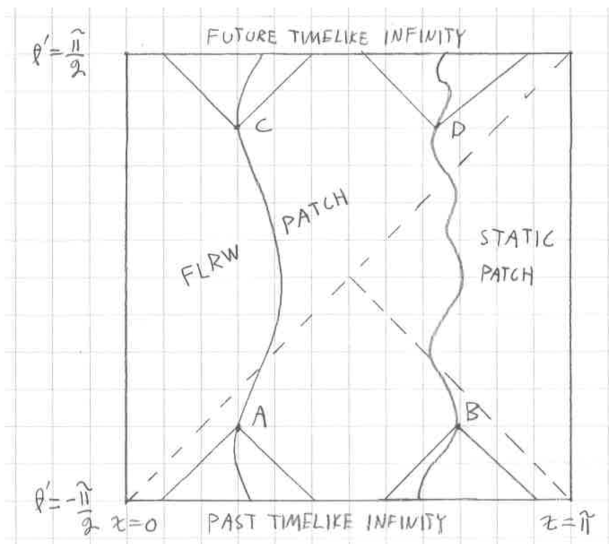

Figure 4: The Penrose diagram of de Sitter space. Each point represents a two-sphere, except those at the edges $\chi = 0, \pi$, where the radius of the two-sphere shrinks to zero. Observers A and B cannot have communicated in the past. Observers at C and D cannot communicate in the future.

The Penrose diagram is shown in figure 4. The FLRW patch covers one half of the spacetime, and corresponds to the upper left triangle. The static patch covers a quarter of the spacetime, and corresponds to the triangle on the right; note the resemblance to the Penrose diagram of region I of the Schwarzschild spacetime (the region outside the black hole covered by Schwarzschild coordinates). For every pair of observers separated by finite proper spacelike distance at some time t (or t'), there is a time t_1 before which they have not been able to send signals to each other, and a time $t_2 > t_1$ after which they cannot send signals to each other. Every observer has their own event horizon.

9.4 Anti-de Sitter space

9.4.1 Hypersurface construction

Let us now consider anti-de Sitter space, which is the maximally symmetric negatively curved spacetime (with exactly one time direction). We can embed d-dimensional AdS space into $d+1$-dimensional flat spacetime, but unlike in the previous cases, the embedding spacetime has to have two time directions:

$$ds^2 = -du^2 - dv^2 + \delta_{ij} dx^i dx^j \ , \tag{9.79}$$

where i and j range from 1 to $d-1$. Anti-de Sitter space is the hypersurface in this spacetime where

$$-u^2 - v^2 + \delta_{ij} x^i x^j = -\alpha^2 \ . \tag{9.80}$$

Let us first consider the case $d = 2$. The embedding space metric is

$$ds^2 = -du^2 - dv^2 + dx^2 \ , \tag{9.81}$$

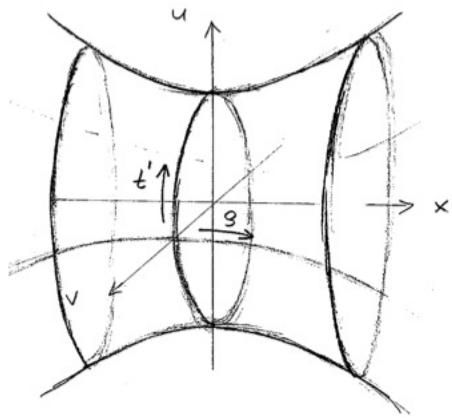

Figure 5: Two-dimensional anti-de Sitter space as a hypersurface.

and the hypersurface is given by

$$-u^2 - v^2 + x^2 = -\alpha^2 \ . \tag{9.82}$$

This is depicted in figure 5. As before, we parametrise the coordinates so that the hypersurface condition is satisfied:

$$\begin{aligned} u &= \alpha \sin t' \cosh \rho \\ v &= \alpha \cos t' \cosh \rho \\ x &= \alpha \sinh \rho \ . \end{aligned} \tag{9.83}$$

The range of the coordinates is $0 \leq t' < 2\pi$, $-\infty < \rho < \infty$. Inserting (9.83) into the metric (9.81), we get the metric of two-dimensional anti-de Sitter space:

$$\mathrm{d}s^2 = \alpha^2(-\cosh^2\rho \mathrm{d}t'^2 + \mathrm{d}\rho^2) \ . \tag{9.84}$$

In the case $d = 4$, we introduce two new angular coordinates, as before:

$$\begin{aligned} u &= \alpha \sin t' \cosh \rho \\ v &= \alpha \cos t' \cosh \rho \\ x &= \alpha \sinh \rho \cos \theta \\ y &= \alpha \sinh \rho \sin \theta \cos \varphi \\ z &= \alpha \sinh \rho \sin \theta \sin \varphi \ . \end{aligned} \tag{9.85}$$

Note that the values $(-\rho, \pi - \theta, \varphi + \pi)$ give the same coordinates (u, v, x, y, z) as the values (ρ, θ, φ). Inserting (9.85) into the five-dimensional embedding space metric

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - \mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{9.86}$$

gives the metric of four-dimensional AdS space:

$$\mathrm{d}s^2 = \alpha^2\Big[-\cosh^2\rho \mathrm{d}t'^2 + \underbrace{\mathrm{d}\rho^2 + \sinh^2\rho(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)}_{\mathbb{H}^3} \Big] \ . \tag{9.87}$$

Compared to the two-dimensional case, every pair of points (t', ρ), $(t', -\rho)$ is replaced with a two-sphere with radius $\alpha \sinh \rho$, and the range of ρ is restricted to $0 < \rho < \infty$. When we originally introduced the time coordinate t', it covered only the range $0 \le t' < 2\pi$. Because of this, we might think we should identify points with coordinates $(t', \rho, \theta, \varphi)$ and $(t'+2\pi, \rho, \theta, \varphi)$. However, there is no longer any need for this. The original coordinates cover only part of AdS: we can now let all values of t' from $-\infty$ to ∞ correspond to different physical points.[8] This gives the full AdS space.

9.4.2 Hyperbolic foliation

The metric (9.87) involves the hyperbolic space $\mathbb{H}^3$, but g_{00} depends on ρ, so the coordinate time runs at different rates at different points on $\mathbb{H}^3$. We can eliminate this feature by going to new coordinates (t, χ) (finding the coordinate transformation is left as an exercise) in terms of which the metric reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \alpha^2 \cos^2 \frac{t}{\alpha} \Big[\mathrm{d}\chi^2 + \sinh^2 \chi (\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\varphi^2) \Big] \ . \tag{9.88}$$

This metric now has the form $\mathbb{H}^3$ times a line segment. This is a FLRW universe with positively curved spatial sections. There is an apparent big bang at $t = -\frac{\pi}{2}\alpha$ and an apparent big crunch at $t = \frac{\pi}{2}\alpha$. As we know, the spacetime curvature is constant everywhere, and these are just coordinate singularities: spacetime continues smoothly across them.

9.4.3 Static coordinates

We can write the metric in the same kind of static coordinates as we did in the de Sitter case:

$$\mathrm{d}s^2 = -\left(1 + \frac{|\Lambda|}{3} r^2\right) \mathrm{d}t^2 + \frac{1}{1 + \frac{|\Lambda|}{3} r^2} \mathrm{d}r^2 + r^2 \big(\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\varphi^2\big) \ , \tag{9.89}$$

where $\alpha = \sqrt{3/|\Lambda|}$. The radial coordinate has the range $0 < r < \infty$, and there is no event horizon. This is an indication of the different causality properties of AdS space compared to de Sitter space.

If we include the mass M, we have the **AdS–Schwarzschild metric** derived in section 5.1.2:

$$\mathrm{d}s^2 = -\left(1 - \frac{r_\mathrm{s}}{r} + \frac{r^2}{\alpha^2}\right) \mathrm{d}t^2 + \frac{1}{1 - \frac{r_\mathrm{s}}{r} + \frac{r^2}{\alpha^2}} \mathrm{d}r^2 + r^2 \big(\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\varphi^2\big) \ , \tag{9.90}$$

where $r_\mathrm{s} = 2G_\mathrm{N} M$ as usual. This metric is important in the **AdS/CFT conjecture**, according to which a gravitational theory in AdS space (or variations thereof, like

[8] You may ask why we didn't do this for the de Sitter space coordinate φ. Indeed, in the two-dimensional case (9.63) you can unroll the spacetime like this. However, this doesn't work for the four-dimensional metric (9.67) because the points at $\theta = 0$ will not be different from each other (on the north pole, the only direction is south). This corresponds to the fact that you can cut two circles and join them smoothly, but it is not possible to cut two spheres and join them smoothly.

the AdS–Schwarzschild space) and a quantum field theory living on the boundary of that space are physically equivalent, i.e. **dual** to each other. The conjecture has become a key element in the study of quantum gravity and its applications. In that context, it is common to use the **Poincaré coordinates**.

9.4.4 Poincaré coordinates

With another change of coordinates, we can write the AdS metric in Poincaré coordinates:

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{\tilde{r}^2}{\alpha^2}\mathrm{d}\tilde{t}^2 + \frac{\alpha^2}{\tilde{r}^2}\mathrm{d}\tilde{r}^2 + \frac{\tilde{r}^2}{\alpha^2}(\mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2) \\ &= \frac{\alpha^2}{R^2}\left(\mathrm{d}R^2 - \mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2\right) \ , \end{aligned} \tag{9.91}$$

where $0 < r < \infty$, and on the second line we have written $R \equiv \alpha^2/\tilde{r}$. In these coordinates, the metric is conformal to the Minkowski metric, and the boundary at $r = \infty$ is moved to a finite coordinate value, corresponding to $R = 0$.

These coordinates make transparent the six-dimensional Poincaré symmetry group that involves the $2+1$-dimensional submanifold spanned by the coordinates $(\tilde{t}, \tilde{x}, \tilde{y})$, which is a subgroup of the full 10-dimensional AdS symmetry group. The Poincaré coordinates do not cover the entire manifold. Let us make the global structure of AdS explicit with a Penrose diagram.

9.4.5 Penrose diagram

To build the Penrose diagram, we start from the coordinates (9.87) that cover the whole AdS space. The angular directions (φ, θ) are trivial, but we need to compactify t' and ρ, which have the ranges $-\infty < t' < \infty$ and $0 < \rho < \infty$. It turns out that it is not possible to simultaneously compactify both coordinates and have light travel at 45°. We can compactify the spatial coordinate and have light travel at 45°, but the range of the time coordinate remains infinite. We define the new radial coordinate ρ' by

$$\cosh\rho = \frac{1}{\cos\rho'} \ , \tag{9.92}$$

so it has the range $0 < \rho' < \pi/2$. Inserting this new coordinate into the metric (9.87), we get

$$\mathrm{d}s^2 = \frac{\alpha^2}{\cos^2\rho'}\Big[-\mathrm{d}t'^2 + \mathrm{d}\rho'^2 + \sin^2\rho'(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\Big] \ . \tag{9.93}$$

The Penrose diagram is shown in figure 6. Each point (t', ρ') represents a two-sphere, except points on the boundary $\rho' = 0$, which represent a point (the origin of the spherical coordinates), and the points at $\rho' = \pi/2$, which represent spatial infinity.

The causal structure of AdS is very different from that of de Sitter space. Every observer can always send and receive signals to any other observer and to and from spatial infinity.

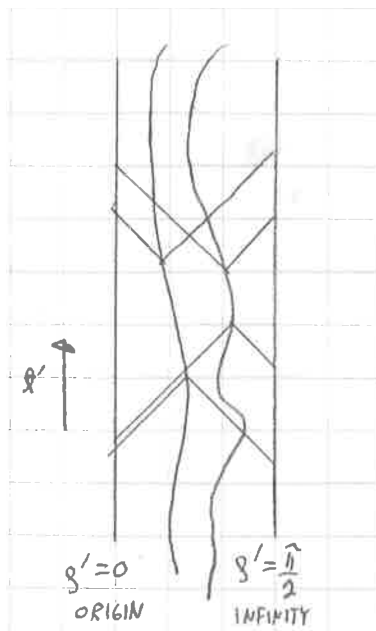

Figure 6: The Penrose diagram of AdS space, with the past and future lightcones of two observers.