Contents

6.1 Action formulation

6.1.1 Action for a scalar field

We now turn to the action formulation of GR. We will derive the Einstein equation from an action, and turn some of our previous assumptions into results. The action formulation is more transparent and involves less assumptions than giving the equation of motion directly. The action is also useful for building a theory of quantum gravity (so far attained only in a limited perturbative way in cosmic inflation). As a warmup exercise, let us consider deriving the equation of motion for a scalar field φ from an action.

In classical mechanics, the degrees of freedom are point particles, and the solution for the system consists of their trajectories as a function of time. In field theory, the variables are fields and the solution for the system consists of the values of the fields everywhere in space as a function of time. When varying the action of a point particle, we keep the position of the particle at the endpoints of the path fixed, and vary the trajectory in between. For a field, we keep the action fixed at the boundary of the spacetime region we consider and vary the field inside. By the principle of least action, the physical field configuration is the one that extremises the action. The action for a scalar field is

$$S = \int \mathrm{d}^4x \sqrt{-g} \mathcal{L}(\varphi, \partial_\alpha \varphi) \ , \tag{6.1}$$

where $\mathcal{L}$ is the **Lagrangian density** (also called the Lagrangian for short). The theory is defined by specifying the scalar function $\mathcal{L}$.

We repeat the steps we followed for the point particle in section 3.2.4. We vary the field as (the metric is an independent field, and so is kept fixed when varying φ)

$$\varphi(x) \to \varphi(x) + \delta\varphi(x) \ , \tag{6.2}$$

where $\delta\varphi(x)$ is an arbitrary infinitesimal function that vanishes on the boundary of the region we consider. The derivative of the field correspondingly changes as $\partial_\alpha\varphi \to \partial_\alpha\varphi + \partial_\alpha \delta\varphi$. The action changes as

$$\begin{aligned} \delta S &= \int_V \mathrm{d}^4x \sqrt{-g} \delta\mathcal{L} \\ &= \int_V \mathrm{d}^4x \sqrt{-g} \left[\frac{\partial\mathcal{L}}{\partial\varphi} \delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\varphi)} \partial_\alpha \delta\varphi \right] \ , \end{aligned} \tag{6.3}$$

where V is the four-volume over which we consider the field. As in the point particle case, we use partial integration to shift from $\partial_\alpha \delta\varphi$ to $\delta\varphi$. There is a subtlety here, related to partial versus covariant derivatives. We replace partial derivatives of the scalar field and its variation in the action with covariant derivatives (they are of course the same thing when operating on a scalar) and write

$$\begin{aligned}
\delta S &= \int_V \mathrm{d}^4x\sqrt{-g}\left[\frac{\partial\mathcal{L}}{\partial\varphi}\,\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)}\nabla_\alpha\,\delta\varphi\right] \\
&= \int_V \mathrm{d}^4x\sqrt{-g}\left\{\frac{\partial\mathcal{L}}{\partial\varphi}\,\delta\varphi - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)}\,\delta\varphi + \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)}\,\delta\varphi\right]\right\} \\
&= \int_V \mathrm{d}^4x\sqrt{-g}\left[\frac{\partial\mathcal{L}}{\partial\varphi} - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)}\right]\delta\varphi + \int_{\partial V}\mathrm{d}^3x\sqrt{|\gamma|}n_\alpha\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)}\delta\varphi\ , \quad (6.4)
\end{aligned}$$

where γ is the determinant of the induced metric on the boundary, and n_α is the unit vector normal to the boundary; ∂V has timelike as well as spacelike parts. Whereas on the first line the covariant derivatives in the second term could be replaced with partial derivatives (as they act on scalars), this is not the case for the individual terms on the second line. There the covariant derivatives act on vectors. The connection coefficients involved in the second and third term cancel so that the action as a whole does not depend on the connection, but this is not true for the individual terms. At this stage, we could use any connection. However, on the third line we have used Stokes' theorem to convert the total derivative into a boundary term. This is only possible if the connection is the Levi–Civita connection. So the fact that the variation gives the equation of motion (requiring us to get rid of $\partial_\alpha\delta\varphi$) fixes the connection to be the Levi–Civita connection. Even if we used some more general connection on the manifold, the connection in the scalar field equation of motion would be the Levi–Civita connection. (This argument is independent of the feature that demanding that straight paths give an extremum of the distance leads to the Levi–Civita connection.)

Taking into account that $\delta\varphi = 0$ on ∂V and demanding $\delta S = 0$, we get the scalar field equation of motion

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)} = 0\ . \quad (6.5)$$

We adopt the simplest Lagrangian density for a scalar field with self-interactions,

$$\mathcal{L} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V(\varphi)\ . \quad (6.6)$$

We thus get from (6.5)

$$\Box\varphi = V'(\varphi)\ , \quad (6.7)$$

where $\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ and $V' \equiv \frac{\mathrm{d}V}{\mathrm{d}\varphi}$.[1] Using the "minimal coupling principle" (which is more a prescription than a principle) to go from the equation of motion for a scalar

[1] Note the analogy between the value of the scalar field in spacetime and the position of a Newtonian point particle in time. The Lagrange function for a point particle with unit mass in classical mechanics is $L = \frac{1}{2}\dot{x}^2 - V(x)$, where $V(x)$ is the potential energy, leading to the equation of motion $\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$.

field in SR would involve taking the equation $\eta^{\alpha\beta}\partial_\alpha\partial_\beta\varphi = V'$, written in Cartesian coordinates in Minkowski space, replacing the partial derivatives with covariant derivatives to go to general coordinates in Minkowski space, and keeping the same equation when going to GR. This would give (6.7). Although covariant derivatives do not commute, there is no ordering ambiguity, because they are contracted with the metric, which is symmetric.[2] For the electromagnetic field, the situation is different.

6.1.2 Action for the electromagnetic field

The dynamical variable for the electromagnetic field is the vector potential A^α, and its Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \ , \tag{6.8}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. We can replace these partial derivatives by covariant derivatives if we want, because contribution of the the connection coefficients cancels due to the antisymmetry (as long as the torsion is zero). This shows that $F_{\alpha\beta}F^{\alpha\beta}$ is a scalar, and independent of the connection. This Lagrangian density is the simplest possibility built from A_α that is invariant under the gauge transformation $A_\alpha \to A_\alpha + \partial_\alpha\sigma$. (The term $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ is a total derivative, so its variation does not contribute to the equation of motion.) Terms with a larger number of factors of $F_{\alpha\beta}$, such as $F_{\alpha\beta}F^{\beta\gamma}F_\gamma{}^\alpha$, have to be divided with a dimensionful parameter in order to be of dimension length^{-4} as required for the Lagrangian density. So $F_{\alpha\beta}F^{\alpha\beta}$ is also the unique term that is invariant under the gauge transformation and does not contain any dimensional parameters.

Including matter other than the electromagnetic field, the total action is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \mathcal{L}_\mathrm{m}(\psi, A^\alpha)\right] \ , \tag{6.9}$$

where the matter Lagrangian $\mathcal{L}_\mathrm{m}$ can depend on A^α as well as on other degrees of freedom, collectively denoted by ψ.

We now extremise this action with respect to the variation $A^\alpha \to A^\alpha + \delta A^\alpha$. Analogously to the scalar field case, the derivative changes as $\partial_\beta A^\alpha \to \partial_\beta A^\alpha + \partial_\beta\delta A^\alpha$. The connection in the equation of motion is fixed to be the Levi–Civita connection by demanding that we get a boundary term in the same way as in the scalar field case, and we get

$$0 \ = \ \frac{\partial\mathcal{L}}{\partial A^\mu} - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha A^\mu)} \ . \tag{6.10}$$

The action (6.9) now gives Maxwell equations in curved spacetime:

$$\nabla_\beta F^{\alpha\beta} = j^\alpha \ , \tag{6.11}$$

[2] If we include in the action a direct coupling between the Riemann tensor and the scalar field, for example $R\varphi^2$, the equation of motion will be different from the minimal coupling prescription result. Such couplings are expected to exist on theoretical grounds (they are generated by quantum corrections if we consider quantum fields in curved spacetime – we will stick to classical fields in this course), though at present there is no observational evidence for them.

where the four-current is defined as

$$j_\alpha \equiv \frac{\partial \mathcal{L}_\mathrm{m}}{\partial A^\alpha} \ . \tag{6.12}$$

Exercise: Derive (6.10) and (6.11).

Writing (6.11) in terms of A^α, we have

$$\begin{aligned} j^\alpha &= \nabla_\beta(\nabla^\alpha A^\beta - \nabla^\beta A^\alpha) \\ &= -\Box A^\alpha + \nabla^\beta \nabla^\alpha A_\beta \\ &= -\Box A^\alpha + \nabla^\alpha \nabla^\beta A_\beta - [\nabla^\alpha, \nabla^\beta] A_\beta \\ &= -\Box A^\alpha + \nabla^\alpha \nabla^\beta A_\beta - R_{\beta\delta}{}^{\alpha\beta} A^\delta \\ &= -\Box A^\alpha + \nabla^\alpha \nabla^\beta A_\beta + R^{\alpha\beta} A_\beta \ , \end{aligned} \tag{6.13}$$

where on the next-to-last line we have used the definition of the Riemann tensor (3.68). Imposing the covariant form of the Lorenz gauge condition, $\nabla_\alpha A^\alpha = 0$, we end up with the equation of motion of the electromagnetic field:

$$\Box A^\alpha = -j^\alpha + R^\alpha{}_\beta A^\beta \ . \tag{6.14}$$

Three observations are in order.

First, because of the term $R^\alpha{}_\beta A^\beta$, (6.14) is not what we would unambiguously get by writing the SR equation of motion in terms of covariant derivatives. The "minimal coupling principle" now has an ordering ambiguity, because the equation of motion involves unsymmetrised second covariant derivatives. This is reminiscent of the operator ordering problem in quantum mechanics: the term xp is equal to px in classical mechanics, but not in the quantum theory; here x is position and p is the corresponding momentum. In GR, there is no problem if we consider the action and it contains only first order derivatives, as then commutators of derivatives do not appear. As equations of motion are often higher order in derivatives than the action (as in the scalar case (6.5) and electromagnetic case (6.11)), going to the action helps resolve such ambiguities and show what is the correct equation of motion.

Second, the equation of motion (6.14) for the electromagnetic field in the Lorenz gauge does not satisfy the strong equivalence principle. If we choose locally inertial coordinates where the connection is zero, the equation of motion does not reduce to the SR equation of motion at a point, because of the presence of the Ricci tensor. Note that the form of the equation of motion (6.13) where the gauge condition is not imposed does satisfy the strong equivalence principle, as the connection coefficients cancel out. The Lorenz gauge condition satisfies the strong equivalence principle as well, but its derivative does not, and it is the derivative of the gauge condition that appears in (6.13). So the violation of the strong equivalence principle is subtle. The principle is violated for the wave equation, but not for the equation of motion where no gauge condition is imposed.

Third, it follows from (6.14) that light rays do not follow null geodesics. Null geodesics are only a first order approximation in the limit where the wavelength of light is much smaller than 1) the scale given by properly normalised components of the Riemann tensor (the spacetime curvature scale), and 2) the scale over which the amplitude of the wave changes (the curvature scale of the wavefront). This is called the **geometrical optics approximation**. In most applications, these conditions are well satisfied, and light travels very close to null geodesics.

Exercise. Show that in the geometrical optics approximation light travels on null geodesics. Hint: Consider an electromagnetic field of the local plane wave form $A_\alpha = \mathrm{Re}(a_\alpha e^{i\theta/\epsilon})$, where $a_\alpha(x)$ and $\theta(x)$ are the amplitude and the phase of the wave, respectively, and $\epsilon \ll 1$ is the ratio of the wavelength to all relevant lengths. The light tangent vector is $k_\alpha = \partial_\alpha\theta$. Consider the equation of motion (6.14) to leading order in ϵ, using the Lorenz gauge. This will give the null condition $k_\alpha k^\alpha = 0$, and its covariant derivative will give the geodesic equation.

6.1.3 Action for the metric

Let us now turn to formulating the action for gravity. We seem to be faced with a problem. The Lagrangian is a scalar, and the Einstein equation is second order in derivatives. Typically equations of motion are higher order in derivatives than the action, as we have seen for the scalar field and electromagnetic field above. However, the only scalar that can be built from the metric and its first derivatives is a constant (the contraction of the metric with itself). There is no tensor that can be built from the first derivatives of the metric alone (as they can be set to zero at a point by a coordinate transformation). The problem will only disappear if higher order derivatives for some reason do not contribute to the equations of motion, and this is indeed what happens in GR.

So we have to go to second derivatives of the metric. We already know that their tensorial representation is given by the Riemann tensor, from which we can form exactly one scalar that is linear in the second derivatives – the Ricci scalar R. It has dimension of length^{-2}, so to get a Lagrangian with dimension length^{-4}, we need to multiply it with $1/G_\mathrm{N}$. Including also the cosmological constant and matter, the action is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi G_\mathrm{N}}(R-2\Lambda) + \mathcal{L}_\mathrm{m}(\psi, g_{\alpha\beta})\right] \ , \tag{6.15}$$

where Λ is the cosmological constant and, as in the electromagnetic case, the matter action $\mathcal{L}_\mathrm{m}$ can depend on $g_{\alpha\beta}$, as well as on any non-gravitational degrees of freedom, collectively denoted by ψ. Here ψ can include scalar fields, electromagnetic fields, and other types of matter. The numerical factor in front of $1/G_\mathrm{N}$ is a matter of convention. The part of the action that includes only the Ricci scalar is called the **Einstein–Hilbert action**.

The field to be varied is the metric, but it is equivalent and more convenient to vary the inverse metric, $g^{\alpha\beta} \to g^{\alpha\beta} + \delta g^{\alpha\beta}$. Unlike in the case of the scalar field and the electromagnetic field, we now have to vary the volume element as well, because the metric changes. The variation of the volume element can be written as

$$\frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}} = \frac{\delta[-\det(g^{\mu\nu})]^{-1/2}}{\delta g^{\alpha\beta}} = -\frac{1}{2}[-\det(g^{\mu\nu})]^{-3/2}\frac{\delta[-\det(g^{\mu\nu})]}{\delta g^{\alpha\beta}} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta} \ , \tag{6.16}$$

where we have applied the result $\delta \det M = \det M \operatorname{tr}\big(M^{-1}\delta M\big)$ (easily derived from the determinant-trace formula) to the matrix $M_{\alpha\beta} = g^{\alpha\beta}$.

We are now ready to vary the action. Writing the Ricci scalar as $R = g^{\alpha\beta}R_{\alpha\beta}$, we have

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x \delta\sqrt{-g}\frac{1}{16\pi G_\mathrm{N}}(R - 2\Lambda) + \int \mathrm{d}^4x \delta(\sqrt{-g}\mathcal{L}_\mathrm{m}) \\
&\quad + \int \mathrm{d}^4x \sqrt{-g}\frac{1}{16\pi G_\mathrm{N}}(\delta g^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}) \\
&= \int \mathrm{d}^4x \sqrt{-g}\left[\frac{1}{16\pi G_\mathrm{N}}\left(-\frac{1}{2}R + \Lambda\right)g_{\alpha\beta} + \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\mathrm{m})}{\delta g^{\alpha\beta}}\right]\delta g^{\alpha\beta} \\
&\quad + \int \mathrm{d}^4x \sqrt{-g}\frac{1}{16\pi G_\mathrm{N}}(\delta g^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}) \\
&= \int \mathrm{d}^4x \sqrt{-g}\left[\frac{1}{16\pi G_\mathrm{N}}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta}\right) + \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\mathrm{m})}{\delta g^{\alpha\beta}}\right]\delta g^{\alpha\beta} \\
&\quad + \int \mathrm{d}^4x \sqrt{-g}\frac{1}{16\pi G_\mathrm{N}}g^{\alpha\beta}\delta R_{\alpha\beta} \ , \qquad (6.17)
\end{aligned}$$

where in the second equality we have used (6.16). Now we just have to show that the last line in (6.17) is zero. This is easiest to do by taking advantage of the fact that the Ricci tensor only depends on the metric via the connection. As the connection is Levi–Civita, the variation of the metric induces a change in the connection: $\Gamma^\gamma_{\alpha\beta} \to \Gamma^\gamma_{\alpha\beta} + \delta\Gamma^\gamma_{\alpha\beta}$, with $\delta\Gamma^\gamma_{\alpha\beta} = \frac{\delta\Gamma^\gamma_{\alpha\beta}}{\delta g_{\mu\nu}}\delta g_{\mu\nu}$. Let us see how the Riemann tensor changes under the variation of the connection:

$$\delta R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\delta\Gamma^\alpha_{\delta\beta} + \delta\Gamma^\alpha_{\gamma\mu}\Gamma^\mu_{\delta\beta} + \Gamma^\alpha_{\gamma\mu}\delta\Gamma^\mu_{\delta\beta} - (\gamma \leftrightarrow \delta) \ . \qquad (6.18)$$

Note that $\delta\Gamma^\gamma_{\alpha\beta}$ is the difference between two connections and thus a tensor. As the above expression is linear in $\delta\Gamma^\gamma_{\alpha\beta}$, we can write it in terms of the covariant derivative. Adding and subtracting the term $\Gamma^\mu_{\gamma\delta}\delta\Gamma^\alpha_{\mu\beta}$ to (6.18), we can collect the terms to get

$$\delta R^\alpha{}_{\beta\gamma\delta} = \nabla_\gamma\delta\Gamma^\alpha_{\delta\beta} - \nabla_\delta\delta\Gamma^\alpha_{\gamma\beta} \ , \qquad (6.19)$$

so the variation of the Ricci tensor is

$$\delta R_{\alpha\beta} = \nabla_\gamma\delta\Gamma^\gamma_{\alpha\beta} - \nabla_\beta\delta\Gamma^\gamma_{\gamma\alpha} \ . \qquad (6.20)$$

The last line of (6.17) can then be written as, dropping the constant coefficient and renaming some indices,

$$\begin{aligned}
&\int \mathrm{d}^4x \sqrt{-g}\nabla_\gamma(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta} - g^{\gamma\alpha}\delta\Gamma^\beta_{\beta\alpha}) \\
= &\int \mathrm{d}^3x \sqrt{|\gamma|}n_\gamma(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta} - g^{\gamma\alpha}\delta\Gamma^\beta_{\beta\alpha}) \ , \qquad (6.21)
\end{aligned}$$

where we have used Stokes' theorem. The integral is zero if $\delta\Gamma^\gamma_{\alpha\beta}$ vanishes on the boundary. There is now only a small obstruction to completing the derivation of the

Einstein equation: by the rules of the variational principle, $\delta\Gamma^{\gamma}_{\alpha\beta}$ cannot be taken to vanish on the boundary. The connection is determined by the first derivative of the metric (and the inverse metric), and the field and its derivative cannot be kept fixed on the boundary at the same time.

Consider a Newtonian point particle in 1+1 dimensions with an arbitrary potential. If we give the initial position, we can always find an initial velocity such that the particle ends up in a given final position on a trajectory that solves the equations of motion. So instead of giving the initial position and the initial velocity, we can give the initial and the final position. But if we also keep the initial velocity fixed, then in general there is no trajectory that solves the equation of motion that takes the particle to a given final position. In fact there will be no freedom left at all to choose the final point, because we have exhausted the degrees of freedom when giving both initial position and initial velocity. The same holds for the metric. We could argue that we didn't use the condition that the variation of the metric is zero on the boundary. What if we were to allow the metric to vary on the boundary, and instead keep the connection fixed? However, compare again to the Newtonian point particle. While we can fix both the initial and final position, we cannot in general fix both initial and final velocity. (For example, for a constant potential, the velocity is constant.) This is also the case for the metric.

There are two ways out. (Three, if you count ignoring the problem, historically perhaps the most popular option.) The first is to add a boundary term to the action to cancel the contribution of (6.21). This term is called the **York–Gibbons–Hawking boundary term**, after the people who added it to the action (for other purposes) in 1972 (York) and 1979 (Gibbons and Hawking):

$$S_{\rm YGH} = \frac{1}{8\pi G_{\rm N}} \int {\rm d}^3 y \sqrt{|h|} \epsilon K \ , \tag{6.22}$$

where $K = K^{\alpha}{}_{\alpha}$ is the **extrinsic curvature scalar**, the trace of the **extrinsic curvature tensor** $K_{\alpha\beta} \equiv h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} \nabla_{\nu} n_{\mu}$. The extrinsic curvature tensor measures the curvature of the embedding of a hypersurface, as opposed to the intrinsic curvature measured by the Riemann tensor. Here $h_{\alpha\beta} \equiv g_{\alpha\beta} - \epsilon n_{\alpha} n_{\beta}$ is the tensor that projects orthogonally to the integration surface, which has the normal vector n^{α}, $\epsilon \equiv n^{\alpha} n_{\alpha} = \pm 1$, and h is the determinant of $h_{\alpha\beta}$ restricted to the integration surface. This boundary term has been important in attempts to find a quantum theory of gravity.

The other possibility is to take the connection to be an independent variable instead of assuming the Levi–Civita connection. We will discuss this second possibility below in section 6.1.5. For now, we just assume that the boundary term vanishes. The remaining terms of the variation (6.17) then give the Einstein equation,

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G_{\rm N} T_{\alpha\beta} \ , \tag{6.23}$$

where the energy-momentum tensor is

$$T_{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{\rm m})}{\delta g^{\alpha\beta}} = -2\frac{\delta \mathcal{L}_{\rm m}}{\delta g^{\alpha\beta}} + g_{\alpha\beta}\mathcal{L}_{\rm m} \ . \tag{6.24}$$

The equation of motion is not higher order than the action, because the variation of the derivative terms vanishes: the equation of motion comes completely from varying the metric.

Note the similarity of (6.24) to the charge current (6.12) in the electromagnetic case. In both cases, the source for the fields is given by varying the matter part of the action with respect to the field. This makes it transparent why the source for A^α is the rank 1 tensor j^α, and the source for the metric $g_{\alpha\beta}$ is the symmetric rank 2 tensor $T_{\alpha\beta}$. In the case of gravity we have the coefficient $8\pi G_\mathrm{N}$ multiplying the source term in the Einstein equation, but this is a matter of convention. On the one hand, for some unit conventions, there is also a constant factor in front of j^α in (6.11); on the other hand, in **Planck units** we have $8\pi G_\mathrm{N} = 1$.

Observe that this derivation from the action automatically gives us an energy-momentum tensor whose covariant divergence is zero: applying ∇^α to the left-hand side of (6.23) gives zero, so we must have $\nabla^\alpha T_{\alpha\beta} = 0$. In fact, this would hold even for a more complicated gravitational action. Let us see why.

6.1.4 Energy-momentum tensor from the action

The equation $\nabla_\alpha T^{\alpha\beta} = 0$ for the source of the gravitational field is similar to the relation $\nabla_\alpha j^\alpha = 0$ for the source of the electromagnetic field. (Although, as we have emphasised, the former relation does not give a conservation law in curved spacetime, because $\nabla_\alpha T^{\alpha\beta}$ is not a scalar and cannot be integrated.) Let us therefore first consider electromagnetism, which is simpler, and then apply the same kind of reasoning to the GR case.

We have noted already that because the electromagnetic field strength is antisymmetric, applying ∇_α to the Maxwell equation (6.11) immediately[3] gives $\nabla_\alpha j^\alpha = 0$, similarly to how $\nabla_\alpha T^{\alpha\beta} = 0$ follows from the Einstein equation (6.23). However, we can also deduce both relations directly from the action. As discussed in chapter 1, the field strength and therefore the physics is invariant under the gauge transformation $A_\alpha \to A_\alpha + \partial_\alpha \sigma$, where σ is an arbitrary function. The matter action must also satisfy this symmetry. Under an infinitesimal gauge transformation $A_\alpha \to A_\alpha + \partial_\alpha \delta\sigma$ such that $\delta\sigma$ vanishes on the boundary, the matter action changes as

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x \sqrt{-g} \frac{\partial \mathcal{L}_\mathrm{m}}{\partial A_\alpha} \delta A_\alpha \\
&= \int \mathrm{d}^4x \sqrt{-g} j^\alpha \partial_\alpha \delta\sigma \\
&= \int \mathrm{d}^4x \sqrt{-g} j^\alpha \nabla_\alpha \delta\sigma \\
&= -\int \mathrm{d}^4x \sqrt{-g} \nabla_\alpha j^\alpha \delta\sigma \ ,
\end{aligned} \tag{6.25}$$

where on the second line we have inserted the definition of the charge current (6.12), on the third line we have written the partial derivative as a covariant derivative

3 OK, there is the small detail of commuting the covariant derivatives. With the identity ${F^{\alpha\beta}}_{;\beta} = (\sqrt{-g})^{-1}(\sqrt{-g}F^{\alpha\beta})_{,\beta}$, valid for any antisymmetric tensor $F^{\alpha\beta}$, the result follows immediately. **Exercise:** Derive this identity.

(identical because they operate on a scalar) and on the third line done a partial integration. Because $\delta\sigma$ is arbitrary, we have $\nabla_\alpha j^\alpha = 0$ irrespective of the form of the purely electromagnetic part of the action. This implication that a symmetry of the action leads to a conserved current is an example of **Noether's theorem**.

In GR, the counterpart of gauge invariance is invariance under coordinate transformations. Consider the coordinate transformation $x^\alpha \to x'^\alpha(x) = x^\alpha + \xi^\alpha(x)$, with $|\xi^\alpha| \ll 1$. The Jacobian matrix is $M^\alpha{}_\beta = \delta^\alpha{}_\beta + \partial_\beta \xi^\alpha$, so to first order the inverse matrix is $(M^{-1})^\alpha{}_\beta \simeq \delta^\alpha{}_\beta - \partial_\beta \xi^\alpha$. The metric transforms as

$$\begin{aligned} g_{\alpha\beta}(x) \to g'_{\alpha\beta}[x(x')] &= (M^{-1})^\mu{}_\alpha (M^{-1})^\nu{}_\beta g_{\mu\nu}(x^\gamma) \\ &\simeq (\delta^\mu{}_\alpha - \partial_\alpha \xi^\mu)(\delta^\nu{}_\beta - \partial_\beta \xi^\nu) g_{\mu\nu}(x'^\gamma - \xi^\gamma) \\ &\simeq (\delta^\mu{}_\alpha - \partial_\alpha \xi^\mu)(\delta^\nu{}_\beta - \partial_\beta \xi^\nu)[g_{\mu\nu}(x') - \xi^\gamma \partial_\gamma g_{\mu\nu}] \\ &\simeq g_{\alpha\beta}(x') - (\partial_\alpha \xi^\mu g_{\mu\beta} + \partial_\beta \xi^\nu g_{\alpha\nu} + \xi^\gamma \partial_\gamma g_{\mu\nu}) \\ &= g_{\alpha\beta}(x') - 2\nabla_{(\alpha} \xi_{\beta)} \ , \end{aligned} \tag{6.26}$$

where we have expanded to first order in ξ^α, including in the argument of $g_{\alpha\beta}$. Correspondingly, $g^{\alpha\beta}(x) \simeq g^{\alpha\beta}(x') + 2\nabla^{(\alpha} \xi^{\beta)}$. The coordinate transformation does not affect the physics, it just changes the way we label points on the manifold. Under this relabeling, the matter action transforms as

$$\begin{aligned} \delta S_\mathrm{m} &= \int \mathrm{d}^4 x \frac{\delta(\sqrt{-g}\mathcal{L}_\mathrm{m})}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta} \\ &= -\int \mathrm{d}^4 x \sqrt{-g} T_{\alpha\beta} \nabla^\alpha \xi^\beta \\ &= \int \mathrm{d}^4 x \sqrt{-g} \nabla^\alpha T_{\alpha\beta} \xi^\beta \ , \end{aligned} \tag{6.27}$$

where on the second line we have inserted the definition of the energy-momentum tensor (6.24), and on the third line we have used partial integration and assumed that ξ^α vanishes on the boundary. The action is a scalar, so it is invariant under the coordinate transformation, $\delta S_\mathrm{m} = 0$. As ξ^α inside the volume is arbitrary, we have $\nabla^\alpha T_{\alpha\beta} = 0$.

So the result $\nabla_\alpha T^{\alpha\beta} = 0$ follows simply from the property that physics does not depend on how we label points on the manifold. Let us consider some implications of this result. The most obvious is that in flat spacetime it reduces (in Cartesian coordinates) to $\partial_\alpha T^{\alpha\beta} = 0$, which is equivalent to the conservation of energy and momentum, as we have seen. So energy and momentum conservation is a derived result in GR that follows from invariance under reparametrisations of the manifold and the vanishing of the Riemann tensor. The second condition is not necessary. In particular, if there exists a coordinate system where the metric is independent of time and $g_{0i} = 0$, i.e. the metric is **static** (we will later give a coordinate-independent characterisation of this property), then energy can be meaningfully defined and is conserved.

Let us assume that the energy-momentum tensor has the ideal fluid form (4.12). (It is straightforward to repeat the calculation for general matter, but this adds technical complications while doing little to improve understanding.) Projecting with u^α, we have

$$\begin{aligned} 0 &= u_\beta \nabla_\alpha T^{\alpha\beta} \\ &= u_\beta \nabla_\alpha [(\rho + P) u^\alpha u^\beta + g^{\alpha\beta} P] \\ &= -\dot{\rho} - \theta(\rho + P) , \end{aligned} \tag{6.28}$$

where dot denotes derivative in the time direction given by u^α, $\dot{\rho} \equiv u^\alpha \partial_\alpha \rho$, and we have used the normalisation condition $u^\alpha u_\alpha = -1$. Here $\theta \equiv \nabla_\alpha u^\alpha$ is the expansion rate of space. (The expansion rate is the trace of the extrinsic curvature we introduced when discussing the York–Gibbons–Hawking boundary term.)

The equation (6.28) shows that if the expansion rate of the spatial sections orthogonal to u^α is zero, the energy density is constant along the curve whose tangent vector is u^α (i.e. in the time direction given by u^α). If the spatial sections mesh together to form a three-dimensional submanifold[4], we can integrate ρ over the manifold and call this the energy. If the expansion rate is zero, the energy defined this way is conserved. However, it does not play all of the roles of energy in Newtonian mechanics. We saw an example of this in chapter 5, when we considered the Schwarzschild solution, where $\rho = 0$.

If we project with $h^{\gamma\beta} = g^{\gamma\beta} + u^\gamma u^\beta$ instead, we get (**Exercise:** Show this.)

$$a^\alpha = -\frac{1}{\rho + P} h^{\alpha\beta} \partial_\beta P , \tag{6.29}$$

where $a^\alpha = u^\beta \nabla_\beta u^\alpha$ is the acceleration. This is a generalisation of Newton's second law. If the pressure is zero (i.e. if the matter is dust), the acceleration is zero, and the observer moves on a geodesic. If the pressure is non-zero (and inhomogeneous), the corresponding force pushes the observers off the geodesic. So the assumption we have made earlier that observers move on geodesics can be dropped: it is a derived result that follows from 1) invariance under coordinate transformations and 2) the vanishing of the force. In practice, objects move close to geodesics if the contribution of the inhomogeneous pressure, energy flux and anisotropic stress is small.

Let us consider an example. The scalar field Lagrangian (6.6) and the definition (6.24) gives the energy-momentum tensor

$$T_{\alpha\beta} = \partial_\alpha \varphi \partial_\beta \varphi - g_{\alpha\beta} \left[\frac{1}{2} g^{\gamma\delta} \partial_\gamma \varphi \partial_\delta \varphi + V(\varphi) \right] . \tag{6.30}$$

Decomposing this in terms of a general velocity u^α, we have

$$\rho = u^\alpha u^\beta T_{\alpha\beta} = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\hat{\nabla}_\alpha \varphi \hat{\nabla}^\alpha \varphi + V \tag{6.31}$$

$$P = \frac{1}{3} h^{\alpha\beta} T_{\alpha\beta} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{6}\hat{\nabla}_\alpha \varphi \hat{\nabla}^\alpha \varphi - V \tag{6.32}$$

$$q_\alpha = -h_\alpha{}^\beta u^\gamma T_{\beta\gamma} = -\dot{\varphi} \hat{\nabla}_\alpha \varphi \tag{6.33}$$

$$\Pi_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu T_{\mu\nu} - \frac{1}{3} h^{\mu\nu} T_{\mu\nu} h_{\alpha\beta} = \hat{\nabla}_\alpha \varphi \hat{\nabla}_\beta \varphi - \frac{1}{3} h_{\alpha\beta} \hat{\nabla}_\mu \varphi \hat{\nabla}^\mu \varphi , \tag{6.34}$$

[4] According to **Frobenius' theorem**, this happens if and only if $h^{\alpha\mu} h^{\beta\nu} \nabla_{[\nu} u_{\mu]} = 0$, where $h^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$.

where we have denoted the time derivative as $\dot{\varphi} \equiv u^\alpha \partial_\alpha \varphi$ and the spatial derivative as $\hat{\nabla}_\alpha \varphi \equiv h_\alpha{}^\mu \partial_\mu \varphi$. The above holds for a general velocity u^α. If $\partial_\alpha \varphi$ is timelike, things look particularly simple if we use it as the time direction – i.e. define space as the hypersurface of constant φ. The normalised velocity is

$$u_\alpha = \frac{1}{\sqrt{-g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi}} \partial_\alpha \varphi \ . \tag{6.35}$$

We then have $\hat{\nabla}_\alpha \varphi = 0$, and in terms of this velocity field, the scalar field energy-momentum tensor has the ideal fluid form, with

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V \tag{6.36}$$

$$P = \frac{1}{2}\dot{\varphi}^2 - V \tag{6.37}$$

$$q_\alpha = 0 \tag{6.38}$$

$$\Pi_{\alpha\beta} = 0 \ . \tag{6.39}$$

Inserting (6.36) into the result (6.28) for $u_\alpha \nabla_\beta T^{\alpha\beta} = 0$, we get

$$\ddot{\varphi} + \theta\dot{\varphi} = -V' \ . \tag{6.40}$$

(**Exercise:** Show this.) This just is the general equation of motion (6.7) with a particular choice of decomposition for the energy-momentum tensor. Spatial derivatives do not appear, because we have chosen the time direction so that φ is constant for constant time.

Exercise: Derive the energy-momentum tensor of the electromagnetic field from the Lagrangian density (6.8) and write it in terms of the decomposition (4.7).

6.1.5 The Palatini formulation

With the action formulation, we have managed to get rid of some of the assumptions we made earlier. We had assumed that the gravitational properties of matter are described by a symmetric rank 2 tensor; that energy and momentum are conserved in the flat spacetime limit; that the conservation law written in terms of the energy-momentum tensor generalises to curved spacetime with the minimal coupling principle; that observers move on timelike geodesics; and that photons move on null geodesics. If we want to get the equations of motion for matter without the action, we also have to assume the minimal coupling principle. The action formulation gets rid of all of these assumptions, and turns them into results. (For light moving on null geodesics, we need extra conditions, as it is only approximately true.) The action formulation also gives the energy-momentum tensor directly from the matter Lagrangian. Furthermore, we have shown how the generalisation of Newton's second law follows from coordinate invariance. It is also easier to generalise the theory when starting from the action, because it is simpler to write down scalars than rank 2 tensors.

GR formulated in terms of tensors on a manifold and an action principle is strikingly simple in its assumptions. The manifold description in terms of differential geometry (on which we have tread rather lightly) is the natural language of GR. If we were to write GR as a theory for the components of the metric without using tensors, it would look horribly complicated. (Recall Maxwell did not originally write his equations in terms of vectors.) Likewise, describing Newtonian gravity in the language of a manifold with a metric (which we didn't do) leads to a convoluted construction, which is more complicated than having a gravitational force in flat spacetime; vector calculus is the natural language of Newtonian mechanics.

We can go further and let the variational principle turn one more assumption into a result: that the connection appearing in the Riemann tensor is the Levi–Civita connection. That assumption is part of the **metric formulation** of GR. We can instead take the connection to be an independent variable. This approach is called the **Palatini formulation**[5] of GR, or the **metric-affine formulation**. The construction is simple. We have the same action as before, but make no assumption about the connection:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{16\pi G_{\mathrm{N}}}[g^{\alpha\beta}R_{\alpha\beta}(\Gamma,\partial\Gamma) - 2\Lambda] + \mathcal{L}_{\mathrm{m}}(\psi, g_{\alpha\beta})\right\} \,, \tag{6.41}$$

where we have explicitly noted that $R_{\alpha\beta}$ depends on the connection, but not on the metric. Note that the matter action is assumed not to depend on the connection. Now varying the action with respect to $g^{\alpha\beta}$ gives the Einstein equation (6.23) easily: we only need to vary $\sqrt{-g}$ and $g^{\alpha\beta}$, the action does not contain any derivatives of the metric. In this sense the metric has no dynamics, it is an auxiliary variable. The Einstein equation is an algebraic equation for the components of the metric, not a differential equation. The complexity has been shifted to the connection, which is a dynamical variable. Varying the action with respect to $\Gamma^\gamma_{\alpha\beta}$ gives the connection in terms of the metric. It is a pleasant exercise to show that if we assume that the connection is symmetric or metric compatible, this gives the Levi–Civita connection.[6] Substituting the connection back to the Einstein equation then gives back the same differential equation for the metric as in the metric formulation of GR.

Exercise: Assuming that the connection is symmetric, $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{(\alpha\beta)}$, show that the equation $\frac{\delta S}{\delta\Gamma^\gamma_{\alpha\beta}} = 0$ for the action (6.41) is equivalent to $\nabla_\gamma g_{\alpha\beta} = 0$.

Since the connection is only fixed in terms of the metric by the equations of motion, they are independent in the action, so we can fix both $\delta g^{\alpha\beta} = 0$ and $\delta\Gamma^\gamma_{\alpha\beta} = 0$ on the boundary. There is thus no need to add the York–Gibbons–Hawking boundary term for the variational principle (although it can have other uses). Furthermore, the action involves only first derivatives.

In the Palatini formulation, the connection that appears in the equation of motion for the metric is thus fixed by the variational principle, just as the connection

[5] This formulation was introduced by Einstein. But it was Einstein who named it after Palatini, referring to a paper by Attilio Palatini which first used the trick (6.19) of writing the variation of the Riemann tensor in terms of the covariant derivative of the variation of the connection.

[6] If we leave both torsion and non-metricity free, the equations of motion leave a linear combination of them undetermined. However, this combination does not appear in the action nor in the equations of motion and thus has no physical consequence, so it can be considered to be a gauge degree of freedom.

that appears in the scalar field and electromagnetic field equations of motion is fixed by the variational principle. (The latter is true whether the formulation is metric or Palatini.) For the Einstein–Hilbert action for gravity plus a matter action that does not depend on the connection, the metric formulation and the Palatini formulation are physically equivalent (up to the York–Gibbons–Hawking boundary term). It is then a matter of taste whether one finds it simpler to assume the Levi–Civita connection from the beginning and add a boundary term or to increase the number of a priori independent variables.

If we generalise the theory by either complicating the gravitational action or by writing down a matter action that depends on the connection, the equations of motion for the connection do not, in general, give back the Levi–Civita connection. In particular, this is true if we couple a scalar field φ or the electromagnetic field directly to the connection in the action, for example with a term like $\varphi^2 R$. Then the metric formulation and the Palatini formulation are physically distinct theories of gravity. There are also other formulations of GR that are equivalent for the Einstein–Hilbert action and matter coupled to the metric only, but differ for more complicated cases. So far there is no evidence to observationally distinguish between these formulations, and the term GR is usually tacitly taken to refer to the metric formulation.