Contents

4 Gravitation

4.1 The Einstein equation

4.1.1 Geometrisation of Newtonian gravity

In the previous chapters, we set up the machinery to describe the curvature of manifolds and established how it determines the motion of matter. We are missing one piece: how is the curvature determined? We take the metric to be the fundamental variable that describes spacetime geometry, so we need an equation of motion for the metric, sourced by matter.

We want the equation of motion to be written in terms of tensors (i.e. defined on the manifold), and we want it to be second order. Equations higher than second order generally have violent instabilities, as the Hamiltonian will be linear instead of quadratic in one or more of the canonical momenta. There do exist stable higher derivative theories that extend GR, and they have been extensively studied. We will only consider GR, where the equation of motion is second order and linear in the second derivatives. It is then linear in the Riemann tensor and has the form (part of the Riemann tensor) = (matter source). The word "part" is key here. The entire Riemann tensor cannot be locally sourced by matter. One reason is phenomenological: for such an equation of motion, the curvature would vanish outside matter, and there would be no gravity in vacuum, so the theory does not describe our world. Another, more fundamental, reason is mathematical: the Riemann tensor has 20 independent components while the metric has 10. So only 10 components of the Riemann tensor can enter into the equation of motion of the metric, otherwise it will be overdetermined. (We will also have to factor in the feature that we have four coordinate degrees of freedom.)

To get an idea of which part of the Riemann tensor to pick and why, it is instructive to look at Newtonian gravity in terms of a manifold, in other words to geometrise Newtonian gravity. The Newtonian equation of motion for a particle under the influence of gravity alone is

$$0 = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \delta^{ij}\partial_j \phi \ , \tag{4.1}$$

where ϕ is the gravitational potential. In chapter 3 we used the Euler–Lagrange equations to obtain the connection coefficients by identifying the equation of motion of a free particle with the geodesic equation. We will do the same here in the Newtonian case, but now this is not just a calculational device, but a definition: we demand that (4.1) agrees with the geodesic equation

$$0 = \frac{\mathrm{d}^2 x^\gamma}{\mathrm{d}\tau^2} + \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \ . \tag{4.2}$$

Equation (4.1) does not have a zero component. However, if $\Gamma^0_{\alpha\beta} = 0$, then (4.2) is just $\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0$, which gives $t = A\tau + B$. This is just the statement that Newtonian proper time equals (up to a change of units and origin of time) coordinate time, i.e. time is absolute. The spatial components give $\Gamma^i_{j\alpha} = 0$ and

$$\Gamma^i_{00} = \delta^{ij} \phi_{,j} \ . \tag{4.3}$$

Inserting this into the definition of the Riemann tensor in (3.68), we find that its only non-zero component is

$$R^i{}_{0j0} = \delta^{ik} \phi_{,kj} \ . \tag{4.4}$$

Contracting the first and the third index, we see that the only non-zero component of the Ricci tensor is

$$R_{00} = \nabla^2 \phi \ . \tag{4.5}$$

As we noted in chapter 3, the Riemann tensor is defined in terms of the connection alone, the metric does not make an appearance. This is also true for the Ricci tensor. However, without a metric, we cannot raise or lower indices nor take traces of down-down or up-up indices, so the Ricci scalar and the Weyl tensor are not defined. Newtonian spacetime is an example of a manifold that has a connection but no metric. It is possible to introduce metric structure, but we need one metric with down indices and another with up indices, both being degenerate (i.e. the matrix formed by the components of the metric has zero determinant) and not the inverse of each other. Such a construction actually provides an elegant way to understand Newtonian gravity as the limit of GR where the metric becomes degenerate. But let us not continue in that direction, recalling that we are in the process of finding the equation of motion of GR.

In table 1 we compare various quantities in GR and Newtonian gravity. In Newtonian gravity, the gravitational field is described by the potential ϕ, which enters the particle equation of motion via the gradient $\partial_i \phi$. The equations of motion are second order in derivatives, so they involve $\partial_i \partial_j \phi$. This is a symmetric 3×3 tensor, and thus has 6 independent components. Since the gravitational potential involves only one field, its equation of motion can have only one component (or rather, the number of components minus constraints must be 1). In other words, an equation of the form $\partial_i \partial_j \phi = A_{ij}$ does not have a solution for general symmetric tensor $A_{ij} = A_{(ij)}$ that describes a matter source. It is the trace of the second derivative, $\nabla^2 \phi$, that is proportional to the matter source. The procedure is that we

	GR	Newtonian gravity
field	$g_{\alpha\beta}$	ϕ
appears in particle EOM	$\Gamma^{\gamma}_{\alpha\beta}$	$\partial_i\phi$
curvature	$R^{\alpha}{}_{\beta\gamma\delta}$	$\partial_i\partial_j\phi$
trace of the curvature	$R_{\alpha\beta}$	$\nabla^2\phi$
non-locally determined part of the curvature	$C_{\alpha\beta\gamma\delta}$	$\partial_i\partial_j\phi - \frac{1}{3}\delta_{ij}\nabla^2\phi$
particle EOM	$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} = 0$	$\ddot{x}^i + \delta^{ij}\partial_j\phi = 0$
field equation	$G_{\alpha\beta} = 8\pi G_{\mathrm{N}} T_{\alpha\beta}$	$\nabla^2\phi = 4\pi G_{\mathrm{N}}\rho_{\mathrm{m}}$

Table 1: Comparison of GR in arbitrary coordinates and Newtonian gravity in Cartesian coordinates. EOM stands for equation of motion.

first solve ϕ from the Poisson equation $\nabla^2\phi = 4\pi G_{\mathrm{N}}\rho_{\mathrm{m}}$, then calculate $\partial_i\phi$ which determines particle trajectories, and from that we get the traceless combination $\partial_i\partial_j\phi - \frac{1}{3}\delta_{ij}\nabla^2\phi$, which gives tidal effects.

We will want to use Newtonian gravity as a guide to finding the equation of motion of GR. We will later derive the equation of motion from an action principle à la Hilbert, bypassing the assumptions we make here, but it is useful to also understand this Einsteinian route to the equations of motion.

In GR, the gravitational field is described by the metric $g_{\alpha\beta}$, which has 10 independent components. We want an equation of motion that is linear in the Riemann tensor. It cannot involve all of the components of the Riemann tensor, i.e. it cannot be of the form $R_{\alpha\beta\gamma\delta} = A_{\alpha\beta\gamma\delta}$, where $A_{\alpha\beta\gamma\delta}$ is some tensor that describes matter. Instead, we need an equation with 10 components.

In the Newtonian case, the Poisson equation written in terms of the Ricci tensor reads $R_{00} = 4\pi G_{\mathrm{N}}\rho_{\mathrm{m}}$. As a first try, we could guess that the GR equation of motion would be $R_{\alpha\beta} = \kappa T_{\alpha\beta}$, where κ is a constant and $T_{\alpha\beta}$ is the **energy-momentum tensor** that describes matter. We would then solve for the metric, and take derivatives of it to find the Weyl tensor, which gives tidal effects, in analogy with the Newtonian case. This was Einstein's first attempt. It's wrong, for reasons that will soon become clear. However, the more general ansatz

$$\kappa T_{\alpha\beta} = R_{\alpha\beta} + A g_{\alpha\beta} R \ , \tag{4.6}$$

where A is a constant, works. To determine the value of A, we consider properties of the energy-momentum tensor.

4.1.2 The energy-momentum tensor

Given an arbitrary timelike unit vector field u^{α}, we can without loss of generality decompose the energy-momentum tensor as

$$T_{\alpha\beta} = \rho u_{\alpha} u_{\beta} + P h_{\alpha\beta} + 2 q_{(\alpha} u_{\beta)} + \Pi_{\alpha\beta} \ , \tag{4.7}$$

where $h_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha} u_{\beta}$ projects orthogonally to the time direction given by u^{α}, ρ is the **energy density**, P is the **pressure**, q_{α} is the **energy flux** or **momentum**

density, and $\Pi_{\alpha\beta}$ is the **anisotropic stress** or **anisotropic pressure**. Both q_α and $\Pi_{\alpha\beta}$ are orthogonal to u^α, $q_\alpha u^\alpha = 0$, $\Pi_{\alpha\beta}u^\beta = 0$, and $\Pi_{\alpha\beta}$ is symmetric and traceless, $\Pi_{\alpha\beta} = \Pi_{(\alpha\beta)}$, $g^{\alpha\beta}\Pi_{\alpha\beta} = 0$. Taking different projections of (4.7), we find that these quantities are given in terms of the energy-momentum tensor as

$$\rho = u^\alpha u^\beta T_{\alpha\beta} \tag{4.8}$$

$$P = \frac{1}{3}h^{\alpha\beta}T_{\alpha\beta} \tag{4.9}$$

$$q_\alpha = -h_\alpha{}^\beta u^\gamma T_{\beta\gamma} \tag{4.10}$$

$$\Pi_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu T_{\mu\nu} - \frac{1}{3}h^{\mu\nu}T_{\mu\nu}h_{\alpha\beta} \ . \tag{4.11}$$

Observers with different u^α have different decompositions of the energy-momentum tensor and so measure different values of the energy density and other quantities. (Compare to the fact that energy of a particle with momentum p^α as measured by an observer with four-velocity u^α is $E = -u_\alpha p^\alpha$.) The trace of the energy-momentum tensor is the same for all observers, $T^\alpha{}_\alpha = -\rho + 3P$. If $q_\alpha = 0, \Pi_{\alpha\beta} = 0$, we say that matter is an **ideal fluid**:

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + P h_{\alpha\beta} \ . \tag{4.12}$$

If also $P = 0$, we say that matter is **dust**. Two observations are in order. First, these conditions say nothing about the microscopic properties of matter, in particular it need not consist of a gas of particles in thermal equilibrium. (In chapter 6 we will see that the energy-momentum tensor of a scalar field has the ideal fluid form, for example.) Second, the ideal fluid (and dust) form is observer-dependent. We say that observers for which the energy-momentum tensor has the ideal fluid form are **comoving** with the fluid. Observers who move with respect to the comoving observers do not see the energy-momentum tensor as having the ideal fluid form. So, precisely speaking, matter is an ideal fluid if and only if there exists a timelike unit velocity field such that when the energy-momentum tensor is decomposed with respect to that field, $q_\alpha = 0, \Pi_{\alpha\beta} = 0$.

Consider the energy-momentum tensor in Minkowski space in Cartesian coordinates with the time direction taken to be the coordinate time, $u^\alpha = \delta^{\alpha 0}$. We then have $u_\alpha = -\delta_{\alpha 0}$ and $h_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta j}\delta_{ij}$, so the non-zero components of the decomposition quantities are

$$\rho = T_{00} \tag{4.13}$$

$$P = \frac{1}{3}T^i{}_i \tag{4.14}$$

$$q_i = -T_{0i} \tag{4.15}$$

$$\Pi_{ij} = T_{ij} - \frac{1}{3}T^k{}_k\delta_{ij} \ . \tag{4.16}$$

In matrix form we have

$$T_{\alpha\beta} = \begin{pmatrix} \rho & -q_1 & -q_2 & -q_3 \\ -q_1 & P + \Pi_{11} & \Pi_{12} & \Pi_{13} \\ -q_2 & \Pi_{12} & P + \Pi_{22} & \Pi_{23} \\ -q_3 & \Pi_{13} & \Pi_{23} & P + \Pi_{33} \end{pmatrix} \ , \tag{4.17}$$

with $\delta^{ij}\Pi_{ij} = 0$.

An important property of the energy-momentum tensor in Minkowski space is that (in Cartesian coordinates)

$$\partial_\alpha T^{\alpha\beta} = 0 \iff \text{energy and momentum are conserved} . \tag{4.18}$$

Let us prove (4.18). Assume that the left-hand side holds, take $\beta = 0$ and integrate over a fixed volume V

$$\begin{aligned} 0 &= \int_V \mathrm{d}^3x(\partial_0 T^{00} + \partial_i T^{0i}) \\ &= \partial_0 \int_V \mathrm{d}^3x\rho + \int_{\partial V} \mathrm{d}S n_i q^i , \end{aligned} \tag{4.19}$$

where on the second line we have used (4.13) and Gauss' theorem, and n^i are the components of the unit vector orthogonal to the boundary ∂V. If there is no energy flux through the boundary ∂V (i.e. $q^i = 0$ there), the total energy, defined as $E \equiv \int_V \mathrm{d}x^3\rho$, is conserved. Repeating the exercise for $\beta = i$ and assuming that the momentum flux T_{ij} through the boundary vanishes, we find that the total momentum, defined as $P^i \equiv \int_V \mathrm{d}x^3 q^i$, is conserved. This also clarifies the physical meaning of the components T_{ij}. To prove the implication in the other direction, we assume that the time derivative is zero when the flux through the boundary vanishes, and work backwards. Requiring this to hold regardless of how V is chosen gives the local result.

The property $\partial_\alpha T^{\alpha\beta} = 0$ is sometimes called the energy conservation equation. Note the similarity to the electrodynamics equation $\partial_\alpha j^\alpha = 0$ for the charge current, from which it follows that charge is conserved. Moving from Cartesian coordinates to general coordinates (but still in Minkowski space), the equation $\partial_\alpha T^{\alpha\beta} = 0$ generalises to

$$\nabla_\alpha T^{\alpha\beta} = 0 \qquad \text{in Minkowski space} . \tag{4.20}$$

There is no physics in the above generalisation, just coordinate transformations. However, we now assume that the above equation generalises to the following law:

$$\nabla_\alpha T^{\alpha\beta} = 0 \qquad \text{on a general manifold} . \tag{4.21}$$

The physical assumption in (4.21) is that the geometry of the manifold affects the covariant divergence of the energy-momentum tensor only via the connection. The equation (4.21) is called the **energy-momentum tensor continuity equation** or sometimes the covariant conservation law of the energy-momentum tensor. The latter term is misleading and not recommended: because (4.21) is a tensor equation (of rank higher than zero), it cannot be integrated on the manifold, and hence it does not lead to a conservation law. In fact, the equation quantifies how energy is not conserved in GR, as we will see when we discuss cosmology. (It would be more accurate to say that in GR, total energy as a concept is not in general defined, and when it can be defined, it is not in general conserved.)

Taking SR equations for non-gravitational physics in Cartesian coordinates and replacing the Minkowski metric with a general metric and partial derivatives with covariant derivatives to obtain the equations that apply in GR is called the **minimal coupling principle**. Looking only at the equations of motion, it does not follow from other properties of the theory, it is an extra assumption. For example, were we to add to (4.21) terms that depend on the Riemann tensor, it would still have the desired SR limit. The minimal coupling principle is in fact not much needed in GR once we use the action formulation, to which we come in chapter 6. We will then see how partial derivatives in the action for scalar fields and gauge fields turn into covariant derivatives in the equations of motion automatically via the variational principle, without the need to invoke extra assumptions. (The fermion case is a bit more complicated, and we will not discuss it.) We will also see in which way terms that break the minimal coupling principle are allowed.

4.1.3 Generality of the Einstein equation

Let us now get back to our equation of motion (4.6). Contracting with the covariant derivative gives zero on the left-hand side by the assumption (4.21). Therefore the right-hand side must be zero as well. Comparing to (3.82), we see that we must have $A = -\frac{1}{2}$, i.e. the right-hand side has to be the Einstein tensor. We end up with

$$G_{\alpha\beta} = \kappa T_{\alpha\beta} \ , \tag{4.22}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is the Einstein tensor. The relation (4.22) is the **Einstein equation**. This is the equation of motion of general relativity that we have been seeking.

How general is this equation of motion? Assume that the equation of motion is of the form $\kappa T_{\alpha\beta} = A_{\alpha\beta}$ for some gravitational tensor $A_{\alpha\beta}$. It can be shown that if the following conditions hold,

1) $A_{\alpha\beta} = A_{\alpha\beta}(g, \partial g, \partial^2 g)$,

2) $\nabla_\alpha A^{\alpha\beta} = 0$,

3) $A_{\alpha\beta}$ is linear in $\partial^2 g$,

4) $A_{\alpha\beta} = A_{(\alpha\beta)}$,

then in addition to the Einstein tensor, $A_{\alpha\beta}$ can consist of only one other tensor, namely the metric itself: $A_{\alpha\beta} = \lambda G_{\alpha\beta} + \Lambda g_{\alpha\beta}$, where λ and Λ are constants. The coefficient Λ is called the **cosmological constant**. It was not present in the original formulation of the equation of motion, and the name comes from the fact that it was added by Einstein in 1917 to obtain a static cosmological solution, as we will discuss in chapter 9. If we demand that Minkowski space is a solution when there is no matter, we get $\Lambda = 0$ (although it is not clear whether this is a reasonable demand).

Alternatively, we can replace conditions 3 and 4 with the requirement that there are no more than 4 spacetime dimensions, so that the set of conditions reads

1) $A_{\alpha\beta} = A_{\alpha\beta}(g, \partial g, \partial^2 g)$.

2) $\nabla_\alpha A^{\alpha\beta} = 0$.

3) $d \leq 4$.

At $d = 5$, there is one extra tensor that satisfies the above two conditions and that is quadratic in the Riemann tensor; every other dimension thereafter adds a new term that is one power higher in the Riemann tensor. (The cosmological constant term and the Einstein tensor are part of this progression, appearing at $d = 1$ and $d = 3$, respectively.)

So either way, we end up with the equation of motion

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \ . \tag{4.23}$$

Comparing to the decomposition (4.7) of the energy-momentum tensor, we see that if we shift the cosmological constant term to the right-hand side, it corresponds to a contribution to the energy-momentum tensor of the ideal fluid form with $\rho = \Lambda/\kappa$ and $P = -\rho$. This form of matter with constant energy density and pressure is called **vacuum energy**. We will sometimes drop the cosmological constant term, because it can always be considered to be included in the energy-momentum tensor like this.

The identity $\nabla_\alpha G^{\alpha\beta} = 0$ (which, recall, is just the trace of the Bianchi identity) reduces the number of independent differential equations by 4. So (4.23) consists of 10 equations connected by 4 constraints. This is just as well, because it should give the 10 components of the metric, up to 4 coordinate transformations. If the number of independent equations of motion were equal to the number of components of the metric, the equations of motion would fix the coordinate system in addition to fixing the physics, in violation of diffeomorphism invariance according to which all coordinate systems are physically equivalent.

With the equation of motion (4.23), conservation of energy and momentum in the limit of flat spacetime is a consequence of Bianchi identity. This analogous to how charge conservation follows from the structure of the dynamical part of the Maxwell equations, ${F^{\alpha\beta}}_{,\beta} = j^\alpha$. (Because $F_{\alpha\beta} = F_{[\alpha\beta]}$, $\partial_\alpha j^\alpha = 0$ is a consistency condition for the equation.) Of course, since we used the conservation of energy and momentum to find the equation of motion, we cannot say that we have derived it. This will change once we find the equation of motion from varying an action; then $\nabla_\alpha T^{\alpha\beta} = 0$ and, in the flat spacetime limit, energy conservation become results and not assumptions.

Note how the requirement of diffeomorphism invariance has pinned down the equation of motion. We noted in chapter 1 that the symmetries of Newtonian mechanics do not fix the gravitational force: its magnitude could depend on the distance between particles in an arbitrary manner. In GR, the situation is strikingly different: we have no free functions, and only two constants. Before going to the action, let us look at the Newtonian limit to fix one of them, κ that determines the strength of the curvature sourced by matter.

4.2 Newtonian limit

4.2.1 Weak field and small velocity

Our study of the relativistic theory of gravity so far has a major shortcoming: we haven't shown that it has anything to do with gravity. Let's now demonstrate that

Newtonian gravity is a limit of GR. We expect to need at least two conditions for the Newtonian limit: weak gravitational fields and small velocities. The second condition is necessary, because we know that the validity of Newtonian physics is limited to the region where velocities are much smaller than the speed of light (i.e. unity).

Let us begin by defining that by weak gravitational fields we mean that there exists a coordinate system where the metric is close to the Minkowski metric, $g_{\alpha\beta} = \eta_{\alpha\beta} + \delta g_{\alpha\beta}$, with $|\delta g_{\alpha\beta}| \ll 1$. We will expand to linear order in $\delta g_{\alpha\beta}$, and the derivatives of $\delta g_{\alpha\beta}$ are considered to be of the same order of smallness as $\delta g_{\alpha\beta}$. In chapter 8 we will consider the full decomposition of $\delta g_{\alpha\beta}$ into ten degrees of freedom (and the impact of coordinate transformations on them). For now, we simply take the metric to be diagonal and of the form:

$$\mathrm{d}s^2 = -(1+2\phi)\mathrm{d}t^2 + (1-2\phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ , \tag{4.24}$$

i.e. $g_{\alpha\beta} = \eta_{\alpha\beta} - 2\phi\delta_{\alpha\beta}$, with $|\phi| \ll 1$. To linear order, the inverse metric is

$$g^{\alpha\beta} \simeq \eta_{\alpha\beta} + 2\phi\delta_{\alpha\beta} \ . \tag{4.25}$$

It is straightforward to derive the connection coefficients. To linear order, they are

$$\begin{aligned} \Gamma^{\gamma}_{\alpha\beta} &= \frac{1}{2}g^{\gamma\mu}(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}) \\ &\simeq -\eta_{\gamma\beta}\partial_\alpha\phi - \eta_{\alpha\gamma}\partial_\beta\phi + \delta_{\alpha\beta}\eta^{\gamma\mu}\partial_\mu\phi \ . \end{aligned} \tag{4.26}$$

Writing out the coefficients, we have

$$\Gamma^0_{00} \simeq \dot{\phi} \tag{4.27}$$
$$\Gamma^0_{0i} \simeq \partial_i\phi \tag{4.28}$$
$$\Gamma^0_{ij} \simeq -\delta_{ij}\dot{\phi} \tag{4.29}$$
$$\Gamma^i_{00} \simeq \partial_i\phi \tag{4.30}$$
$$\Gamma^i_{0j} \simeq -\delta_{ij}\dot{\phi} \tag{4.31}$$
$$\Gamma^i_{jk} \simeq -\delta_{ik}\partial_j\phi - \delta_{ij}\partial_k\phi + \delta_{jk}\partial_i\phi \ , \tag{4.32}$$

where dot denotes partial derivative with respect to the coordinate time t. These do not agree with the connection coefficients derived for the exact Newtonian case in section 4.1.1, where the only non-zero coefficient was $\Gamma^i_{00} = \partial_i\phi$. One simple difference is the presence of $\dot{\phi}$, but the exact Newtonian theory is also missing terms that have the same form as Γ^i_{00}, and are not suppressed by any small factor. We will see the reason when we look at the trajectories of observers.

4.2.2 Equation of motion of matter

Consider a timelike geodesic with unit tangent vector $u^\alpha = \delta^{\alpha 0} + \delta u^\alpha$. We assume that the spatial velocity is small: in the coordinate system (4.24), $|\delta u^i| \ll 1$. The geodesic equation is

$$
\begin{aligned}
0 &= u^\beta \nabla_\beta u^\alpha \\
&= u^\beta \partial_\beta u^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma \\
&\simeq \partial_0 u^\alpha + \Gamma^\alpha_{00} \ ,
\end{aligned} \tag{4.33}
$$

where we discarded all terms that are nonlinear in ϕ and/or $u^i = \delta u^i$. The component $\alpha = i$ gives, using (4.27) and noting that $u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}t} \simeq \frac{\mathrm{d}x^i}{\mathrm{d}t}$,

$$
0 = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \delta^{ij}\partial_j \phi \ , \tag{4.34}
$$

the Newtonian equation of motion under gravity. We now see why the exact Newtonian theory is missing the non-zero parts of the terms Γ^i_{jk} (the same applies to Γ^0_{0i}). In the geodesic equation, they couple to the velocity, and their effect is suppressed when the velocity is small. They are thus invisible in the Newtonian limit.

The component $\alpha = 0$ gives, again using (4.27), $\delta u^0 \simeq -\phi$ (demanding that $\delta u^0 = 0$ when $\phi = 0$). Recalling that $u^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau}$, this gives gravitational time dilation. Instead of using the geodesic equation, we could get the time dilation from the normalisation condition:

$$
\begin{aligned}
-1 &= g_{\alpha\beta} u^\alpha u^\beta \\
&\simeq g_{00} u^0 u^0 \\
&= -(1+2\phi) u^0 u^0 \ ,
\end{aligned} \tag{4.35}
$$

and inputting $u^0 = 1 + \delta u^0$ gives $\delta u^0 \simeq -\phi$ to linear order.

4.2.3 Equation of motion of the gravitational field

We have shown that in the limit of weak gravitational fields and small velocities gravity affects matter in the same way as in Newtonian theory. Let us now turn to how matter generates gravity. For that we need the Riemann tensor. To linear order we have, using (4.26)

$$
\begin{aligned}
R^\alpha{}_{\beta\gamma\delta} &\simeq \partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} \\
&\simeq \eta_{\beta\delta}\partial_\alpha\partial_\gamma\phi - \eta_{\alpha\delta}\partial_\beta\partial_\gamma\phi + \eta_{\alpha\gamma}\partial_\beta\partial_\delta\phi - \eta_{\beta\gamma}\partial_\alpha\partial_\delta\phi \ .
\end{aligned} \tag{4.36}
$$

Writing the Riemann tensor component by component, we have

$$
R^i{}_{0j0} \simeq \partial_i\partial_j\phi + \delta_{ij}\ddot{\phi} \tag{4.37}
$$

$$
R^i{}_{0jk} \simeq \delta_{ij}\partial_k\dot{\phi} - \delta_{ik}\partial_j\dot{\phi} \tag{4.38}
$$

$$
R^i{}_{jkl} \simeq \delta_{ik}\partial_j\partial_l\phi - \delta_{jk}\partial_i\partial_l\phi + \delta_{jl}\partial_i\partial_k\phi - \delta_{il}\partial_j\partial_k\phi \ . \tag{4.39}
$$

We thus get for the Ricci tensor ($R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$)

$$
R_{00} \simeq \nabla^2\phi + 3\ddot{\phi} \tag{4.40}
$$

$$
R_{0i} \simeq 2\partial_i\dot{\phi} \tag{4.41}
$$

$$
R_{ij} \simeq \delta_{ij}(\nabla^2\phi - \ddot{\phi}) \ , \tag{4.42}
$$

and the Ricci scalar is

$$R = g^{\alpha\beta}R_{\alpha\beta} \simeq \eta^{\alpha\beta}R_{\alpha\beta} \simeq 2\nabla^2\phi - 6\ddot{\phi} \ . \tag{4.43}$$

Finally, the Einstein tensor is

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \simeq R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R \ , \tag{4.44}$$

which gives

$$G_{00} \simeq 2\nabla^2\phi \tag{4.45}$$
$$G_{0i} \simeq 2\partial_i\dot{\phi} \tag{4.46}$$
$$G_{ij} \simeq 2\delta_{ij}\ddot{\phi} \ . \tag{4.47}$$

Note how the Einstein tensor is simpler than the Ricci tensor.

We should now look at the Einstein equation. We consider a general energy-momentum tensor and write it using the decomposition (4.7) for the observer velocity u^α. Because we treat the derivatives of ϕ as of the same order of smallness as ϕ and the velocity, it follows that the source terms in the energy momentum tensor must also be small, so we linearise with respect to them. To linear order, as δu^i is small, the orthogonality of u^α with q^α and $\Pi_{\alpha\beta}$ implies $q_0 \simeq 0$, $\Pi_{0\alpha} \simeq 0$. Linearising the Einstein equation (4.23) (and dropping the cosmological constant), we get

$$2\nabla^2\phi \simeq \kappa\rho \tag{4.48}$$
$$2\partial_i\dot{\phi} \simeq -\kappa q_i \tag{4.49}$$
$$2\delta_{ij}\ddot{\phi} \simeq \kappa(P\delta_{ij} + \Pi_{ij}) \ . \tag{4.50}$$

We immediately note that (4.50) is inconsistent unless Π_{ij} is smaller than the other source terms so that it can be neglected, $\Pi_{ij} \simeq 0$. (We will see later that Π_{ij} would generate a difference between δg_{00} and δg_{ij}, which we have not accounted for, and which is not present in the Newtonian limit, where there is only one gravitational potential.) The equation (4.48) gives the Poisson equation if $\kappa = 8\pi G_{\rm N}$ and if the energy density is dominated by the mass density, i.e. if matter consists of a gas of particles whose masses are much larger than their kinetic energies, $\rho \simeq \rho_{\rm m}$. In general, not only rest energy due to mass but also other forms of energy contribute to the energy density. When we discuss cosmology in chapter 9, we will see how massless particles contribute to the energy density.

What about the other two equations, (4.49) and (4.50) (with the latter sourced only by pressure)? These equations are not independent: taking a time derivative of (4.48) and a spatial derivative of (4.49) gives $\dot{\rho} + \partial_i q^i = 0$. (Note the analogy between the energy density of GR and charge density of electrodynamics, and the energy flux of GR and charge current of electrodynamics.) From (4.49) and (4.50) we likewise get $\partial_i P + \dot{q}_i = 0$. These relations are in fact just the 0 and i components of the linearised continuity equation $\nabla_\alpha T^{\alpha\beta} \simeq \partial_\alpha T^{\alpha\beta}$. This is related to our earlier comment that not all components of the Einstein equation are independent, because they are related by the Bianchi identity.

If we want the Newtonian limit to include the condition that only the mass density has an effect on gravity (as in Newtonian theory), then we should impose the conditions $P \simeq 0, q_i \simeq 0$. Together with $\Pi_{ij} \simeq 0$, this amounts to demanding that the energy density is much larger than any other contribution to the energy-momentum tensor. In this case $\dot{\phi} \simeq 0$, i.e. the gravitational potential varies slowly in time, and so does the energy density, $\dot{\rho} \simeq 0$. So this version of the Newtonian limit corresponds to not only slow motion of sources, but also to slow evolution of the metric. We see that the Newtonian limit implies conditions not just on the strength of the gravitational fields and observer velocities, but also on the energy-momentum tensor. For matter that consists of a gas of particles, the conditions that Π_{ij}, q_i and P are small reduce to the demand that the particle velocities are small, which also guarantees $\rho \simeq \rho_{\mathrm{m}}$. So they can be considered part of the small velocity assumption, if it is extended to cover not only the velocity of the observer but also the velocity of the sources.

The Poisson equation, which we end up with, is elliptic: time derivatives have disappeared. This is because in Newtonian theory, the gravitational field does not have its own degrees of freedom, it is fixed by the matter via a constraint equation. This is not true in GR: for suitable conditions on the matter content, the Einstein equation is hyperbolic and has a well-defined initial value problem. This difference means that the Newtonian limit is singular: solving the Einstein equation and taking the Newtonian limit do not commute. There are solutions of the Newtonian equations that are not the limit of any GR solution, even if the velocities are small. (There are also, less surprisingly, GR solutions that have no Newtonian limit.) Describing the Newtonian limit in detail, correctly accounting for this feature, is an interesting problem that we cannot stop to further discuss, as our itinerary calls for us to move on to the orbit of Mercury and the bending of light in the Schwarzschild solution.