# Contents



# 3 Curvature

## 3.1 Connection and covariant derivative

### 3.1.1 General connection

We have established that the second derivatives of the metric contain (in 4 dimensions) a total of 100 functions, of which 80 are coordinate degrees of freedom. The remaining 20 functions describe coordinate-independent information, called curvature, that distinguishes the manifold from Minkowski space at a point. We will see that the curvature in fact contains all information about that difference in the sense that if it vanishes, the spacetime is Minkowski. We want to express this information in a way that (unlike partial derivatives of the components of the metric) is independent of the coordinates, in other words we seek a tensor that describes spacetime curvature. A straightforward way to do this would be to find the combination of the second derivatives of the metric (and the metric and its first derivatives) that transforms like a tensor. However, the combination turns out to be quite messy. It is easier to first introduce a bit more structure on the manifold, and then express the curvature in terms of that structure. The structure in question is the **covariant derivative**.

In the previous chapter, we noted that the partial derivatives of the components of a tensor field (of rank $\geq 1$) are not the components of a tensor field. In other words, a partial derivative is defined only in a coordinate system, not on the manifold: it operates on components, not on tensors. Let us now define a derivative that instead operates on tensors, the covariant derivative $\nabla$. Formally, it is an operation that maps a tensor field to another tensor field:

$$\nabla \ : \text{tensor field of type } (r,s) \ \rightarrow \ \text{tensor field of type } (r,s+1) \ . \tag{3.1}$$

In order to be identified as a derivative operator that generalises the concept of partial derivative to manifolds, we demand the map $\nabla$ to satisfy the following conditions ($A$ and $B$ are tensors, $a$ and $b$ are real numbers):

1) Linearity: $\nabla(aA + bB) = a\nabla A + b\nabla B$ .
2) Leibniz rule: $\nabla(A \otimes B) = \nabla A \otimes B + A \otimes \nabla B$ .
3) The covariant derivative of a scalar function is the partial derivative.
4) The covariant derivative commutes with contraction of indices.

The components of the covariant derivative are denoted by

$$(\nabla T)_\mu{}^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} \equiv \nabla_\mu T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} \equiv T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta;\mu} \ . \tag{3.2}$$

Note that $\nabla_\mu$ comes before the other indices, whereas $;\mu$ comes after them. Condition 4 then reads

$$\nabla_\mu(T^{\alpha_1\ldots\nu\ldots\alpha_r}{}_{\beta_1\ldots\nu\ldots\beta_s}) = (\nabla T)_\mu{}^{\alpha_1\ldots\nu\ldots\alpha_r}{}_{\beta_1\ldots\nu\ldots\beta_s} \ . \tag{3.3}$$

For a vector field $\underline{U}$ we have

$$(\nabla\underline{U})_\beta{}^\alpha \equiv \nabla_\beta U^\alpha \equiv U^\alpha{}_{;\beta} \ . \tag{3.4}$$

It follows from properties 1 to 4 that the components of the covariant derivative are given by a partial derivative of the components of the tensor it operates on, plus a linear combination of those components. (This is not obvious; we skip the proof.) For a vector, we have

$$\nabla_\beta U^\alpha = \partial_\beta U^\alpha + \Gamma^\alpha_{\beta\gamma} U^\gamma \ , \tag{3.5}$$

where the functions $\Gamma^\alpha_{\beta\gamma}$ are called the **connection coefficients**, or simply the **connection**. In $d$ dimensions, there are $d^3$ of them, so 64 for $d = 4$. These functions define the covariant derivative. The connection coefficients are not the components of a tensor. From the condition that $\nabla\underline{U}$ is a type $(1,1)$ tensor, we find that the connection coefficients transform as (**Exercise:** show this.)

$$\Gamma'^\gamma{}_{\alpha\beta} \to \Gamma'^\gamma{}_{\alpha\beta} = M^\gamma{}_\rho (M^{-1})^\mu{}_\alpha (M^{-1})^\nu{}_\beta \Gamma^\rho_{\mu\nu} - (M^{-1})^\mu{}_\alpha (M^{-1})^\nu{}_\beta M^\gamma{}_{\mu,\nu} \ , \tag{3.6}$$

where $M^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}$ is the Jacobian matrix as usual.

If the tensor has more than one contravariant index, we get a corresponding linear combination (3.5) of all of them. For a covector $\tilde{\omega}$ we have

$$\nabla_\beta \omega_\alpha \equiv \partial_\beta \omega_\alpha + \tilde{\Gamma}^\gamma_{\beta\alpha} \omega_\gamma \equiv \omega_{\alpha;\beta} \ . \tag{3.7}$$

where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are some coefficients that are a priori independent of $\Gamma^\alpha_{\beta\gamma}$. From property 4 it follows that $\nabla_\beta(V^\alpha\omega_\alpha) = \partial_\beta(V^\alpha\omega_\alpha)$, and applying the Leibniz rule on both sides of the equation gives (**Exercise:** show this.)

$$\tilde{\Gamma}^\gamma_{\alpha\beta} = -\Gamma^\gamma_{\alpha\beta} \ . \tag{3.8}$$

The components of the covariant derivative for a tensor of any type are therefore

$$\begin{aligned}\nabla_\mu T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} &= \partial_\mu T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} + \Gamma^{\alpha_1}_{\mu\gamma} T^{\gamma\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} + \ldots + \Gamma^{\alpha_r}_{\mu\gamma} T^{\alpha_1\ldots\gamma}{}_{\beta_1\ldots\beta_s} \\ &\quad - \Gamma^{\gamma}_{\mu\beta_1} T^{\alpha_1\ldots\alpha_r}{}_{\gamma\ldots\beta_s} - \ldots - \Gamma^{\gamma}_{\mu\beta_s} T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\gamma} \ . \end{aligned} \tag{3.9}$$

This rule is easy to remember: we contract one tensor index at a time with the connection, with a plus sign for up indices, and a minus sign for down indices. The rules for tensors up to rank 2 are listed below.

$$\begin{array}{|ll|} f_{;\alpha} = f_{,\alpha} & T^{\alpha\beta}{}_{;\gamma} = T^{\alpha\beta}{}_{,\gamma} + \Gamma^\alpha_{\gamma\delta} T^{\delta\beta} + \Gamma^\beta_{\gamma\delta} T^{\alpha\delta} \\ U^\alpha{}_{;\beta} = U^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} U^\gamma & T^\alpha{}_{\beta;\gamma} = T^\alpha{}_{\beta,\gamma} + \Gamma^\alpha_{\gamma\delta} T^\delta{}_\beta - \Gamma^\delta_{\gamma\beta} T^\alpha{}_\delta \\ \omega_{\alpha;\beta} = \omega_{\alpha,\beta} - \Gamma^\gamma_{\beta\alpha}\omega_\gamma & T_{\alpha\beta;\gamma} = T_{\alpha\beta,\gamma} - \Gamma^\delta_{\gamma\alpha} T_{\delta\beta} - \Gamma^\delta_{\gamma\beta} T_{\alpha\delta} \end{array} \tag{3.10}$$

### 3.1.2 Levi–Civita connection

So far, the connection has been left general. It is an an extra structure, in addition to the metric, that is part of the definition of the manifold. For example, we could give any 64 functions in some coordinate system and say that they are the connection coefficients, and their values in other coordinate systems are given by the rule (3.6). (Of course, such a definition would be restricted to one coordinate patch.) The Lie derivative mentioned in chapter 2 is in a sense a simpler object than the covariant derivative, because it does not require the existence of a connection (nor the metric), just a vector field. We instead want to derive the connection from structure that already exists on the manifold.

From the transformation rule (3.6) for the connection coefficients, it follows that the difference between two connections transforms like a tensor, because the homogeneous part drops out. And anything that transforms like a tensor is a tensor. So, given two connections $\Gamma^\gamma_{\alpha\beta}$ and $\hat\Gamma^\gamma_{\alpha\beta}$, the difference $\Gamma^\gamma_{\alpha\beta} - \hat\Gamma^\gamma_{\alpha\beta}$ is a tensor. In particular, we can take $\hat\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$; the resulting tensor with components

$$T^\gamma{}_{\alpha\beta} \equiv \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} = 2\Gamma^\gamma_{[\alpha\beta]} \tag{3.11}$$

is called the **torsion**. In 4 dimensions, it has $4 \times 6 = 24$ independent components (4 for the up index, times 6 for the two antisymmetric down indices). The other 40 independent functions needed to define the connection are given by the **non-metricity tensor**, defined in terms of the components as

$$Q_{\gamma\alpha\beta} \equiv \nabla_\gamma g_{\alpha\beta} \ . \tag{3.12}$$

Non-metricity has $4 \times 10 = 40$ independent components (4 for the first index, times 10 for the two symmetric indices).

We started with a manifold $M$ and added the metric $g_{\alpha\beta}$. Now we also have the torsion and the non-metricity. The latter two tensors could be left as functions to be determined by the equations of motion (as we will do for the metric). We will in chapter 6 discuss the **Palatini** formulation, also called the **metric-affine** formulation, where the metric and the connection are taken to be independent variables. In

GR (more precisely, the metric formulation of GR), we instead make the simplest possible choice in the sense of not including any tensors apart from the metric, but fix the connection in terms of the metric by making two assumptions:

5) The connection is **torsion-free**: $T^{\gamma}{}_{\alpha\beta} = 0 \iff \Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{(\alpha\beta)}$ .

6) The connection is **metric-compatible**: $\nabla_{\gamma} g_{\alpha\beta} = 0$ .

The conditions $T_{\gamma\alpha\beta} = 0, Q_{\gamma\alpha\beta} = 0$ are a set of 64 independent equations (defined in terms of tensors on the manifold), which fix the 64 components of the connection uniquely in terms of the metric. It is easy to show this. Let us write the condition $Q_{\gamma\alpha\beta} = \nabla_{\gamma} g_{\alpha\beta} = 0$ three times, permuting the indices and sum the equations, assuming $\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{(\alpha\beta)}$:

$$
\begin{aligned}
+ \nabla_{\gamma} g_{\alpha\beta} &= \partial_{\gamma} g_{\alpha\beta} - \cancel{\Gamma^{\mu}_{\gamma\alpha} g_{\mu\beta}} - \cancel{\Gamma^{\mu}_{\gamma\beta} g_{\alpha\mu}} = 0 \\
- \nabla_{\alpha} g_{\beta\gamma} &= -\partial_{\alpha} g_{\beta\gamma} + \Gamma^{\mu}_{\alpha\beta} g_{\mu\gamma} + \cancel{\Gamma^{\mu}_{\alpha\gamma} g_{\beta\mu}} = 0 \\
- \nabla_{\beta} g_{\gamma\alpha} &= -\partial_{\beta} g_{\gamma\alpha} + \cancel{\Gamma^{\mu}_{\beta\gamma} g_{\mu\alpha}} + \Gamma^{\mu}_{\beta\alpha} g_{\gamma\mu} = 0 \\
\hline
\partial_{\gamma} g_{\alpha\beta} &- \partial_{\alpha} g_{\beta\gamma} - \partial_{\beta} g_{\gamma\alpha} + 2\Gamma^{\mu}_{\alpha\beta} g_{\mu\gamma} = 0
\end{aligned}
$$

Contracting with $g^{\nu\gamma}$ and solving for $\Gamma^{\nu}_{\alpha\beta}$, we get

$$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2} g^{\nu\gamma} (\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta}) \ . \tag{3.13}$$

This is the **Levi–Civita connection**[1], and its connection coefficients are known as the **Christoffel symbols**. This is the connection of GR. (From the transformation rule of the metric, you can show that the Levi–Civita connection indeed transforms according to the transformation rule (3.6).) It is obtained from a tensor (the metric), but is is not a tensor (as partial derivatives of tensor components are not components of a tensor). From now on, we assume that the connection is the Levi–Civita connection, unless otherwise noted. Because we can set the first derivative of the metric to zero at a point, the connection can be put to zero at a point. So the first covariant derivative at a point can be reduced to a partial derivative, just as the metric can be reduced to the Minkowski metric.

Because the covariant derivative of the metric is zero (we say that the metric is "covariantly constant"), this is also true for the inverse metric,

$$\nabla_{\gamma} g^{\alpha\beta} = 0 \ . \tag{3.14}$$

We also define the covariant derivative of the determinant of the metric as the covariant derivative in terms of the components of the covariant derivative of the metric using the Leibniz rule. This is an exception, as usually the covariant derivative is not defined for quantities that are not tensors:

$$\nabla_{\gamma} g = 0 \ . \tag{3.15}$$

[1] No relation to the Levi–Civita tensor, except that both are named after Tullio Levi–Civita.

It follows from (3.15) that the Levi–Civita tensor is also covariantly constant,

$$\nabla_\mu \epsilon_{\alpha\beta\gamma\delta} = 0 \,. \tag{3.16}$$

Metric compatibility implies that the covariant derivative commutes with raising and lowering indices:

$$g_{\alpha\mu}\nabla_\beta A^\mu = \nabla_\beta (g_{\alpha\mu} A^\mu) = \nabla_\beta A_\alpha \,. \tag{3.17}$$

In contrast, partial derivatives do not commute with raising and lowering indices: $g_{\alpha\mu}\partial_\beta A^\mu \neq \partial_\beta (g_{\alpha\mu} A^\mu) = \partial_\beta A_\alpha$.

As the Levi–Civita connection is symmetric, it drops out of the antisymmetrisation of the covariant derivative of a covector. Therefore the antisymmetrised partial derivative of a covector, called the **exterior derivative**, can be equivalently written in terms of the covariant derivative:

$$\partial_{[\alpha}\omega_{\beta]} = \nabla_{[\alpha}\omega_{\beta]} \,. \tag{3.18}$$

The Levi–Civita connection also drops out from the commutator of two vector fields,

$$[\underline{U}, \underline{V}]^\alpha = U^\beta V^\alpha{}_{,\beta} - V^\beta U^\alpha{}_{,\beta} = U^\beta V^\alpha{}_{;\beta} - V^\beta U^\alpha{}_{;\beta} \,. \tag{3.19}$$

For a connection with non-zero torsion, these equalities do not hold, but the exterior derivative and the commutator of two vector fields are still tensors. They are defined with partial derivatives, with no need to involve the connection.

The Levi–Civita connection satisfies (**Exercise:** Show this. Hint: Use the relation between the determinant and the trace of a matrix.)

$$\Gamma^\alpha_{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\beta\sqrt{-g} \,. \tag{3.20}$$

This leads to a simple result for the covariant divergence of a vector:

$$\nabla_\alpha V^\alpha = \partial_\alpha V^\alpha + \Gamma^\alpha_{\alpha\beta} V^\beta = \partial_\alpha V^\alpha + \frac{1}{\sqrt{-g}}\partial_\beta\sqrt{-g}V^\beta = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}V^\alpha\right) \,. \tag{3.21}$$

The result (3.21) plays an important role in Stokes' theorem.

### 3.1.3 Stokes' theorem

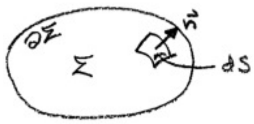
In three-dimensional Euclidean space in Cartesian coordinates, Gauss' theorem relates the integral of the divergence of a vector field $\vec{V}$ over the volume $\Sigma$ to the integral of the vector field projected onto the surface $\partial\Sigma$ of the volume (see figure 1a):

$$\int_\Sigma \mathrm{d}^3x \nabla\cdot\vec{V} = \int_{\partial\Sigma} \mathrm{d}S\vec{n}\cdot\vec{V} \,, \tag{3.22}$$

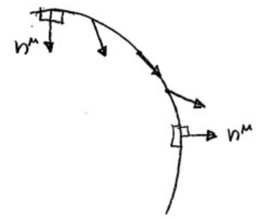
where $\mathrm{d}S$ is the area element and $\vec{n}$ is the unit vector orthogonal to the surface.

In 4-dimensional Minkowski space in Cartesian coordinates, the corresponding result is (see fig. 1b)

$$\int_\Sigma \mathrm{d}^4x \partial_\mu U^\mu = \int_{\partial\Sigma} \mathrm{d}^3\sigma n_\mu U^\mu \,, \tag{3.23}$$




(a) Volume, surface, area element and orthogonal unit vector.




(b) The unit vector $n^\mu$ is orthogonal to the "surface" element $d^3\sigma$. The vector points inward when it is timelike and outward when it is spacelike.

Figure 1: For Gauss' theorem in Euclidean space and Minkowski space.

where $d^3\sigma$ is the three-volume element on the boundary.

On a general manifold in general coordinates, we have Stokes' Theorem:

$$\int_\Sigma d^n x \sqrt{|g|} \nabla_\mu U^\mu = \int_{\partial\Sigma} d^{n-1} x \sqrt{|\gamma|} n_\mu U^\mu \ , \tag{3.24}$$

where we have used (3.21), and $\gamma$ is the determinant of the **induced metric** on the boundary $\partial\Sigma$. The induced metric is obtained from the full metric by inputting the condition $f(x) =$ constant that defines the boundary into the metric. For the surface $x^0 =$ constant, we simply put $dx^0 = 0$. For a general case, we have $df = dx^\alpha \partial_\alpha f = 0$. We will not go into details, as the only thing important for us is that the proper volume integral of the divergence of a vector field vanishes if the vector field vanishes on the boundary. We will need this result when we come to the variational principle in chapter 6.

The condition (3.21), and hence Stokes' theorem, holds only for the Levi–Civita connection. For a general connection, torsion and non-metricity make an appearance, and the integral over the total divergence of a vector does not reduce to a boundary term.

## 3.2 Parallel transport and geodesics

### 3.2.1 Parallel transport

The covariant derivative measures the change of a tensor on the manifold in a given direction, generalising the way a partial derivative measures the change of a scalar function in a given direction. A coordinate-independent way of looking at this is to consider the rate of change of a vector field along a curve on the manifold. In Minkowski space or Euclidean space in Cartesian coordinates, a tensor $T = T^{\alpha\beta} \underline{e}_\alpha \underline{e}_\beta$ being constant along a curve with coordinates $x^\alpha(\lambda)$ just means that its components do not change when moving along the curve:

$$\frac{d}{d\lambda} T^{\alpha\beta} = \frac{dx^\mu}{d\lambda} \partial_\mu T^{\alpha\beta} = 0 \ , \tag{3.25}$$

where we have used the chain rule.

If we consider Minkowski space or Euclidean space in general coordinates, we have the covariant derivative instead, so the same condition reads

$$\frac{\mathrm{D}}{\mathrm{d}\lambda}T^{\alpha\beta} \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\nabla_\mu T^{\alpha\beta} = 0 \ . \tag{3.26}$$

We now promote this equation to hold also on a general manifold in general coordinates, defining the directional covariant derivative (a map from tensors of type $(r,s)$ to tensors of type $(r,s)$)

$$\frac{\mathrm{D}}{\mathrm{d}\lambda} \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\nabla_\mu \ . \tag{3.27}$$

So being constant along a curve means that (using a type $(2,0)$ tensor as an example)

$$\left(\frac{\mathrm{D}}{\mathrm{d}\lambda}T\right)^{\alpha\beta} \equiv \frac{\mathrm{D}}{\mathrm{d}\lambda}T^{\alpha\beta} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\nabla_\mu T^{\alpha\beta} = 0 \ . \tag{3.28}$$

The condition (3.28) is the **parallel transport equation**. A tensor that satisfies this equation for the curve whose tangent vector has the components $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ is constant when transported along the curve.

For a vector field, the parallel transport equation reads

$$\left(\frac{\mathrm{D}}{\mathrm{d}\lambda}\underline{V}\right)^\gamma = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\left(\partial_\alpha V^\gamma + \Gamma^\gamma_{\alpha\beta}V^\beta\right) = \frac{\mathrm{d}V^\gamma}{\mathrm{d}\lambda} + \Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}V^\beta = 0 \ . \tag{3.29}$$

We can think of the parallel transport equation in terms of an initial value problem. Given a tensor at initial point $p$, and a curve $\gamma$ with coordinates $x^\alpha(\lambda)$ that passes through $p$, the first order differential equation (3.29) gives the parallel-transported tensor along the curve.

Parallel transport thus provides a map between any two tangent (and cotangent) spaces on any points on the manifold joined by a curve. The mapping is in general not unique, but depends on the curve. This is illustrated in figure 2 for the two-sphere. If we take a vector that is orthogonal to the equator and parallel transport it directly to the north pole along a great circle, the result is different than if we first transport it along the equator and then take it to the North pole along a different great circle.

From metric compatibility it follows that the metric satisfies the parallel transport equation,

$$\frac{\mathrm{D}}{\mathrm{d}\lambda}g_{\alpha\beta} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\nabla_\mu g_{\alpha\beta} = 0 \ . \tag{3.30}$$

Therefore parallel transport conserves the dot product:

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{d}\lambda}\left(\underline{U}\cdot\underline{V}\right) &= \frac{\mathrm{D}}{\mathrm{d}\lambda}\left(g_{\alpha\beta}U^\alpha V^\beta\right) \\ &= g_{\alpha\beta}\frac{\mathrm{D}}{\mathrm{d}\lambda}\left(U^\alpha V^\beta\right) \\ &= g_{\alpha\beta}U^\alpha\frac{\mathrm{D}}{\mathrm{d}\lambda}V^\beta + g_{\alpha\beta}V^\beta\frac{\mathrm{D}}{\mathrm{d}\lambda}U^\alpha = 0 \ ,\end{aligned} \tag{3.31}$$

if both $\underline{U}$ and $\underline{V}$ are parallel transported. In particular, the norm of a vector is unchanged by parallel transport. So parallel transport is a rigid bijective map from one tangent space to another in the sense that it preserves the lengths and relative directions of vectors.

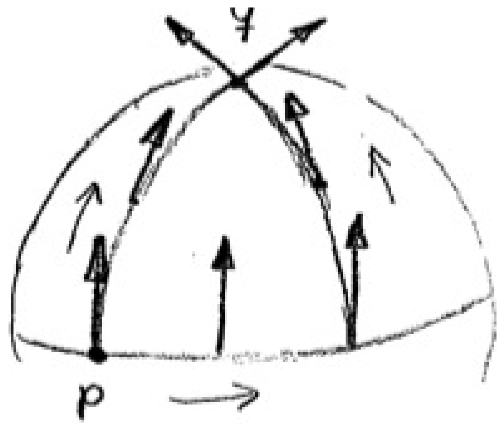



Figure 2: Parallel transport on the two-sphere.

### 3.2.2 Example of parallel transport

Let us consider an example of parallel transport on the two-sphere. We get the metric of the two-sphere of radius $a$ by putting $r = a$ in (2.48), the metric of $\mathbb{R}^3$:

$$\mathrm{d}s^2 = a^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \ . \tag{3.32}$$

It is straighforward to calculate the Levi–Civita connection from (3.13). The non-zero coefficients are

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \tag{3.33}$$

$$\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \frac{\cos\theta}{\sin\theta} \ . \tag{3.34}$$

Consider the unit vector $\underline{V}$ that points along the meridian at point $p$, which is the intersection of the meridian and the equator (where $(\theta, \varphi) = (\frac{\pi}{2}, 0)$). We will transport it to point $q$, which is on the meridian line halfway up to the North pole (where $(\theta, \varphi) = (\frac{\pi}{4}, 0)$). (We don't go all the way to the North pole because it is not covered by our coordinate system.) We do the transport along two different routes. Route $A$ goes along the meridian. Route $B$ goes $\frac{\pi}{2}$ radians forward on the equator, then up $\frac{\pi}{4}$ radians on a great circle towards the North Pole, and finally back $\frac{\pi}{2}$ radians on a great circle to point $q$. At $p$, we have $\underline{V}(p) = V^\theta(p)\partial_\theta = a^{-1}\partial_\theta$.

Route $A$ is given by the curve $(\theta(\lambda), \varphi(\lambda)) = (\lambda, 0)$, where $\lambda$ goes from $\frac{\pi}{2}$ to $\frac{\pi}{4}$. The curve corresponding to $B$ consists of three sections. First, $(\theta(\lambda), \varphi(\lambda)) = (\frac{\pi}{2}, \lambda)$, where $\lambda$ goes from 0 to $\frac{\pi}{2}$. Second, $(\theta(\lambda), \varphi(\lambda)) = (\pi - \lambda, \frac{\pi}{2})$, where $\lambda$ goes from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$. Third, $(\theta(\lambda), \varphi(\lambda)) = (\frac{\pi}{4}, \frac{5\pi}{4} - \lambda)$, where $\lambda$ goes from $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$. Now we just have to solve the parallel transport equation (3.29) along these two curves, with the connection coefficients (3.33) and the given initial condition.

On route $A$,

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = (1, 0) \ , \tag{3.35}$$

so

$$0 = \frac{dV^\gamma}{d\lambda} + \Gamma^\gamma_{\alpha\beta}\frac{dx^\alpha}{d\lambda}V^\beta = \frac{dV^\gamma}{d\lambda} + \Gamma^\gamma_{\theta\varphi}V^\varphi \ , \tag{3.36}$$

where we have used (3.33). Therefore, again using (3.33), we have $\frac{dV^\theta}{d\lambda} = 0$, while for the $\varphi$ component we get

$$0 = \frac{dV^\varphi}{d\lambda} + \frac{\cos\theta}{\sin\theta}V^\varphi \ . \tag{3.37}$$

We could integrate (3.37) using the fact that along route $A$ we have $\theta = \lambda$. However, because the initial $V^\varphi$ is zero, we see that it will remain zero. So all in all,

$$V^\alpha = (a^{-1}, 0) \ . \tag{3.38}$$

This result is expected. Because both the initial vector and the tangent vector of the curve along which it is parallel transported point along the meridian, the resulting vector will also point along the meridian. And given that parallel transport conserves the norm of vectors, the parallel transported vector has unit norm.

Let us now consider route $B$. Along section 1, $\varphi = \lambda$, and

$$\frac{dx^\alpha}{d\lambda} = (0, 1) \ . \tag{3.39}$$

Writing the parallel transport equation (3.29) terms of the components, we have (still along section 1)

$$\begin{aligned} 0 &= \frac{dV^\theta}{d\lambda} - \sin\theta\cos\theta V^\varphi \\ 0 &= \frac{dV^\varphi}{d\lambda} + \frac{\cos\theta}{\sin\theta}V^\theta \ . \end{aligned} \tag{3.40}$$

On section 1 we have $\theta = \frac{\pi}{2}$, so $\frac{dV^\alpha}{d\lambda} = 0$: there is no change in the components.

Along section 2, $\theta = \pi - \lambda$, and

$$\frac{dx^\alpha}{d\lambda} = (-1, 0) \ . \tag{3.41}$$

We know from calculation of route A that there is no change in the components. That leaves section 3, where $\varphi = \frac{5\pi}{4} - \lambda$, and

$$\frac{dx^\alpha}{d\lambda} = (0, -1) \ . \tag{3.42}$$

We get

$$\begin{aligned} 0 &= \frac{dV^\theta}{d\lambda} + \sin\theta\cos\theta V^\varphi = \frac{dV^\theta}{d\lambda} + \frac{1}{2}V^\varphi \\ 0 &= \frac{dV^\varphi}{d\lambda} - \frac{\cos\theta}{\sin\theta}V^\theta = \frac{dV^\varphi}{d\lambda} - V^\theta \ , \end{aligned} \tag{3.43}$$

where in the second equality we have taken into account that $\theta = \frac{\pi}{4}$. We can separate the equations by taking a derivative with respect to $\lambda$, getting

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2 V^\theta}{\mathrm{d}\lambda^2} + \frac{1}{2} V^\theta \\ 0 &= \frac{\mathrm{d}^2 V^\varphi}{\mathrm{d}\lambda^2} + \frac{1}{2} V^\varphi \ . \end{aligned} \tag{3.44}$$

With the initial condition $V^\alpha = (a^{-1}, 0)$, the solution is

$$\begin{aligned} V^\theta &= a^{-1} \cos\left[\frac{1}{\sqrt{2}}\left(\lambda - \frac{3\pi}{4}\right)\right] \\ V^\varphi &= \sqrt{2} a^{-1} \sin\left[\frac{1}{\sqrt{2}}\left(\lambda - \frac{3\pi}{4}\right)\right] \ . \end{aligned} \tag{3.45}$$

Using the metric (3.32), it is easy to check that these are the components of a unit vector. Putting $\lambda = \frac{5\pi}{4}$ now gives us the vector at point $q$, and shows that the result is different from that obtained along route $A$:

$$\begin{aligned} V^\theta(q) &= a^{-1} \cos\left(\frac{\pi}{2\sqrt{2}}\right) \\ V^\varphi(q) &= \sqrt{2} a^{-1} \sin\left(\frac{\pi}{2\sqrt{2}}\right) \ . \end{aligned} \tag{3.46}$$

### 3.2.3 Geodesics

In the above example, we discussed great circles, which have a special role on the two-sphere. We now use parallel transport to define straight lines for a general manifold; great circles will turn out to be straight lines on two-spheres. To do so, we promote a result from Euclidean space into a definition on a general manifold. In Euclidean space, a straight line is characterised by two properties:

1) A straight line is the shortest path between any two points on it.

2) A straight line parallel transports its own tangent vector.

The first property has to do with distance, the second with direction. For a general connection, distance and direction are independent properties. Distance is given by the metric, direction is defined by the connection. The way we defined constant direction with the parallel transport equation above involves only the connection, the metric makes no direct appearance. The Levi–Civita connection gives the connection in terms of the metric, relating direction and distance.

We choose property 2 above as the definition of a straight line, called a **geodesic**, on the manifold. A geodesic is a curve whose tangent vector $\underline{u}$ is parallel transported with respect to itself, i.e. the curve is **autoparallel**. Then the parallel transport equation (3.28) reads

$$\frac{\mathrm{D}}{\mathrm{d}\lambda}\underline{u} = \underline{u}^\alpha \nabla_\alpha \underline{u} = 0 \ . \tag{3.47}$$

This is the **geodesic equation**, and a curve is a geodesic if and only if its tangent vector satisfies this equation. In the definition (3.47) we could have opted for the seemingly weaker requirement that the tangent vector changes only in the direction proportional to itself, i.e. its length changes but the direction does not. However, the change of length can be undone by a redefinition $\lambda \to \lambda'(\lambda)$. We assume that the parameter $\lambda$ has been chosen in this way. Such a parameter $\lambda$ along the curve is called an **affine parameter**, and the parametrisation of a curve in terms of it is called an **affine parametrisation**. The only freedom in changing $\lambda$ is then $\lambda \to \lambda'(\lambda) = a\lambda + b$, where $a \neq 0$ and $b$ are constants.

In terms of components, the geodesic equation reads

$$\begin{aligned} 0 &= u^\alpha \nabla_\alpha u^\gamma \\ &= u^\alpha \partial_\alpha u^\gamma + \Gamma^\gamma_{\alpha\beta} u^\alpha u^\beta \\ &= \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\partial}{\partial x^\alpha} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} + \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \\ &= \frac{\mathrm{d}^2 x^\gamma}{\mathrm{d}\lambda^2} + \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \\ &\equiv \ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \ . \end{aligned} \tag{3.48}$$

On the last line of (3.48) we have denoted the derivative with respect to $\lambda$ by an overdot. If the spacetime is such that we can set the connection to zero everywhere[2] and we do so, the geodesic equation reduces to $\ddot{x}^\gamma = 0$. This is Newton's second law for inertial observers in the absence of forces, which says that particles have constant velocity. (We will come back to this in more detail in chapter 4.)

Even in Minkowski space or Euclidean space we can choose coordinates other than Cartesian coordinates, such as spherical coordinates or the merry-go-round coordinates introduced in section 1.5.1., relevant for an observer attached to a body that rotates with constant angular velocity, such as the Earth. In Newtonian mechanics, the connection is non-zero in non-Cartesian coordinate systems, and Newton's second law does not hold. The connection terms give the corrections to Newton's second law. If we move them to the right-hand side of the equation, we can call them "apparent forces", as is sometimes done in discussions of Newtonian mechanics, although this can be somewhat misleading. In the language of manifold, metric and connection, these contributions are conceptually simple. The partial derivative of a vector field does not give a coordinate-invariant description of the direction and rate of change of the vector field, because the coordinates change as one moves on the manifold, in addition to the vector field changing. The Levi–Civita connection is the connection that precisely corrects for these changes in the coordinate system as described by the metric.

**Exercise:** Find the connection for the merry-go-round coordinates defined in section 1.5.1.

In GR, we have now generalised the geodesic equation to describe the motion of free particles (meaning particles that are not under the influence of any force –

[2] From (3.13) we see that the connection vanishes precisely when the metric is constant; we will soon give a coordinate-independent characterisation of when it is possible to choose such a coordinate system.

gravity not being a force but an aspect of spacetime geometry) even when the manifold is non-trivial, i.e. when the metric cannot be taken to be constant everywhere. This gives a precise meaning to the weak equivalence principle, according to which particles fall in the same way: they move in the same way because they all move on straight lines. Thus GR unifies motion at constant velocity and motion in "free fall" (meaning under the influence of gravity alone), like Newtonian mechanics unified being at rest and moving at constant velocity. This is a unification of inertia and gravity: gravity only affects motion through the connection, which is derived from the metric. So the connection, like the metric, has a double role: it encodes information both about the geometry of the manifold and the coordinate system used to describe it.

In chapter 6, we will derive the property that free particles move on geodesics starting from more fundamental properties, so it will be a result, not an assumption. (We will also derive the generalisation of the geodesic equation with the force included, i.e. the generalisation of Newton's second law in full.) But for now we just take it as given that free particles move on straight lines in GR, as they do in Newtonian mechanics and SR.

### 3.2.4 Geodesics as curves of extremal length

Defining straight lines as autoparallel curves is a local condition: given a position on the manifold and the initial direction of the curve, the geodesic equation (3.48) allows us to construct the full geodesic piece by piece. In contrast, property 1 that identifies straight lines as curves of minimum length is a global statement. It will not be true on a general manifold, and to see whether it holds for a particular manifold we would need to check all possible curves joining all possible pairs of points. However, for the Levi–Civita connection, a weaker version holds: a geodesic gives a local extremum of the path length. For a timelike curve, it gives the maximum proper time; for a spacelike curve, it gives the minimum proper length. (A null line always has zero length, regardless whether or not it is geodesic.)

Let us show that timelike geodesics give a local extremum of the distance. Consider a timelike curve from point $p$ to point $q$, parametrised in a given coordinate system as $x^\alpha(\lambda)$. The proper time from $p$ to $q$ along the curve is

$$\tau_{pq} = \int_p^q \mathrm{d}\tau = \int_{\lambda_p}^{\lambda_q} \mathrm{d}\lambda \underbrace{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}_{=\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}} , \tag{3.49}$$

where $\dot{x}^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$. Let us now consider the change of $\tau_{pq}$ under a small variation of of the curve, keeping the endpoints fixed. Under a variation of the path

$$x^\alpha(\lambda) \to x^\alpha(\lambda) + \delta x^\alpha(\lambda) , \tag{3.50}$$

with $\delta x^\alpha(\lambda_p) = \delta x^\alpha(\lambda_q) = 0$, the metric changes as (working to first order in the small quantity $\delta x^\alpha(\lambda)$)

$$\begin{aligned} g_{\alpha\beta}[x^\mu(\lambda)] &\to g_{\alpha\beta}[x^\mu(\lambda) + \delta x^\mu(\lambda)] \\ &\simeq g_{\alpha\beta}[x^\mu(\lambda)] + g_{\alpha\beta,\gamma}[x^\mu(\lambda)]\delta x^\gamma(\lambda) . \end{aligned} \tag{3.51}$$

Inputting these changes into (3.49), the change of $\tau_{pq}$ is

$$\begin{aligned}\delta\tau_{pq} &= -\frac{1}{2}\int_p^q \mathrm{d}\lambda \underbrace{\frac{1}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}}_{=\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}} \delta(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) \\ &= -\frac{1}{2}\int_p^q \mathrm{d}\tau(\delta g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + 2g_{\alpha\beta}\dot{x}^\alpha\delta\dot{x}^\beta) \\ &= -\frac{1}{2}\int_p^q \mathrm{d}\tau(g_{\alpha\beta,\gamma}\delta x^\gamma\dot{x}^\alpha\dot{x}^\beta + 2g_{\alpha\gamma}\dot{x}^\alpha\delta\dot{x}^\gamma) \ , \qquad (3.52)\end{aligned}$$

where on the second line we have chosen $\lambda = \tau$. Integrating the last term by parts and using the property that that the variation $\delta x^\beta$ vanishes at the endpoints, we get

$$\begin{aligned}\delta\tau_{pq} &= \int_p^q \mathrm{d}\tau \left(-\frac{1}{2}g_{\alpha\beta,\gamma}\dot{x}^\alpha\dot{x}^\beta + g_{\alpha\gamma,\beta}\dot{x}^\beta\dot{x}^\alpha + g_{\alpha\gamma}\ddot{x}^\alpha\right)\delta x^\gamma \\ &= \int_p^q \mathrm{d}\tau \left[-\frac{1}{2}g_{\alpha\beta,\gamma}\dot{x}^\alpha\dot{x}^\beta + \frac{1}{2}(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha})\dot{x}^\alpha\dot{x}^\beta + g_{\mu\gamma}\ddot{x}^\mu\right]\delta x^\gamma \\ &= \int_p^q \mathrm{d}\tau \Bigg[\ddot{x}^\mu + \underbrace{\frac{1}{2}g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu})}_{\Gamma^\mu_{\alpha\beta}}\dot{x}^\alpha\dot{x}^\beta\Bigg] g_{\mu\gamma}\delta x^\gamma \ . \qquad (3.53)\end{aligned}$$

We now demand that the curve gives an extremum of the proper time, i.e. that $\delta\tau_{pq} = 0$ for all $\delta x^\gamma$. The term inside the square brackets then has to vanish, and we get the geodesic equation (3.48). So the curve is a geodesic, assuming that the connection is Levi–Civita. Were we to use a connection that is not Levi–Civita, straight lines and curves of local extremum length would not coincide. Viewed from another direction, had we opted to define geodesics as lines of local extrema of path length, we would have ended up with the Levi–Civita connection.

For the spacelike case, the calculation goes the same way, apart from some sign differences. Considering the second variation of the path length shows that a timelike geodesic gives a local maximum of the proper time, and a spacelike geodesic gives a local minimum of the path length. Recall the "twin paradox": the twin who stays home moves on a geodesic, so their proper time is longer than the proper time of the the twin who undergoes acceleration i.e. is pushed off a geodesic. Note that geodesics give only local extrema: the above result says nothing about whether there may be a longer/shorter (in the timelike/spacelike case, respectively) path that is not close to a geodesic. When we come to black holes in chapter 7, we will encounter an example of a timelike path that is longer than the geodesic connecting two points.

### 3.2.5 Calculating the connection with the Euler–Lagrange equation

In the expression (3.49) for the proper time, the integrand is proportional to the square root of $L \equiv \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ (where $\cdot \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}$), which is the Lagrangian of a free particle moving in a spacetime with metric $g_{\alpha\beta}$. The variation that gives the locally longest duration also gives the path of a classical particle, because free particles move on geodesics. Therefore we can use the classical equations of motion of the particle to find the connection coefficients. Consider the action

$$S = \int \mathrm{d}\tau L(x^\alpha, \dot{x}^\alpha) = \frac{1}{2}\int \mathrm{d}\tau g_{\alpha\beta}(x^\gamma)\dot{x}^\alpha \dot{x}^\beta \ . \tag{3.54}$$

By the usual variational principle of classical mechanics, variation of the above action gives the **Euler–Lagrange equations**

$$\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}^\alpha} = 0 \ . \tag{3.55}$$

These equations are often a quicker way to calculate the geodesic equation and the connection coefficients for a given metric $g_{\alpha\beta}$ than the definition (3.13).

Let us consider the spatially flat **Friedmann–Lemaître–Robertson–Walker (FLRW) metric** as an example. It is one of the simplest –and most useful– metrics of GR. It describes a spacetime that has Euclidean spatial sections. It is a subcase of the spatially homogeneous and isotropic solutions, called the Friedmann–Lemaître–Robertson–Walker solutions, which we will discuss in more detail in chapter 9. The spatially flat FLRW metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ , \tag{3.56}$$

where $a(t)$ is called the **scale factor**.

The free particle Lagrangian reads

$$L = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\frac{1}{2}\dot{t}^2 + \frac{1}{2}a(t)^2\delta_{ij}\dot{x}^i\dot{x}^j = -\frac{1}{2}\dot{t}^2 + \frac{1}{2}a(t)^2\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \ , \tag{3.57}$$

so variation with respect to $t$ and $\dot{t}$ gives

$$\frac{\partial L}{\partial \dot{t}} = -\dot{t} \ , \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{t}} = -\ddot{t} \ , \qquad \frac{\partial L}{\partial t} = aa'\delta_{ij}\dot{x}^i\dot{x}^j \ , \tag{3.58}$$

where $\dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}$ and $' \equiv \frac{\mathrm{d}}{\mathrm{d}t}$. The 0 component of the Euler–Lagrange equation is therefore

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = -\ddot{t} - aa'\delta_{ij}\dot{x}^i\dot{x}^j = -\ddot{t} - aa'\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = 0 \ . \tag{3.59}$$

Comparing to the 0 component of the geodesic equation,

$$\ddot{t} + \Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \ , \tag{3.60}$$

we find

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = 0 \ , \quad \Gamma^0_{ij} = aa'\delta_{ij} \ . \tag{3.61}$$

Varying now the Lagrangian (3.57) with respect to $x^i$ and $\dot{x}^i$, we get

$$\frac{\partial L}{\partial \dot{x}^i} = a^2\dot{x}^i \ , \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}}^i = a^2\ddot{x}^i + 2aa'\dot{t}\dot{x}^i \ , \qquad \frac{\partial L}{\partial x^i} = 0 \ , \tag{3.62}$$

so the $x^i$ component of the Euler–Lagrange equation reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = a^2\ddot{x}^i + 2aa'\dot{t}\dot{x}^i = 0 \ . \tag{3.63}$$

Dividing by $a^2$ to make the coefficient of the second derivative term unity, and again comparing to the geodesic equation

$$\ddot{x}^i + \Gamma^i_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \ , \tag{3.64}$$

we find the connection coefficients

$$\Gamma^i_{00} = \Gamma^i_{jk} = 0 \ , \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{a'}{a}\delta^i{}_j \ . \tag{3.65}$$

All in all, the non-zero connection coefficients for the spatially flat FLRW metric are (not writing down $\Gamma^i_{j0} = \Gamma^i_{0j}$)

$$\Gamma^0_{ij} = aa'\delta_{ij} \ , \quad \Gamma^i_{0j} = \frac{a'}{a}\delta^i{}_j \ . \tag{3.66}$$

### 3.2.6 Null geodesics and redshift

As an example, let us consider null geodesics and photon energy measured by an observer in the spatially flat FLRW universe described by the metric (3.56). The four-velocity of an observer who observes the universe to be symmetric and homogeneous has the components $u^\alpha = \delta^{\alpha 0}$. Photon momentum is $\underline{k}$, and the energy measured by the observer is $E = -\underline{u}\cdot\underline{k} = -u_\alpha k^\alpha = k^0$. The geodesic equation gives

$$\begin{aligned} 0 &= k^\alpha \nabla_\alpha k^0 \\ &= k^\alpha(\partial_\alpha k^0 + \Gamma^0_{\alpha\beta}k^\beta) \\ &= k^\alpha\partial_\alpha k^0 + a\partial_0 a\delta_{ij}k^i k^j \\ &= k^0\partial_0 k^0 + \frac{\partial_0 a}{a}k^0 k^0 \ , \end{aligned} \tag{3.67}$$

where we have on the third line used the connection coefficients (3.66), and on the fourth line used the null condition $g_{\alpha\beta}k^\alpha k^\beta = -k^0k^0 + a^2\delta_{ij}k^ik^j = 0$. We have also taken into account that for symmetry reasons $k^0$ can only depend on time. Dividing (3.67) by $k^0$ and integrating, we get the result $k^0 \propto 1/a$. This decrease of photon energy in an expanding universe inversely proportional to the scale factor is called the cosmological **redshift**. This is an example of the feature that energy is not conserved in GR. Expansion is related to redshift this way more generally than in the highly symmetric FLRW spacetimes, although in other spacetimes terms related to violation of homogeneity and isotropy will also affect the redshift.

## 3.3 The Riemann tensor

### 3.3.1 What is the Riemann tensor

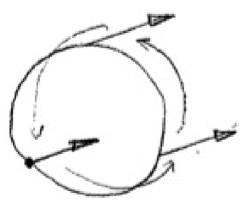
Let us come back to the problem with which we closed chapter 2: what is the tensor representation of the 20 physical degrees of freedom in the second derivatives of the metric? In other words, what is the tensor that represents the curvature of the manifold?

Consider Euclidean space or Minkowski space. On these manifolds, the following three properties hold:

1) Parallel transport around a closed loop leaves vectors unchanged (see figure 3a).

2) Covariant derivatives commute, $\nabla_\beta \nabla_\alpha U^\gamma = \nabla_\alpha \nabla_\beta U^\gamma$.

3) Geodesics that are initially parallel remain parallel (see figure 3b).



(a)

(b)

Figure 3: Properties of flat manifolds.

Condition 1 implies that parallel transport is independent of the path, and only depends on the endpoints. This can be seen as follows. Pick any two points $p$ and $q$ on a closed loop. The loop defines two curves from $p$ to $q$, $\gamma_1$ and $\gamma_2$. The change in the tensor when parallel transported from $p$ to $q$ along $\gamma_1$ and from $q$ to $p$ along $\gamma_2$ is zero, so the change in the segment $\gamma_1$ equals minus the change in the segment $\gamma_2$. Switching the direction of the second segment to be from $p$ to $q$ switches the sign, proving the result.

In the case of straight lines, we listed two results that hold in Euclidean space (minimum length and parallel tangent vector curve), and picked one to serve as the definition of straightness on a general manifold. It turned out that the two conditions are not in general equivalent (geodesics give only the local extrema of length). In the same vein, the above three properties are results on **flat** manifolds, and we can pick one of them as the definition of **flatness** on a general manifold. Its violation will then be a measure of curvature. Unlike in the case of straight lines, it turns out that all the three properties are equivalent on a general manifold, so it doesn't make a difference which one we choose.

The least geometrical (and therefore perhaps the least intuitive) of the three properties, number 2, is the algebraically most straightforward way to define the curvature tensor. Consider the commutator of two covariant derivatives:

$$\begin{aligned}
[\nabla_\gamma, \nabla_\delta]\, U^\alpha &\equiv \nabla_\gamma \nabla_\delta U^\alpha - \nabla_\delta \nabla_\gamma U^\alpha \\
&= \partial_\gamma \nabla_\delta U^\alpha - \cancel{\Gamma^\mu_{\gamma\delta} \nabla_\mu U^\alpha} + \Gamma^\alpha_{\gamma\mu} \nabla_\delta U^\mu - (\gamma \leftrightarrow \delta) \\
&= \partial_\gamma \left( \cancel{\partial_\delta U^\alpha} + \Gamma^\alpha_{\delta\mu} U^\mu \right) + \Gamma^\alpha_{\gamma\mu} (\partial_\delta U^\mu + \Gamma^\mu_{\delta\beta} U^\beta) - (\gamma \leftrightarrow \delta) \\
&= \partial_\gamma \Gamma^\alpha_{\delta\mu} U^\mu + \cancel{\Gamma^\alpha_{\delta\mu} \partial_\gamma U^\mu + \Gamma^\alpha_{\gamma\mu} \partial_\delta U^\mu} + \Gamma^\alpha_{\gamma\mu} \Gamma^\mu_{\delta\beta} U^\beta - (\gamma \leftrightarrow \delta) \\
&= (\partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} + \Gamma^\alpha_{\gamma\mu} \Gamma^\mu_{\delta\beta} - \Gamma^\alpha_{\delta\mu} \Gamma^\mu_{\gamma\beta}) U^\beta \\
&\equiv R^\alpha{}_{\beta\gamma\delta} U^\beta \ ,
\end{aligned} \tag{3.68}$$

where we have slashed the terms that vanish due to the antisymmetry in $\gamma\delta$. Because the coefficients on the left-hand side are the components of a tensor and $U^\beta$ are the components of a tensor, $R^\alpha{}_{\beta\gamma\delta}$ are also the components of a tensor, called the **Riemann curvature tensor** or simply the **Riemann tensor**.

**Exercise:** Show that condition 1 above leads to the same definition for curvature. (Find how a vector changes when it is when parallel transported around a closed loop and show that the change is zero precisely when $R^\alpha{}_{\beta\gamma\delta} = 0$.)

**Exercise:** Show that condition 3 above leads to the same definition for curvature. (Find how initially parallel geodesics change and show that they remain parallel precisely when $R^\alpha{}_{\beta\gamma\delta} = 0$.)

If the Riemann tensor were defined in terms of a general connection, it would have nothing to do with the metric. Curvature as defined by the Riemann tensor is related to straight lines, not distances. (The curvature we have used above is not completely general, we assumed in the derivation that the connection is symmetric, i.e. torsion is zero.) The Levi–Civita connection relates directions and distances, as we have noted, and connects the Riemann tensor to the second derivatives of the metric and their 20 physical degrees of freedom. Note how complicated the components of the Riemann tensor are when written in terms of the components of the metric: we need to find the components of the inverse metric to write the connection (3.13), and take various derivatives and sums over indices.

For the Levi–Civita connection, the following result holds:

$$\boxed{\exists \text{ coordinate system where } g_{\alpha\beta} = \text{constant everywhere} \iff R^\alpha{}_{\beta\gamma\delta} = 0 \text{ everywhere}} \, . \tag{3.69}$$

In one direction, the implication is trivial: if $g_{\alpha\beta}$ is constant, its derivatives are zero, so the connection is zero, so the Riemann tensor is zero. (If the components of a tensor are zero in one coordinate system, they are zero in all coordinate systems.) The proof in the other direction is a bit more involved, and we will not go through it. The idea is to introduce a locally inertial coordinate system at one point, parallel transport the basis vectors to an arbitrary point on the manifold, and show that they constitute a coordinate basis.

We say that the manifold is flat if and only if the Riemann tensor is zero everywhere. If the manifold is not flat, it is **curved**. Given any metric in any coordinate system, we can determine whether or not it describes a flat manifold by calculating the Riemann tensor.

Let us return to the parallel transport equation (3.28). We have said that parallel transport is path independent if and only if the curvature is zero. With the equivalence (3.69), we see this as follows. If the Riemann tensor vanishes, we can choose the connection to be zero everywhere. Then the covariant derivative in the parallel transport equation reduces to a partial derivative, and by the chain rule we get a total derivative with respect to the parameter along the curve:

$$0 = \frac{D}{\mathrm{d}\lambda} T_{\alpha\beta} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \nabla_\mu T_{\alpha\beta} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \partial_\mu T_{\alpha\beta} = \frac{\mathrm{d}}{\mathrm{d}\lambda} T_{\alpha\beta} \ . \tag{3.70}$$

If we integrate over $\lambda$, the result now depends only on the value of $T_{\alpha\beta}$ at the endpoints, i.e. parallel transport is independent of the path when the spacetime is flat. (We have just shown that this is a sufficient condition. It is easy to show that this condition is also necessary.) This means there is a unique way to compare vectors at different points if and only if the curvature is zero.

So, strictly speaking the question"at which velocity is the airplane overhead moving with respect to me?" is meaningless unless you specify along which curve its velocity is transported to your location. The same holds for the velocity of a person walking one meter away from you. In practice, the path-dependence of the result is small if the path along which the velocity is transported only goes through a region where the curvature is small. On Earth (in fact, everywhere in the Solar system) in the vicinity of the present time the path-dependence of parallel transport is tiny.

### 3.3.2 Symmetries of the Riemann tensor

The Riemann tensor has 4 indices, so it has $d^4$ components; for $d = 4$ we have 256 components. However, not all of them independent. Let us see how they are related to the 20 independent degrees of freedom of the metric. The Riemann tensor is by construction antisymmetric in the last two indices. A Riemann tensor corresponding to a general connection has no other symmetries, so it has $6 \times 4^2 = 96$ independent components. The extra symmetries of the Levi–Civita connection reduce the number to 20. The symmetries are most transparent in the version of the Riemann tensor where all indices are down, $R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^\mu{}_{\beta\gamma\delta}$. Symmetries of tensors are independent of coordinates, as they are equality relations between tensors. We can therefore simplify the problem by using local inertial coordinates at point $p$, so

$$\Gamma^\gamma_{\alpha\beta}\big|_p = 0 \qquad \text{(but in general} \quad \Gamma^\gamma_{\alpha\beta,\delta}\big|_p \neq 0) \ .$$

The expression (3.68) for the Riemann tensor now reads

$$\begin{aligned} R_{\alpha\beta\gamma\delta}|_p &= g_{\alpha\mu}(\partial_\gamma \Gamma^\mu_{\delta\beta} - \partial_\delta \Gamma^\mu_{\gamma\beta}) \\ &= \frac{1}{2} g_{\alpha\mu} g^{\mu\nu} \partial_\gamma (g_{\delta\nu,\beta} + g_{\nu\beta,\delta} - g_{\delta\beta,\nu}) \\ &\quad - \frac{1}{2} g_{\alpha\mu} g^{\mu\nu} \partial_\delta (g_{\gamma\nu,\beta} + g_{\nu\beta,\gamma} - g_{\gamma\beta,\nu}) \\ &= \frac{1}{2} (g_{\delta\alpha,\beta\gamma} - g_{\delta\beta,\alpha\gamma} - g_{\gamma\alpha,\beta\delta} + g_{\gamma\beta,\alpha\delta}) \\ &= g_{\delta[\alpha,\beta]\gamma} - g_{\gamma[\alpha,\beta]\delta} \ , \end{aligned} \tag{3.71}$$

where on the second line we have inserted the Levi–Civita connection (3.13) and taken into account $g_{\alpha\beta,\gamma}|_p = 0$. We observe that the Riemann tensor is antisymmetric in the first two indices (in addition to the last two). Furthermore, we see that it is is symmetric under the interchange of the first and the second pair of indices. As tensor equalities do not depend on the coordinate system and the point $p$ is arbitrary, the result holds at all points. To summarise:

$$\begin{array}{ll} \text{1) antisymmetric under} \quad \alpha \leftrightarrow \beta & R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \\ \text{2) antisymmetric under} \quad \gamma \leftrightarrow \delta & R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \\ \text{3) symmetric under} \quad \alpha\beta \leftrightarrow \gamma\delta & R_{\alpha\beta\gamma\delta} = +R_{\gamma\delta\alpha\beta} \end{array} \tag{3.72}$$

For $d = 4$ each antisymmetric pair $\alpha\beta$ and $\gamma\delta$ can take 6 different values: $(4 \times 3)/2 = 6$. The symmetry under pair exchange means that the Riemann tensor is effectively a symmetric $6 \times 6$ tensor. Such a tensor has 6 independent diagonal components and $(6 \times 5)/2 = 15$ independent off-diagonal components, for a total of 21.

A less obvious symmetry of the Riemann tensor that can be read off (3.71) is that the Riemann tensor antisymmetrised in the last three indices vanishes:

$$\text{4)} \quad \boxed{R_{\alpha[\beta\gamma\delta]} = 0} \; . \tag{3.73}$$

Because of the antisymmetry of the last two indices, this condition is equivalent to the vanishing of the cyclic permutation of the last three indices,

$$R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\gamma\delta\beta} + R^{\alpha}{}_{\delta\beta\gamma} = 0 \; . \tag{3.74}$$

The condition (3.73), known as the **first Bianchi identity**, is the last algebraic symmetry of the Riemann tensor. It reduces the number of independent components from 21 to 20, which we know is the maximum possible number.

In addition to these purely algebraic symmetries, the Riemann tensor satisfies an important differential identity, the **second Bianchi identity**, often called just the **Bianchi identity**,

$$\boxed{R^{\alpha}{}_{\beta[\gamma\delta;\epsilon]} = 0} \; . \tag{3.75}$$

Again, due to the antisymmetry of the last two indices of the Riemann tensor, this condition is equivalent to the vanishing of the cyclic permutation of the last three indices,

$$R^{\alpha}{}_{\beta\gamma\delta;\epsilon} + R^{\alpha}{}_{\beta\epsilon\gamma;\delta} + R^{\alpha}{}_{\beta\delta\epsilon;\gamma} = 0 \; . \tag{3.76}$$

This result is analogous to the equation $F_{[\alpha\beta,\gamma]} = 0$ in electromagnetism. It is easy to prove the Bianchi identity. Let us again adopt local inertial coordinates at $p$, so the connection is zero at $p$. We then have

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta;\epsilon} &= (\partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\mu}_{\beta\delta}\Gamma^{\alpha}_{\mu\gamma} - \Gamma^{\mu}_{\beta\gamma}\Gamma^{\alpha}_{\mu\delta})_{;\epsilon} \\ &= (\qquad)_{,\epsilon} + \underbrace{\Gamma(\,) - \Gamma(\,) - \Gamma(\,) - \Gamma(\,)}_{\text{vanish at } p} \\ &= \partial_\epsilon \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\epsilon \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \underbrace{\text{terms of the form } \Gamma\partial\Gamma}_{\text{vanish at } p} \; . \end{aligned} \tag{3.77}$$

We thus have:

$$
\begin{aligned}
&(R^{\alpha}{}_{\beta\gamma\delta;\epsilon} + R^{\alpha}{}_{\beta\delta\epsilon;\gamma} + R^{\alpha}{}_{\beta\epsilon\gamma;\delta})|_p \\
&\quad = \cancel{\partial_\epsilon\partial_\gamma\Gamma^{\alpha}_{\beta\delta}} - \cancel{\partial_\epsilon\partial_\delta\Gamma^{\alpha}_{\beta\gamma}} + \cancel{\partial_\gamma\partial_\delta\Gamma^{\alpha}_{\beta\epsilon}} - \cancel{\partial_\gamma\partial_\epsilon\Gamma^{\alpha}_{\beta\delta}} + \cancel{\partial_\delta\partial_\epsilon\Gamma^{\alpha}_{\beta\gamma}} + \cancel{\partial_\delta\partial_\gamma\Gamma^{\alpha}_{\beta\epsilon}} \\
&\quad = 0 \ . \qquad (3.78)
\end{aligned}
$$

Again, as $p$ is arbitrary, this holds at all points.

### 3.3.3 Ricci tensor and Weyl tensor

Let us decompose the Riemann tensor into its trace and the traceless part. Due to its symmetries, the Riemann tensor has only one independent non-zero first trace, called the **Ricci tensor**, defined as the contraction of the up index and the third down index[3]:

$$
R_{\alpha\beta} \equiv R^{\gamma}{}_{\alpha\gamma\beta} \ . \qquad (3.79)
$$

The Ricci tensor is symmetric, $R_{\beta\alpha} = R_{\alpha\beta}$. The trace of the Ricci tensor (the full trace of the Riemann tensor) is called the **Ricci curvature scalar**, or the **Ricci scalar**, or the **curvature scalar**:

$$
R \equiv g^{\alpha\beta} R_{\alpha\beta} \ . \qquad (3.80)
$$

In 4 dimensions the Ricci tensor has 10 degrees of freedom. The other 10 degrees of freedom of the Riemann tensor are contained in its traceless part, called the **Weyl tensor**. In $d$ dimensions it is

$$
C_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - \frac{2}{d-2}\left(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\right) + \frac{2}{(d-1)(d-2)} g_{\alpha[\gamma}g_{\delta]\beta}R \ . \qquad (3.81)
$$

The Weyl tensor is only defined for $d \geq 3$, and it is identically zero for $d = 3$. It has the same algebraic symmetries as the Riemann tensor, but all of its traces are zero.

The differential symmetry of the Riemann tensor, the (second) Bianchi identity, is reflected in the Ricci tensor as follows. Summing over the indices $\alpha$ and $\gamma$ in (3.76) (i.e. contracting with $\delta^{\gamma}{}_{\alpha}$) and contracting with $g^{\beta\epsilon}$ gives

$$
\begin{aligned}
0 &= 2\nabla_\beta R^{\beta}{}_{\delta} - \nabla_\delta R \\
&= 2\nabla^{\beta}\left(R_{\beta\delta} - \frac{1}{2}g_{\beta\delta}R\right) \ . \qquad (3.82)
\end{aligned}
$$

The combination inside the parenthesis is called the **Einstein tensor**:

$$
G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \ , \qquad (3.83)
$$

for which $\nabla_\alpha G^{\alpha}{}_{\beta} = 0$ by definition.

We have found the tensorial expression for the coordinate-independent information contained in the second derivatives of the metric. From here it is a small step to find the equation of motion for the metric, sourced by matter. This is the topic of the next chapter.

---

[3] Beware: different authors have different conventions for which indices to contract, leading to sign differences.