General relativity I Homework 5 Spring 2026

Due on Monday February 16 by 14.15.

1. **Parallel transport and the Riemann tensor.** Let A, B, C, and D be four points on a manifold, with coordinates x^μ, $x^\mu + \xi^\mu$, $x^\mu + \xi^\mu + \eta^\mu$, $x^\mu + \eta^\mu$, where ξ^μ and η^μ are small, such that ABCDA is a small parallelogram. Parallel transport a vector λ^μ from A to C along the sides of the parallelogram via the two different routes ABC and ADC, and show that the difference between the two resulting vectors is (to leading order in the small quantities)

$$\Delta\lambda^\alpha = -R^\alpha{}_{\beta\gamma\delta}\lambda^\beta\xi^\gamma\eta^\delta \ ,$$

where $R^\alpha{}_{\beta\gamma\delta}$ agrees with the expression in the lecture notes.

2. **Expansion of the metric in local inertial coordinates.** Consider coordinates that are locally inertial at point p. Find the expansion of the metric close to point p, $g_{\alpha\beta}(x^\gamma|_p + \delta x^\gamma)$, to second order in δx^γ in terms of the Riemann tensor. (Hint: use an ansatz for the second derivatives in terms of the Riemann tensor inspired by the expression (3.70).)

3. **Curvature of the spatially flat FLRW universe.** Consider the spatially flat FLRW universe with the metric (3.55).

   a) Calculate the components R_{0i0j} and R_{ijkl} of the Riemann tensor.

   b) Explain by symmetry arguments why these are the only non-zero components. (Apart from components related to the above components by symmetry, like R_{i0j0}.)

   c) Calculate all components of the Ricci tensor, and the Ricci scalar.

   d) Explain by symmetry arguments why the Weyl tensor is zero.

4. **Parallel transport in a FLRW universe.** Let's stay in the spatially flat FLRW universe with the metric (3.55). Take a galaxy that we see today, at time t_0, on the sky. The light left the galaxy at time t, when the galaxy was at coordinate distance x from us. (Note that x and t are not unrelated!) Assume that both the galaxy and we are comoving observers, i.e. at rest with respect to the frame of homogeneity and isotropy. So $u_g^\alpha = \delta^{\alpha 0}$, where g refers to the galaxy, in the tangent space at the galaxy's location, and $u_o^\alpha = \delta^{\alpha 0}$, where o refers to the observer (that's us), in the tangent space at our location.

   a) Parallel transport the galaxy four-velocity to our location via a curve composed of two parts: A) the straight line on the hypersurface of constant time coordinate t to our spatial coordinates, and B) the line of constant spatial coordinates to our time coordinate t_0.

   b) What is the (parallel transported) galaxy three-velocity measured by us?

   c) If you were to do the parallel transport in the reverse order (first along B and then along A), would the result be the same or different? Why?

   d) Parallel transport the galaxy four-velocity instead along the null line connecting the galaxy and us. What is the resulting galaxy three-velocity?

5. **Bonus problem: tides.** (This problem is worth double the normal points, none of which count against the maximum.) Calculate the height of spring tides and neap tides using Newtonian gravity. Neglect hydrodynamic effects, i.e. assume that the Earth is composed of a fluid with no internal non-gravitational forces, so its contours follow the equipotential surfaces of gravity. (In particular, the Earth then does not rotate rigidly.)

(Hint: given the force law $\ddot{x}_i = -\phi_{,i}$, where ϕ is the Newtonian gravitational potential, you need to consider the tensor $\phi_{,ij}$ with contributions from the Sun and the Moon.)

Terminology: Spring tides occur during the full moon or the new moon, and neap tides during the half moon (with the Sun and the Moon at a right angle).

Numbers: Mass and radius of the Earth are 5.97×10^{24} kg and 6 370 km; mass and distance of the Moon are 7.35×10^{22} kg and 384 000 km; mass and distance of the Sun are 1.99×10^{30} kg and 150×10^{6} km. (Approximate the orbits as circular, i.e. neglect time variation in the distance to the Moon and the Sun.)