General relativity I Homework 4 Spring 2026

Due on Monday February 9 by 14.15.

1. **Transformation rule of the connection coefficients.** Derive the transformation rule (3.6) of the connection coefficients.

2. **Rules for derivatives.** Show the following identities in a coordinate basis ($g \equiv \det(g_{\alpha\beta})$).

   a) $g_{,\alpha} = g g^{\mu\nu} g_{\mu\nu,\alpha}$

   b) $\Gamma^{\alpha}_{\alpha\beta} = \frac{1}{2} g^{-1} g_{,\beta} = (\ln \sqrt{-g})_{,\beta}$

   c) $U^{\alpha}{}_{;\alpha} = (\sqrt{-g})^{-1} (\sqrt{-g} U^{\alpha})_{,\alpha}$

   d) $F^{\alpha\beta}{}_{;\beta} = (\sqrt{-g})^{-1} (\sqrt{-g} F^{\alpha\beta})_{,\beta}$, where $F^{\alpha\beta}$ is antisymmetric. Is this true if $F^{\alpha\beta}$ is symmetric?

   e) $\Box f \equiv g^{\alpha\beta} f_{;\alpha\beta} \equiv g^{\alpha\beta} f_{;\alpha;\beta} = (\sqrt{-g})^{-1} (\sqrt{-g} g^{\alpha\beta} f_{,\alpha})_{,\beta}$

3. **Euclidean connection.** The Euclidean metric is

$$\mathrm{d}s^2 = \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

   in Cartesian coordinates, i.e. $g_{ij} = \delta_{ij}$. Consider other coordinates $x'^i(x)$.

   a) Express g'_{ij} in terms of the partial derivatives

$$\frac{\partial x^i}{\partial x'^j} \, .$$

   b) The equation for a straight line is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} = 0 \, .$$

   in Cartesian coordinates. (Are there any requirements for the curve parameter λ?) Transform this equation into the new coordinates x'^i and compare to the geodesic equation. What are Γ'^i_{jk}?

4. **The merry-go-round connection.** Find the connection coefficients for the rotating coordinate system (for Minkowski space) of exercise 1.4.

5. **Parallel transport on a 2-sphere.** Consider a 2-sphere with the metric (a is a constant)

$$\mathrm{d}s^2 = a^2 (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \, .$$

   a) Show that lines of constant longitude ($\varphi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$). (Hint: start by calculating the Christoffel symbols.)

   b) Take a vector with components $U^{\alpha} = (1, 0)$ and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector, as a function of θ?