

Due on Monday February 2 by 14.15.

1. **Commutator of vector fields.** Show that the commutator $[\underline{U}, \underline{V}]$ is a vector field, while $\underline{U} \circ \underline{V}$ is not.
2. **Coordinate transformations.** Consider general coordinate transformations $x^\alpha \rightarrow x'^\alpha(x)$. (We assume that the determinant of the Jacobian matrix is non-zero, as usual.) Show that coordinate transformations on the components $U^\alpha(x)$ of a vector field form a group.
3. **Coordinate patches.** Consider the two-sphere with the metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \frac{dr^2}{1-(r/a)^2} + r^2 d\theta^2$, where a is a positive constant with the dimension of length, $0 < r < a$, and $0 \leq \theta < 2\pi$.
 - a) Show that the coordinates do not cover the full two-sphere. (Hint: consider the area $S = \int d^2x \sqrt{\det g_{\alpha\beta}}$. You can take it as known that the area of the unit two-sphere is 4π .)
 - b) Find a new metric with the coordinate change $r = a \sin \varphi$. Show that the new metric can be extended to smoothly cover the full sphere regularly, apart from the poles and the meridian line.
 - c) What is the problem at the poles? How many coordinate charts do you need to also regularly cover the poles?
4. **Orthonormal coordinates.** Consider Euclidean space in spherical coordinates, with the line element (2.48).
 - a) Write down the orthonormal basis vectors $\underline{e}_{\hat{\theta}}$ and $\underline{e}_{\hat{\varphi}}$.
 - b) Calculate the commutator $[\underline{e}_{\hat{\theta}}, \underline{e}_{\hat{\varphi}}]$.
 - c) What is the commutator of the coordinate basis vectors \underline{e}_θ and \underline{e}_φ ?