General relativity I Homework 3 Spring 2025

Due on Monday February 3 by 14.15.

1. **Commutator of vector fields.** Show that the commutator $[\underline{U}, \underline{V}]$ is a vector field, while $\underline{U} \circ \underline{V}$ is not.

2. **Coordinate transformations.** Consider general coordinate transformations $x^\alpha \to x'^\alpha(x)$. (We assume that the determinant of the Jacobian matrix is non-zero, as usual.) Show that coordinate transformations on the components U^α of a vector form a group.

3. **Coordinate patches.** Consider the two-sphere with the metric $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = \frac{\mathrm{d}r^2}{1-(r/a)^2} + r^2 \mathrm{d}\theta^2$, where a is a positive constant with the dimension of length, $0 < r < a$, and $0 \leq \theta < 2\pi$.

   a) Show that the coordinates do not cover the full two-sphere. (Hint: consider the area $S = \int \mathrm{d}^2x \sqrt{\det g_{\alpha\beta}}$. You can take it as known that the area of the unit two-sphere is 4π.)

   b) Find a new metric with the coordinate change $r = a \sin\varphi$. Show that the new metric can be extended to smoothly cover the full sphere regularly, apart from the poles and the meridian line.

   c) What is the problem at the poles? How many coordinate charts do you need to also regularly cover the poles?

4. **Orthonormal coordinates.** Consider Euclidean space in spherical coordinates, with the line element (2.48).

   a) Write down the orthonormal basis vectors $\underline{e}_{\hat{\theta}}$ and $\underline{e}_{\hat{\varphi}}$.

   b) Calculate the commutator $[\underline{e}_{\hat{\theta}}, \underline{e}_{\hat{\varphi}}]$.

   c) What is the commutator of the coordinate basis vectors $\underline{e}_\theta$ and $\underline{e}_\varphi$?