General relativity I Homework 1 Spring 2025

Due on Monday January 20 by 14.15. Returned via Moodle.

1. **Boost.** Show that the matrix

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   satisfies the condition $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$.

2. **The twin non-paradox.** Consider the twins Alice and Betty. Alice stays on Earth. Betty leaves Alice and travels to Alpha Centauri (distance 4 light years) at the speed $v = 0.8$, turns instantly around, and returns at the same speed. How much have Alice and Betty aged when they meet again? Draw a spacetime diagram of the worldlines of Alice and Betty

   a) in the frame of Alice (K),

   b) in the frames of Betty. Note that at midpoint Betty changes to a frame of constant speed coming back (K''), while the frame where she travelled towards αCen (K') keeps on going.

3. **Null and timelike vectors.**

   a) Show that the sum of two future-pointing null vectors is a future-pointing timelike vector (except when the null vectors are parallel).

   b) Show that any timelike vector can be expressed as a sum of two null vectors. For a given timelike vector, the null vectors are not uniquely determined; what is the freedom in choosing them?

4. **Rotating coordinate system.** Minkowski space metric in Cartesian coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

   Find the metric in the merry-go-round coordinates defined by the transformation

$$\begin{aligned} t &= t' \\ x' &= \sqrt{x^2+y^2}\cos(\varphi - \omega t) \\ y' &= \sqrt{x^2+y^2}\sin(\varphi - \omega t) \\ z' &= z \ , \end{aligned}$$

   where $\varphi \equiv \arctan(y/x)$, and ω is a constant.