**General relativity II** Homework 5 Spring 2026

Due on Monday April 20 by 14.15.

1. **LIGO gravitational wave detector.** A simplified* picture of the operation of the LIGO detector is as follows. Laser light is sent down two straight tunnels. It is reflected back by mirrors at the end of the tunnels. The phase difference of the returned light waves as a function of time is observed. Let's find how it is affected by gravitational waves.

   Take the points where the light rays are combined to be at the origin $(x, y, z) = (0, 0, 0)$, and the mirrors to be at $(L, 0, 0)$ and $(0, L, 0)$. Take the initial beams sent along the arms, with wavelength $\lambda$, to be perfectly in phase. Take the background spacetime to be Minkowski. A gravitational plane wave, with $+$ polarization, amplitude $h_+$, and frequency $f = \omega/(2\pi)$ travels in the $z$ direction, passing through the instrument. We use the transverse gauge.

   a) What is the observed phase shift $\Delta\phi$? (The light travels on a geodesic.) Note that the answer depends on the phase of the gravitational wave at the time $t_e$ when the light beams are sent out.

   b) Take $L = 4$ km, $\lambda = 1064$ nm, and assume that $\Delta\phi = 10^{-10}$ is the smallest phase shift you can observe. What is the smallest amplitude $h_+$ that can be detected? For which frequency $f$ does the instrument achieve this sensitivity?

   (* Here $L$ and $\lambda$ are the real numbers, but the phase shift is in reality amplified by bouncing the light beams back and forth in the tunnels about 300 times before they are rejoined.)

2. **Gravitational waves from a binary system.** On September 14, 2015, LIGO made the first direct detection of gravitational waves. The source was a black hole binary system. Both black holes had mass $30M_\odot$, and the distance to the system was 1 billion light years.

   Take the black holes to be on a circular Newtonian orbit with radius $r = \lambda r_\mathrm{s}$, where $r_\mathrm{s}$ is the Schwarzschild radius and $\lambda > 1$. Take the orbit to be on the $xy$-plane, its centre to be at the origin and the observer to be on the $z$-axis. Approximate the black holes as pointlike and non-rotating, and take the background spacetime to be Minkowski.

   Find the frequency and the amplitude of the emitted gravitational waves as a function of $\lambda$.

3. **Energy loss of a binary system.** (This problem is worth double the usual points.) Consider the binary system of the previous problem.

   a) Starting from (8.129), derive (8.130).

   b) Approximating that the orbit remains circular, find the decay of $\lambda$ as a function of time due to gravitational wave emission. What is the lifetime of the system –defined here as the time to reach $\lambda = 1$, where our approximation must break down– if the initial radius is 1) one astronomical unit or 2) $10r_\mathrm{s}$? How close do the black holes have to start from in order to merge within $10^{10}$ years?

   c) Find the velocity as a function of $\lambda$. Given that we use Newtonian orbits, is there a point before $\lambda = 1$ when the approximation is no longer reliable?

   d) What is the total radiated energy (from the initial radius to $\lambda = 1$) in cases 1) and 2), in units of $M_\odot$?