**General relativity II** Homework 3 Spring 2025

Due on Monday March 31 by 14.15.

1. **Hawking radiation.** In Hawking evaporation total energy is conserved, so black hole mass decreases.

   a) Using the Hawking temperature given in (7.32), find the lifetime of a black hole with initial mass $M$.

   b) What is the temperature and lifetime of a stellar mass ($M = 10M_\odot$) black hole? What is the mass and lifetime of a room temperature ($T = 295$ K) black hole?

   c) If primordial black holes with different masses were produced in the very early universe ($14 \times 10^9$ years ago), what is the minimum initial mass that would have survived to the present?

2. **Time travel.** In the game Quantum Break, you may come across the following metric that describes a general stationary, axisymmetric spacetime with rotation, in cylindrical coordinates ($\phi$ has a period of $2\pi$ as usual):

$$\mathrm{d}s^2 = -A(r,z)\mathrm{d}t^2 + B(r,z)^2\mathrm{d}r^2 + C(r,z)\mathrm{d}\phi^2 + 2D(r,z)\mathrm{d}\phi\mathrm{d}t + B(r,z)^2\mathrm{d}z^2 \ .$$

   a) Take $A > 0$. Show that an observer moving on a circular curve (with constant $r$ and $z$) can go backward in time if certain conditions on the metric functions $A, C$ and $D$ are satisfied.

   b) Show that the maximum amount of time that an observer can travel into the past when going around one full circle is $\Delta t = \frac{2\pi|C|}{D+\sqrt{D^2-A|C|}}$.

   (Hints: Start by demanding that the metric is Lorentzian, $\det(g_{\alpha\beta}) < 0$. Demand then that the curve is timelike, to obtain a condition for the possible angular velocities $\mathrm{d}\phi/\mathrm{d}t$. Show that for certain metric functions the time gain $\mathrm{d}t$ always has the same sign as $\mathrm{d}\phi$, so going in the negative direction in $\phi$ means travelling back in time. Find the most negative value of $\Delta t$.)

3. **Gauge transformation of gravitoelectric and gravitomagnetic fields.**

   a) How do the gravitoelectric and gravitomagnetic fields $\vec{G}$ and $\vec{H}$ transform in a gauge transformation?

   b) There is a subclass of gauge transformations that leave $\vec{G}$ and $\vec{H}$ invariant. What is the condition for such gauge transformations?

4. **Effect of vector perturbations on geodesic motion**. Consider a metric where the scalar and tensor perturbations are zero, but not the vector perturbation. Take the vector perturbation to be constant in time and aligned with the $z$-axis, $w_i = f(x^k)\delta_{iz}$. Find the motion of a test particle that is **not** assumed to be initially at rest. (Assume that the test particle is moving at non-relativistic velocity, so that you can neglect terms that are non-linear in the velocity, but not crossterms between the velocity and the metric perturbation.)

5. **Bonus problem: Alcubierre warp drive.** (This problem is worth double the normal points, none of which are counted against the maximum. It's not terribly difficult, though!) In GR it is possible to design spacetimes with arbitrarily short travel times without exceeding the speed of light. One example is the warp drive proposed by Miguel Alcubierre in 1994. Read his article http://arxiv.org/abs/gr-qc/0009013 and try to understand its main features. Verify the following steps:

   a) Show property (7) of (6). (Note that for $r_s = \pm R$ the result is actually $\frac{1}{2}$.)

   b) Write down the covariant and contravariant form of the metric (8) and its determinant.

   c) Show that the extrinsic curvature $K_{ij}$ defined in (9) gives, for the hypersurface of constant $t$, the result (10).

   d) The volume expansion rate of a hypersurface orthogonal to $n^\alpha$ is $\theta = \nabla_\alpha n^\alpha$. Verify (11) and (12). (Note that in (12), $x_s$ should be $x - x_s$.)

   e) Verify (13) and (15).