Due on Monday March 17 by 14.15.

1. **Action principle for electrodynamics.** The electromagnetic field tensor can be written in terms of the vector potential as

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \ .$$

Starting from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \ ,$$

derive the equation of motion ${F^{\alpha\beta}}_{;\beta} = 0$ by requiring that A_α extremizes the action.

2. **Geometrical optics.** The geometrical optics approximation is valid when the wavelength of light is much smaller than 1) the scale $|R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}|^{-1/2}$ given by components of the Riemann tensor in a local orthonormal frame and 2) the scale over which the amplitude of the wave changes. Show that under these assumptions light travels on null geodesics.

(Hint: Consider an electromagnetic field of the local plane wave form $A_\alpha = \mathrm{Re}(a_\alpha e^{i\theta/\epsilon})$, where $a_\alpha(x)$ and $\theta(x)$ are the amplitude and the phase of the wave, respectively, and $\epsilon \ll 1$ is the ratio of the wavelength to all relevant lengths. The light tangent vector is $k_\alpha = \partial_\alpha \theta$. Consider the equation of motion derived in problem 1 to leading order in ϵ, taking the Lorenz gauge condition $\nabla_\alpha A^\alpha = 0$. This gives the null condition $k_\alpha k^\alpha = 0$, and its covariant derivative gives the geodesic equation.)

3. **Deriving Newton's second law.** Consider an ideal fluid, with $T_{\alpha\beta} = (\rho + P) u_\alpha u_\beta + P g_{\alpha\beta}$. Consider observers comoving with the fluid (i.e. with four-velocity u^α). Show that their four-acceleration $a^\alpha = u^\beta \nabla_\beta u^\alpha$ is

$$a^\alpha = -\frac{1}{\rho + P} h^{\alpha\beta} \nabla_\beta P \ ,$$

where $h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta$. (Hint: start from the continuity equation.)

4. **Energy conditions.** Energy conditions, which set some 'reasonableness criteria' on the matter content, play an important role in general relativity. Let's consider two of them.

1) Weak energy condition: $T^{\alpha\beta} v_\alpha v_\beta \geq 0$, where $\underline{v}$ is an arbitrary future oriented timelike unit vector field (meaning $v^0 > 0$ and $\underline{v} \cdot \underline{v} = -1$).

2) Dominant energy condition: $T^{\alpha\beta} v_\alpha w_\beta \geq 0$, where $\underline{v}$ and $\underline{w}$ are two timelike unit vector fields that are co-oriented, but otherwise arbitrary. (Co-orientation means, for timelike vectors, $\underline{v} \cdot \underline{w} < 0$.)

a) Show that for an ideal fluid 1) is equivalent to to the conditions $\rho \geq 0$ and $\rho + p \geq 0$; and that 2) is equivalent to 1) plus the condition $\rho \geq |p|$.

(Hint: write the energy-momentum tensor as $T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + p g_{\alpha\beta}$ and the four-vectors as $v^\alpha = \gamma(u^\alpha + r^\alpha)$, with $\underline{u} \cdot \underline{r} = 0$ and $0 < \underline{r} \cdot \underline{r} < 1$, and similarly $w^\alpha = \tilde\gamma(u^\alpha + \tilde r^\alpha)$.)

b) Explain the physical meaning of these conditions.

5. **Bonus problem: Palatini formulation.** (This problem is worth double the normal points, none of which count against the maximum.) In the Palatini formulation, the Einstein–Hilbert action is

$$S = \frac{1}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^4 x \sqrt{-g} g^{\alpha\beta} R_{\alpha\beta}(\Gamma, \partial\Gamma) \ ,$$

where the metric $g^{\alpha\beta}$ and the connection $\Gamma^\gamma_{\alpha\beta}$ are independent variables. Assuming that the connection is torsion-free, $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$, it contains 40 degrees of freedom. Show that varying the action with respect to the metric and the connection independently gives the Einstein equation and the condition that $\Gamma^{\gamma}_{\alpha\beta}$ is the Levi-Civita connection. Remember that Stokes' theorem does not apply with the general covariant derivative, only with the covariant derivative defined with the Levi–Civita connection.

(Hint: for the connection calculation, you may find it useful to write $\Gamma^{\gamma}_{\alpha\beta} = \mathring{\Gamma}^{\gamma}_{\alpha\beta} + L^{\gamma}{}_{\alpha\beta}$, where $\mathring{\Gamma}^{\gamma}_{\alpha\beta}$ is the Levi–Civita connection and $L^{\gamma}{}_{\alpha\beta}$ is a tensor, and show that the equation of motion gives $L^{\gamma}{}_{\alpha\beta} = 0$.)