GENERAL RELATIVITY I Homework 1 Spring 2024

Due on Monday January 22 by 14.15. Return via Moodle, or to the metal box marked "General relativity" in the A corridor on the second floor.

1. **Boost.** Show that the matrix

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

satisfies the condition $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$.

2. **The twin non-paradox.** Consider the twins Alice and Betty. Alice stays on Earth. Betty leaves Alice and travels to Alpha Centauri (distance 4 light years) at the speed $v = 0.8$, turns around, and returns at the same speed. How much have Alice and Betty aged when they meet again? Draw a spacetime diagram of the worldlines of Alice and Betty

   a) in the frame of Alice (K),

   b) in the frame of Betty traveling towards αCen (K'),

   c) in the frame of Betty returning (K'').

3. **Null and timelike vectors.**

   a) Show that the sum of two future-pointing null vectors is a future-pointing timelike vector (except when the null vectors are parallel).

   b) Show that any timelike vector can be expressed as a sum of two null vectors. For a given timelike vector, the null vectors are not uniquely determined; what is the freedom in choosing them?

4. **Rotating coordinate system.** Minkowski space metric in Cartesian coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

Find the metric in the merry-go-round coordinates defined by the transformation

$$\begin{aligned} t &= t' \\ x' &= \sqrt{x^2+y^2}\cos(\varphi - \omega t) \\ y' &= \sqrt{x^2+y^2}\sin(\varphi - \omega t) \\ z' &= z \ , \end{aligned}$$

where $\varphi \equiv \arctan(y/x)$, and ω is a constant.