Università degli studi di Pavia
Facoltà di scienze matematiche, fisiche e naturali
CORSO DI LAUREA IN FISICA

Dust Solutions of Einstein Field Equation in Non-Tilted Cosmological Models

Tesi di laurea di
Micol Previde Massara

Relatori
Prof. Mauro Carfora
(Università degli studi di Pavia)
Dr. Syksy Räsänen
(University of Helsinki)

Anno Accademico 2012/2013

ABSTRACT

The early universe is characterized by a model based on three hypothesis: homogeneity and isotropy, ordinary matter and standard gravity. But, in this way, we obtain predictions that are in conflict with observations because they cannot explain the late time expansion of the universe. Then at least one of the three assumptions must be wrong. According to ΛCDM, the agreement with data is restored by introducing a new term in the Einstein equation. Depending on which side of the Einstein equation this term is put in, we have two possibilities: to modify gravity or assume the existence of dark energy. However, a valid alternative is provided by constructing models in which inhomogeneities and/or anisotropies are present. The effect of inhomogeneities/anisotropies on the expansion of the universe is referred to as backreaction. For this reason, we study inhomogeneous and anisotropic Szekeres models and spatially homogeneous but anisotropic Bianchi universes.

In this work we start by introducing the covariant formalism. Then we describe the averaging procedure in the framework of Buchert's approach (considering only the dust case) and we obtain the Buchert equations, from which we can define the backreaction variable. Finally we study dust solutions of the Einstein field equation. First we analyze the Szekeres metric, then we pass to the Bianchi models. In a recent work by M. Lavinto, S. Räsänen and S.J. Szybka the Szekeres metric has been used to construct a Swiss Cheese dust model with Szekeres holes and to prove that, under certain assumption, this model can provide a large backreaction, i.e. inhomogeneity has a significant effect on the average expansion rate of the universe.

Contents

Chapter 1

Introduction

Cosmology is the branch of physics that studies the universe as a whole. About 100 years ago, the prevailing view of the cosmos was that the universe was static and consisted entirely of the Milky Way galaxy, but soon started a revolution by means of observations by Leavitt, Hubble and others which showed that the universe is actually expanding and contains many distant galaxies.

In the twenties and thirties Friedmann, Lemaître and other theorists managed to explain the expansion of the universe by using an exact solution of the Einstein field equation: the *Friedmann-Lemaître-Robertson-Walker* (FLRW) *metric*, that is based on the hypothesis of *exact* homogeneity and isotropy of the universe.

This expansion pointed to an extremely hot origin of the universe, the Big Bang, and Gamow and others showed how this should leave a thermal relic radiation, the cosmic microwave background (CMB), and also how nucleosynthesis of the lights element would take place in the hot early universe. However, it took many decades of observations to catch up and confirm the theory and to lay the basis for further developments.

In the last few decades, cosmology has become an observationally based physical science due to the growth in data from increasingly high-precision experiments, e.g. the Cosmic Background Explorer (COBE), that detected the large-angle anisotropies in the CMB temperature, and its successors, the Wilkinson Microwave Anisotropy Probe (WMAP) and the Planck telescope.

While the measurements seem to indicate an early universe close to homogeneity and isotropy (that can be described by a FLRW model with linear perturbations), the observations of type Ia supernovae, CMB anisotropies and large-scale structure provide data in disagreement with the predictions obtained for the late time universe using the FLRW model (this disagreement rests on observations of cosmological distances and average expansion rate). The FLRW models can account for these observations only either consider modified gravity or introducing a new form of energy with negative pressure, *dark energy*.

In general relativity the matter that fills the universe on a cosmic scale is described through fluid dynamics.

Since Einstein's theory does not tell us which kind of matter (i.e. fluid) and which metric better characterize the universe, but we know only that Einstein equation must be valid, then the particular *cosmological model* provides us with this information.

The current "standard model of cosmology" is the inflationary cold dark matter model with cosmological constant (ΛCDM). It describes dark matter as being cold (i.e. made up of non-relativistic particles) and it is based on a remarkably small number of cosmological parameters.

In this picture the universe looks like this. Approximately 13.8 billion years ago the universe was in a very hot, dense, rapidly expanding state. This initial state is often referred to as the *Big Bang*. Then a very short period of rapid expansion, called *inflation*, follows. After the inflationary epoch, the universe is described with ΛCDM model, using the FLRW metric.

From the end of inflation up to the universe was 50000 years old we had the *radiation dominated era*, because the density of radiation exceed the density of matter. When the universe was around 380000 years old *recombination* occurred (i.e. electrons and protons started to form hydrogen atoms) and soon the photons decoupled from baryonic matter. Then the universe became transparent to photons, which constitute the radiation we observe today, the CMB. After recombination era the growth of baryonic structures could begin.

The epoch started when the universe was about 10 billion years old (that lasts until today) is called the *dark energy dominated* era, because, according to ΛCDM, the density of dark energy exceeds the density of matter.

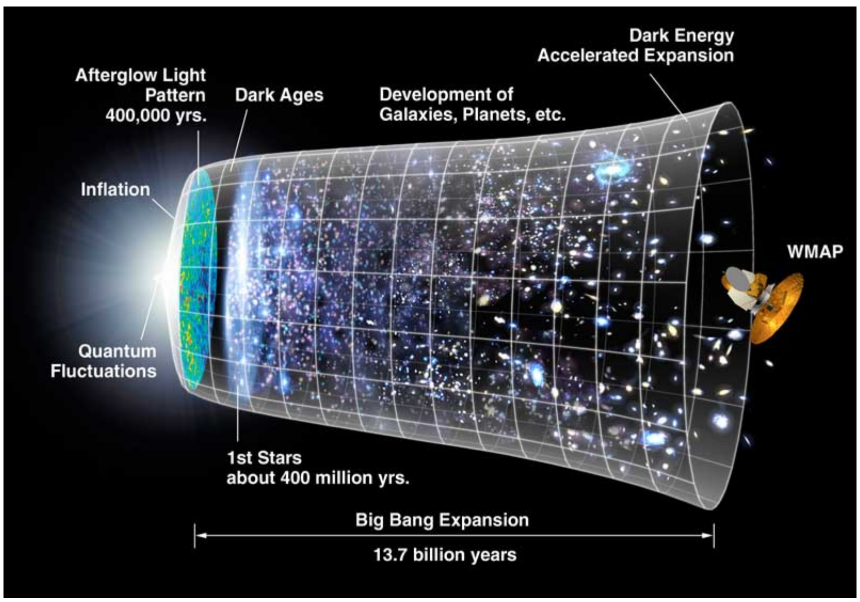

Figure 1.1: Illustration of evolution of the universe from the Big Bang (left) according to WMAP observations. The time-coordinate increases to the right.

In the framework of ΛCDM, dark energy is currently estimated to be about 68% of the energy density of the present universe, dark matter is considered to constitute the 27%, whereas the remaining 5% is due to ordinary matter.

The early universe is characterized by a model based on three hypothesis: homogeneity and isotropy (FLRW metric), ordinary matter (matter with non-negative pressure) and standard gravity. But, in this way, we obtain predictions that are in conflict with observations because they cannot explain the late time expansion of the universe. Then at least one of the three assumptions must be wrong.

According to ΛCDM, the agreement with data is restored by introducing a new term in the Einstein equation. We have two possibilities: we can insert it in the geometric side of the equation and consider it a modification of the law of gravity, or we can include it in the matter side. Both the aforementioned choices mean we have to discard one of the three initial hypothesis: we must modify standard gravity or add exotic matter, with negative pressure (dark energy), in addition to ordinary matter. Several methods to modify gravity have been studied (even if the use of cosmological constant is the easiest way of doing it) and we have many candidates for dark energy too (for more details see [8] and the references in it).

However, while trying to account for the late time expansion (which is accelerating) of the universe, we did not consider a third hypothesis: to construct models in which inhomogeneities and/or anisotropies are present. In this way we can make predictions that fit observations without modifying gravity or introducing dark energy.

This idea emerges from the fact that the late time (real) universe is far from exact homogeneity and isotropy due to the formation of non-linear structures, i.e. galaxies, clusters of galaxies, voids, etc. The effect of inhomogeneities/anisotropies on the expansion of the universe is referred to as *backreaction* and is what we study in this work. Because of backreaction, the average expansion rate can accelerate even in a dust universe where the local expansion rate decelerates everywhere. The fact that backreaction can provide acceleration is proved in some works by S. Räsänen and others (see for example [35], [46], [47], [49] and [50]).

Any mathematical description of a physical system depends on an averaging scale that is usually not made explicit and the world looks completely different depending on the scale of representation we choose. For example, when a fluid is described as a continuum we are using an averaging scale large enough to neglect the individual molecules, but, if we shrink the averaging scale to the molecular one, then we should change the description of the fluid and consider the individual molecules.

The same is true for cosmology and astronomy: the universe may be statistically homogeneous and isotropic above a certain scale, but on small and intermediate scales (up to several hundreds of Mpcs [38]) it is highly inhomogeneous, quite unlike a FLRW universe (see figure 1.2). Large-scale models of the cosmos (such as the standard model of cosmology) are coarse-grained representations of what is actually there. Then, an important issue for cosmology is how to relate the observations with the homogeneous and isotropic models and whether the smaller scale structures influence the dynamics of the universe on larger scales.

In cosmology usually we talk about an exactly homogeneous and isotropic "background" on which galaxies and structures are seen as perturbations. Then the following question arise: is this the same as starting with a more detailed truly inhomogeneous metric of spacetime and progressively smoothing it until we get to this background? The answer is no. In fact the non-linearity of the Einstein equation ensures they are not. Then the problem is whether the difference is important and this has been subject of controversy. For example, Buchert's approach (see [4] and [39]) shows backreaction from inhomogeneity can potentially explain the observed accelerating expansion of the universe without introducing dark energy, but we should also mention that other physicists claim that backreaction effect is completely negligible (see for instance [40], [41] and [5]).

There are different approaches to compute the effects of backreaction. Some of them consist on starting with an inhomogeneous model and using an averaging process to produce an approximately FLRW model, others calculate the backreaction effect perturbatively, by averaging over structure in the standard model.

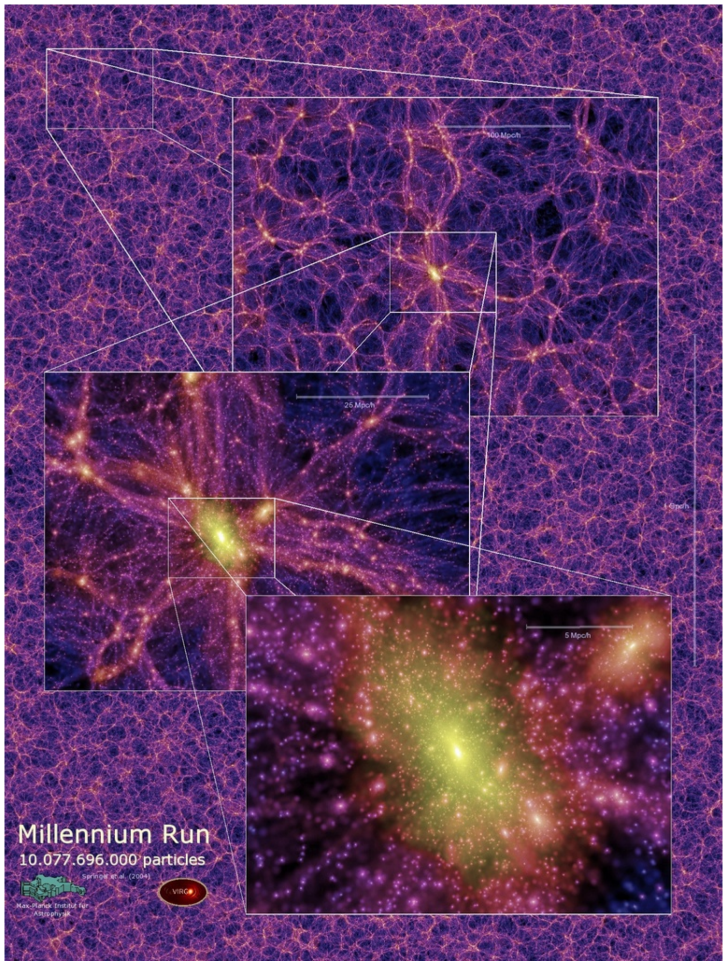

Figure 1.2: Millenium simulation. The figure shows a projected density field for a 15 Mpc/h thick slice of the redshift z=0 output. The overlaid panels zoom in by factors of 4 in each case, enlarging the regions indicated by the white squares. Yardsticks are included as well.

Now, let us give a description of the averaging problem.

The key difference between the averaged models and the standard model is due to the fact the spatial average of a tensor field over a domain comoving with the fluid does not commute with the time evolution of the tensor field, then we run into the non commutativity of the averaging procedure and the introduction of the inhomogeneous quantities in the Einstein equation.

In order to better explain this concept, we can start with a metric $g_{\mu\nu}^{local}$ that gives a realistic description of the universe on a small scale (we might represent individual stars and planets in the universe). By averaging it (by means of a smoothing procedure) we find a metric $g_{\mu\nu}^{gal}$ at a bigger scale (for instance where galaxies are well represented but individual stars are invisible). If we average this metric again to a further bigger scale we obtain $g_{\mu\nu}^{lss}$ (in our example now only large scale structures are considered). The largest scale possible (completely smoothed) will be described by a FLRW model, with a metric $g_{\mu\nu}^{cosm}$, where all traces of inhomogeneity have been removed. We will have similar averages for the stress energy tensor $T_{\mu\nu}$.

Now, we can assume that Einstein equation holds at the "local scale", i.e. we have

$$R_{\mu\nu}^{local} - \frac{1}{2}R^{local}g_{\mu\nu}^{local} + \Lambda g_{\mu\nu}^{local} = T_{\mu\nu}^{local}\,.$$

Since the averaging process does not commute with the Einstein equation, then at a larger (e.g. galaxies) scale we find

$$R_{\mu\nu}^{gal} - \frac{1}{2}R^{gal}g_{\mu\nu}^{gal} + \Lambda g_{\mu\nu}^{gal} = T_{\mu\nu}^{gal} + E_{\mu\nu}^{gal}\,,$$

where the new term $E_{\mu\nu}^{gal}$ is due to the non commutativity and it represents the effect of averaging out smaller scale structures.

For the other bigger scales there will appear similar terms $E_{\mu\nu}^{lss}$ and $E_{\mu\nu}^{cosm}$. This is the backreaction from the small scales to the larger scales.

As outlined before, to compute the average of a quantity in the theory of general relativity is a very involved problem: there are no preferred time-slices we can average over, non-linearity of the Einstein equation and the fact that we only know how to do the average of scalar quantities.

For this, the problem of averaging inhomogeneous cosmologies has been studied by many authors.

In this thesis we follow Buchert's approach, by concentrating on averaging scalars on spatial hypersurfaces[1]. In Newtonian gravity, an averaging procedure is developed by T. Buchert and J. Ehlers in [42]. Another approach to covariantly averaging was started and explored by R.M. Zalaletdinov, see [5]. A further rigorous approach is based on the deformation of the spatial metric of initial data sets along its Ricci flow (see [43] and [44]). By following the work by M. Carfora and K. Piotrkowska [7], in which a renormalization group approach to coarse-graining in cosmology has been studied, this method is a way of smoothing a spacetime that can be linked to the standard renormalization group of effective field theories.

Other methods to study backreaction (model building, perturbative approaches) can be found in [5].

In this thesis we start (chapter 2) by introducing the $3+1$ covariant formalism and by giving a fluid dynamical description. We also summarize some important results of general relativity. These are the basis for what we study in the rest of the work.

The third chapter is dedicated to explain in detail what backreation is in the Buchert's approach. Here we treat only the dust case, since this is what we are interested in. After a brief introduction, we describe the averaging procedure and give the Buchert equations, from which we obtain the expression for the backreaction variable. Finally we derive the integrability condition, i.e. the relation between the evolution of the spatial curvature and the backreaction.

Then we pass to describe dust solutions of the Einstein field equation.

In chapter 4, we study the Szekeres metric. This is the most general known inhomogeneous dust solution: it describes an inhomogeneous and anisotropic spacetime. The Lemaître-Tolman-Bondi (LTB) model

[1] We will give a complete description of this averaging method in chapter 3.

is contained in the Szekeres metric as the spherically symmetric special case. We first give the solutions of the Einstein equation for the Szekeres metric and then its physical interpretation. After that, we calculate the dynamical quantities. Finally, after a brief discussion of the LTB model, we compute the average quantities and the backreaction variable.

The last chapter is focused on Bianchi models, which describe homogeneous but anisotropic universes. We start giving a modern classification of Bianchi models. Then we specialize in non-tilted Bianchi models: after a general discussion of the spatially homogeneous model in the synchronous system, we study all the different types of dust metrics, with their properties.

The two appendices are dedicated to a classification of cosmological models by means of their symmetries and to the original Bianchi's classification of metrics respectively.

Notations

- Spacetime indices are denoted by Greek letters (α, β, γ, ...) and run from 0 to 3.
- Spatial indices are denoted by Latin letters (a, b, c, ...) and run from 1 to 3.
- The metric tensor is denoted by $g_{\mu\nu}$.
- The signature of the metric is $(-+++)$.
- We use Einstein summation convention.
- We employ units such that the speed of light and the Newton gravitational constant satisfy $c = 1 = \frac{8\pi G_N}{c^2}$.
- Symmetrization is denoted by round brackets, e.g. $T_{(\mu\nu)} \doteq \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$.

- Antisimmetrization is denoted by square brackets, e.g. $T_{[\mu\nu]} \doteq \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$.
- The partial derivative with respect to x^μ is denoted by ∂_μ or by a comma, e.g. $u_{\nu,\mu} = \partial_\mu u_\nu$.
- The covariant derivative with respect to x^μ is denoted by ∇_μ or by a semicolon, e.g. $u_{\nu;\mu} = \nabla_\mu u_\nu$.

Chapter 2

The covariant formalism

The goal of cosmology is to find a model in order to describe the universe in the most suitable way. Moreover from any such model the observational predictions have to be extracted. Since Einstein field equations are not particularly intuitive, they are usually splitted in a new set of equations (by keeping their covariant character), introducing at the same time directly measurable quantities.

In order to have a complete cosmological model, in addition to a metric defined on a manifold, we must specify a family of observers spread out in spacetime. The velocity of these observers is described by a velocity vector field that gives rise to a family of preferred world lines representing their motion.

By means of the velocity vector field we can split a tensor in its parallel and orthogonal to the world lines parts. This is the so called $3+1$ splitting of quantities. Moreover, the splitting is covariant because we can define the velocity vector field uniquely and without any coordinates.

In this chapter, by following [1], [2], [3], [8] and [37], we summarize the $3+1$ covariant approach together with the evolution and constraint equations that arise from Einstein equation (here we consider only the dust case, for a more general description see the references given above).

2.1 Curvature tensor

In general relativity, we describe spacetime as a manifold M on which is defined a Lorentzian metric $g_{\mu\nu}$.

The curvature of spacetime is described by the *Riemann curvature tensor*, defined as

$$R^{\alpha}{}_{\beta\gamma\delta} v^{\beta} \doteq \left(\nabla_{\gamma}\nabla_{\delta} - \nabla_{\delta}\nabla_{\gamma}\right) v^{\alpha} \,, \tag{2.1}$$

where v^{α} is an arbitrary vector field. The Riemann tensor's components satisfy the following properties:

$$\begin{aligned} R_{[\alpha\beta][\gamma\delta]} &= R_{\alpha\beta\gamma\delta} \,, & (2.2a) \\ R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta} \,, & (2.2b) \\ R_{\alpha[\beta\gamma\delta]} &= 0 \,. & (2.2c) \end{aligned}$$

The Riemann tensor has twenty independent components[1]. Of these, ten are the Ricci tensor's components, while the remaining ten are the Weyl tensor's components, where the Ricci and Weyl tensors are defined as

$$\begin{aligned} R_{\beta\delta} &\doteq R^{\alpha}{}_{\beta\alpha\delta} = -R^{\alpha}{}_{\beta\delta\alpha} \,, & (2.3) \\ C_{\alpha\beta\gamma\delta} &\doteq R_{\alpha\beta\gamma\delta} - \left(g_{\alpha[\gamma} R_{\delta]\beta} + g_{\beta[\gamma} R_{\delta]\alpha}\right) + \frac{1}{3} R g_{\alpha[\gamma} g_{\delta]\beta} \,. & (2.4) \end{aligned}$$

In the last equation R is the Ricci scalar:

$$R \doteq R^{\alpha}{}_{\alpha} \,. \tag{2.5}$$

The Ricci tensor is symmetric, i.e.

$$R_{\beta\delta} = R_{\delta\beta} \,. \tag{2.6}$$

[1] In a n-dimensional manifold the number of algebraically independent components of the Riemann tensor is $\frac{1}{12} n^2 (n^2 - 1)$.

We can think of it as the trace part of the Riemann tensor $R_{\alpha\beta\gamma\delta}$.

The Weyl tensor $C_{\alpha\beta\gamma\delta}$ has all the symmetries of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ and the following property:

$$C^{\alpha\beta}{}_{\alpha\gamma} = 0\,. \tag{2.7}$$

We can think of it as the trace-free part of the Riemann tensor.

In general relativity, the Weyl tensor describes that part of the gravitational field that propagates into vacuum (is a solution of the vacuum Einstein field equation) and is detectable outside the sources, gravitational waves among other things.

The Ricci tensor $R_{\beta\delta}$, instead, is determined locally at each point by the energy-momentum tensor through the Einstein equation and it vanishes identically in the vacuum case[2].

The two tensors $C_{\alpha\beta\gamma\delta}$ and $R_{\beta\delta}$ completely represent the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$, which can be decomposed as

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - g_{\alpha[\delta}R_{\gamma]\beta} - g_{\beta[\gamma}R_{\delta]\alpha} - \frac{R}{3}g_{\alpha[\gamma}g_{\delta]\beta}\,. \tag{2.8}$$

2.2 Comoving coordinates

In cosmology we can make a physically motivated choice of preferred motion due to the matter components (e.g. the CMB frame). Such a choice corresponds to a preferred four-velocity field u^μ that generates a family of preferred world lines[3].

To describe the spacetime geometry it is convenient to use comoving coordinates, adapted to the fundamental world lines.

Comoving coordinates $\left(t,\, x^i\right)$ are defined by choosing a surface S that intersects each world line only once and labelling the intersections

[2]As we can see from Einstein field equation (2.68), this is true only for vanishing cosmological constant, otherwise, in the vacuum case, the Ricci tensor is proportional to the metric tensor.

[3]This family of preferred world lines is called a *congruence* (for the definition see [2, sec. 9.2]).

by coordinates x^i. Then we extend this labelling off the surface S by maintaining the same tagging for the world lines at later and earlier times. In this way the flow lines in spacetime are the curves $x^i = const$ and t defines the time coordinate along the world lines.

There are two coordinate freedoms which preserve the form: time transformations $\left(t' = t'\left(t,\, x^i\right),\, x^{i'} = x^i\right)$, corresponding to a new choice of time surfaces, and relabelling of the world lines $\left(t' = t,\, x^{i'} = x^{i'}\left(x^i\right)\right)$, by choosing new coordinates on the initial surface.

A particular and often convenient choice for the time coordinate t is the normalized time $s = s_0 + \tau$, where τ is the proper time measured along the fundamental world lines from the surface S and s_0 is an arbitrary constant. With this choice, $x^\mu = \left(s,\, x^i\right)$ are normalized comoving coordinates, s measuring the proper time from the surface S.

It is worth giving some quantities in terms of comoving coordinates.

2.2.1 Four-velocity

The preferred matter motion implies a preferred four-velocity at each point. If the preferred world lines are given in terms of local coordinates $\left(x^\mu = x^\mu\left(\tau\right)\right)$, then the preferred four-velocity is the unit timelike vector

$$u^\mu \doteq \frac{dx^\mu}{d\tau}\,, \tag{2.9}$$

where τ is the proper time along the world lines. The normalization is

$$u_\mu u^\mu = -1\,. \tag{2.10}$$

In normalized comoving coordinates[4] $x^\mu = \left(s,\, x^i\right)$ equation (2.9) becomes

$$u^\mu = \delta^\mu{}_0\,, \tag{2.11}$$

[4]In the rest of this work we will refer to *normalized comoving coordinates* simply as *comoving coordinates*, omitting the word "normalized".

where $\delta^{\mu}{}_{0}$ is the Kronecker delta.

Throughout this work we use this particular choice of coordinates.

2.2.2 Time derivative and acceleration vector

The *covariant time derivative* along the flow lines of any tensor $S^{\alpha\ldots\beta}{}_{\gamma\ldots\delta}$ is

$$\dot{S}^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} \doteq u^{\mu}\nabla_{\mu}S^{\alpha\ldots\beta}{}_{\gamma\ldots\delta}\,. \tag{2.12}$$

Using the definition of covariant derivative, this can be written as

$$\begin{aligned}\dot{S}^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} &= \frac{\partial}{\partial\tau}S^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} + u^{\mu}\Gamma^{\alpha}{}_{\mu\nu}S^{\nu\ldots\beta}{}_{\gamma\ldots\delta} + \ldots + u^{\mu}\Gamma^{\beta}{}_{\mu\nu}S^{\alpha\ldots\nu}{}_{\gamma\ldots\delta} \\ &\quad - u^{\mu}\Gamma^{\nu}{}_{\mu\gamma}S^{\alpha\ldots\beta}{}_{\nu\ldots\delta} - \ldots - u^{\mu}\Gamma^{\nu}{}_{\mu\delta}S^{\alpha\ldots\beta}{}_{\gamma\ldots\nu}\,. \end{aligned} \tag{2.13}$$

We define the *acceleration vector* $\dot{u}^{\mu}$ as

$$\begin{aligned}\dot{u}^{\mu} &= u^{\nu}\nabla_{\nu}u^{\mu} \qquad (2.14)\\ &= \frac{\partial}{\partial\tau}u^{\mu} + u^{\nu}\Gamma^{\mu}{}_{\nu\rho}u^{\rho}\,.\end{aligned}$$

The acceleration vector represents the degree to which the matter moves under the influence of non-gravitational interaction and (2.14) vanishes iff the flow lines are geodesic, if matter moves under inertia and gravity alone (matter is in free fall).

In addition, from (2.14) and the normalization condition (2.10) it follows that

$$\dot{u}^{\mu}u_{\mu} = 0\,. \tag{2.15}$$

In comoving coordinates, since $u^{\mu} = \delta^{\mu}{}_{0}$, the acceleration vector can be expressed in terms of the Christoffel symbols:

$$\dot{u}^{\mu} = \Gamma^{\mu}{}_{00}\,. \tag{2.16}$$

2.2.3 Volume element

Since $(M,\, g)$ is a pseudo-Riemannian manifold by construction, we can write the natural volume form as

$$\begin{aligned} \eta &= \frac{1}{4!}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\sqrt{|g|}dx^{\alpha_1}\wedge dx^{\alpha_2}\wedge dx^{\alpha_3}\wedge dx^{\alpha_4} \\ &= \sqrt{|g|}dx^1\wedge dx^2\wedge dx^3\wedge dx^4\,, \end{aligned} \tag{2.17}$$

where $g \doteq \det(g_{\mu\nu}) < 0$ and $\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}$ is the four-dimensional Levi-Civita symbol.

Thus the four-dimensional volume element is

$$\eta_{\alpha\beta\gamma\delta} = \frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}\sqrt{|g|}\,, \tag{2.18}$$

so $\eta_{\alpha\beta\gamma\delta} = \eta_{[\alpha\beta\gamma\delta]}$.
From (2.18) we define the three-dimensional volume element:

$$\eta_{\alpha\beta\gamma} \doteq \eta_{\alpha\beta\gamma\delta}u^{\delta}\,, \tag{2.19}$$

with the following properties

$$\eta_{\alpha\beta\gamma} = \eta_{[\alpha\beta\gamma]} \quad \text{and} \quad \eta_{\alpha\beta\gamma}u^{\gamma} = 0\,. \tag{2.20}$$

2.3 Projection tensors

The existence of a preferred velocity at each point implies the existence of preferred rest frames at each point; locally these define surfaces of simultaneity for the fundamental observers.

We define unique *projection tensors*:

$$U_{\mu\nu} \doteq -u_\mu u_\nu \tag{2.21}$$

and

$$h_{\mu\nu} \doteq g_{\mu\nu} + u_\mu u_\nu\,. \tag{2.22}$$

Tensor (2.21) projects parallel to the four-velocity vector u^μ, while (2.22) projects onto the instantaneous rest space of an observer moving with four-velocity[5] u^μ.

The two tensors above satisfy the following properties:

$$U^\mu{}_\gamma U^\gamma{}_\nu = U^\mu{}_\nu\,, \qquad U^\mu{}_\mu = 1\,, \qquad U_{\mu\nu}u^\nu = u_\mu\,, \tag{2.23}$$

$$h^\mu{}_\gamma h^\gamma{}_\nu = h^\mu{}_\nu\,, \qquad h^\mu{}_\mu = 3\,, \qquad h_{\mu\nu}u^\nu = 0\,. \tag{2.24}$$

We also define a *fully orthogonally projected covariant derivative* $\hat{\nabla}$ for any tensor $S^{\alpha\ldots\beta}{}_{\gamma\ldots\delta}$ as

$$\hat{\nabla}_\mu S^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} \doteq h^\alpha{}_\rho h^\lambda{}_\gamma \ldots h^\beta{}_\sigma h^\kappa{}_\delta h^\tau{}_\mu \nabla_\tau S^{\rho\ldots\sigma}{}_{\lambda\ldots\kappa}\,, \tag{2.25}$$

with total projection on all free indices. The three-dimensional covariant derivative $\hat{\nabla}$ is defined as operator in a three-dimensional manifold only if the vorticity vanishes[6]. When vorticity in non-zero, $\hat{\nabla}$ is only an operator in the tangent hyperplane at each point and not on a manifold.

Because of the definition (2.22) we can split the metric tensor in its parallel and perpendicular to u^μ parts, given by

$$g_{\mu\nu} = h_{\mu\nu} + U_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu\,. \tag{2.26}$$

In this way the interval ds^2 can be decomposed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = h_{\mu\nu}dx^\mu dx^\nu - (u_\mu dx^\mu)^2\,. \tag{2.27}$$

Finally we introduce the brackets $\langle\cdot\rangle$ to denote the *orthogonal projections of vectors* and the *orthogonally projected symmetric trace-free*

[5] If we consider another four-velocity vector n^μdifferent from u^μ, we can construct two analogous projectors (2.21) and (2.22). In general, the two observers, moving with four-velocity n^μ and u^μ respectively, measure different values of the same physical quantity because of the different projectors.

[6] For the definition of vorticity see section 2.4 and for an exhaustive discussion about its role see [8, chapter 3].

part of tensors

$$v^{\langle\mu\rangle} \doteq h^{\mu}{}_{\nu} v^{\nu}, \qquad S^{\langle\mu\nu\rangle} \doteq \left[h^{(\mu}{}_{\gamma} h^{\nu)}{}_{\delta} - \frac{1}{3} h^{\mu\nu} h_{\gamma\delta} \right] S^{\gamma\delta} . \tag{2.28}$$

The same brackets are also used to indicate *orthogonal projections of covariant time derivatives* along u^μ (called *Fermi derivatives*):

$$\dot{v}^{\langle\mu\rangle} \doteq h^{\mu}{}_{\nu} \dot{v}^{\nu}, \quad \dot{S}^{\langle\mu\nu\rangle} \doteq \left[h^{(\mu}{}_{\gamma} h^{\mu)}{}_{\delta} - \frac{1}{3} h^{\mu\nu} h_{\gamma\delta} \right] \dot{S}^{\gamma\delta} . \tag{2.29}$$

2.4 The kinematic quantities

Referring to the previous section, we are now able to split the covariant derivative of the four-velocity u^μ into its irreducible parts, defined by their symmetry properties[7]:

$$u_{\mu;\nu} = \nabla_\nu u_\mu = -u_\nu \dot{u}_\mu + \hat{\nabla}_\nu u_\mu = -u_\nu \dot{u}_\mu + \frac{1}{3} \Theta h_{\mu\nu} + \omega_{\mu\nu} + \sigma_{\mu\nu} , \tag{2.30}$$

where Θ is the *expansion rate*, $\omega_{\alpha\beta}$ is the *vorticity tensor* and $\sigma_{\alpha\beta}$ is the *shear tensor*.

Before defining and giving the properties of the new quantities introduced above, it is worth deriving a couple of useful equations (the *generalized Hubble law* and the *rate of change of relative distance*).

Let us start denoting by $\gamma_s(t)$ a smooth one-parameter family of timelike curves (for each $s \in \mathbb{R}$, the curve γ_s is a timelike curve, parametrized by affine parameter t, see figure 2.1). The map $(t, s) \longrightarrow \gamma_s(t)$ is smooth, one-to-one and has smooth inverse.

---

[7] It is worth noting that in this work we define $\omega_{\mu\nu} \doteq \hat{\nabla}_{[\nu} u_{\mu]}$ (like [1] and [8]), while some authors use a different convention. For instance, [3] uses $\omega_{\mu\nu} \doteq \hat{\nabla}_{[\mu} u_{\nu]}$ and in this case instead of (2.30) we have $\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + \hat{\nabla}_\mu u_\nu = -u_\mu \dot{u}_\nu + \frac{1}{3}\Theta h_{\mu\nu} + \omega_{\mu\nu} + \sigma_{\mu\nu}$. Also in some other equations the signs differ from ours.

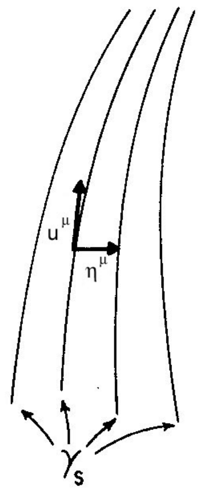

Figure 2.1: A one-parameter family of geodesics γ_s, with tangent vector u^μ and deviation vector η^μ.

Let Σ denote the two-dimensional submanifold spanned by the curves $\gamma_s(t)$. We may choose s and t as coordinates of Σ. Then the vector field $u^\mu = \left(\frac{\partial}{\partial t}\right)^\mu$ is tangent to the family of timelike curves and, thus, satisfies $u^\mu \nabla_\mu u^\nu = 0$. The vector field $\eta^\mu = \left(\frac{\partial}{\partial s}\right)^\mu$ represents the displacement to an infinitesimally nearby timelike curve and is called the deviation vector. These two vectors satisfy[8]:

$$u^\mu \nabla_\mu \eta^\nu = \eta^\mu \nabla_\mu u^\nu \,. \tag{2.31}$$

We obtain a relative position vector by using the projector (2.22):

$$\eta_\perp^\mu = h^\mu{}_\nu \eta^\nu \tag{2.32}$$

and decompose it as

$$\eta_\perp^\mu = \delta l e^\mu \,, \tag{2.33}$$

[8]For a full description see [2, p. 46].

where δl is the relative distance for the family of world lines and e^μ is the relative direction vector, for which $e_\mu e^\mu = 1$ (thus $\dot{e}_\mu e^\mu = 0$) and $e_\mu u^\mu = 0$.

At this point we can express $\dot{\eta}^\mu_\perp$ in two ways:

$$\dot{\eta}^\mu_\perp = u^\mu \nabla_\mu \eta^\nu = \eta^\mu \nabla_\mu u^\nu \tag{2.34}$$

and

$$\dot{\eta}^\mu_\perp = \delta l \dot{e}^\mu + \dot{\delta l} e^\mu \,. \tag{2.35}$$

By comparing these two expressions and projecting with e^μ we get the *rate of change of relative distance* (*generalized Hubble law*):

$$\frac{\dot{\delta l}}{\delta l} = \frac{1}{3}\Theta + \sigma_{\mu\nu} e^\mu e^\nu \,. \tag{2.36}$$

This equation, considered in a cosmological model, describes both isotropic and anisotropic expansions.

Finally, matching (2.34) with (2.35) and using (2.30) and (2.36) we obtain the following propagation equation:

$$\dot{e}^{\langle\mu\rangle} = \left[\sigma^\mu{}_\nu - \left(\sigma_{\alpha\beta} e^\alpha e^\beta\right) h^\mu{}_\nu - \omega^\mu{}_\nu\right] e^\nu \,. \tag{2.37}$$

This last expression is the *rate of change of direction* and it gives the rate of change of position in the space of neighboring clusters of galaxies with respect to an observer at rest in a local inertial frame.

Now we come back to the kinematic quantities introduced at the beginning of this section.

2.4.1 Expansion rate

The scalar quantity Θ (expansion rate) is defined as the trace of the velocity gradient:

$$\Theta \doteq \nabla_\mu u^\mu = \hat{\nabla}_\mu u^\mu \,, \tag{2.38}$$

where the last term comes out from (2.10) and (2.15).

In the case of *pure expansion* ($\sigma_{\mu\nu} = 0$ and $\omega_{\mu\nu} = 0$), from (2.36) the rate of change of relative distance becomes

$$\frac{\dot{l}}{l} = \frac{1}{3}\Theta\,, \tag{2.39}$$

where we have defined a representative length l, while the rate of change of direction becomes

$$\dot{e}^{\langle\mu\rangle} = 0\,. \tag{2.40}$$

So, if we consider a sphere of galaxies of radius dl at time t, at time $t + \delta t$, from (2.39), the distances to all of the galaxies have increased by

$$dl = \frac{\Theta \delta l \delta t}{3}\,, \tag{2.41}$$

while their directions have all remained unchanged. Thus the galaxies form a larger sphere (assuming $\Theta > 0$) with each galaxy lying in the same direction as before. Hence we have distortion-free expansion without any rotation.

In other words, thinking of a sphere of fluid particles that changes according to (2.39) during a small increment of proper time, Θ describes the *isotropic* volume expansion of that sphere (see figure 2.2).

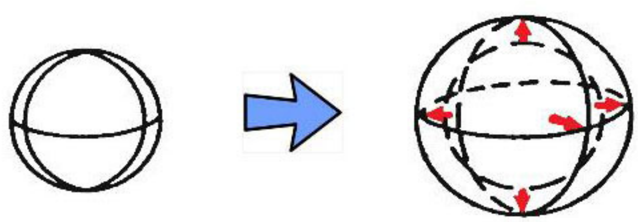

Figure 2.2: The expansion rate Θ describes the *isotropic* volume expansion of a sphere of fluid particles that changes according to (2.39) during a small increment of proper time.

The quantity l, which represents completely the volume behavior of the fluid, in a Friedmann-Lemaître-Robertson-Walker (FLRW) model (where isotropy holds by construction) corresponds to the scale factor $a(t)$[9].
From it we can define the *Hubble parameter* as

$$H(t) \doteq \frac{\dot{l}}{l} = \frac{1}{3}\Theta\,. \tag{2.42}$$

So the the Hubble parameter $H(t)$ is the slope of the curve $l(t)$. Let us call *Hubble constant* the value $H_0 \doteq H(t_0)$ assumed today by this parameter (t_0 being the present age of the universe).

Finally, it is worth giving the expression of the expansion rate Θ in comoving coordinates[10], i.e.

$$\Theta = \frac{1}{\sqrt{|g|}}\partial_0\left(\sqrt{|g|}\right) = \partial_0\left(\ln\sqrt{|g|}\right)\,, \tag{2.43}$$

or, using Christoffel symbols[11],

$$\Theta = \Gamma^{\mu}{}_{\mu 0}\,. \tag{2.44}$$

These two last expressions of Θ will be used in chapters 4 and 5.

Equation (2.43) is particularly useful for calculations when the metric $g_{\mu\nu}$ is non-diagonal.

[9] For a review of FLRW model see [9].

[10] We can obtain (2.43) using either the expression for the divergence of a four-vector $\nabla_\mu v^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}v^\mu\right)$ or the expression of the covariant derivative in terms of the Christoffel symbols and the property $\Gamma^{\mu}{}_{\nu\mu} = \frac{\partial_\nu g}{2g} = \partial_\nu\left(\ln\sqrt{|g|}\right)$.

[11] Equation (2.44) is obtained using the definition of covariant derivative in terms of Christoffel symbols, i.e. $\nabla_\mu u^\nu = \partial_\mu u^\nu + u^\rho\Gamma^{\nu}{}_{\mu\rho}$.

2.4.2 Vorticity

The vorticity tensor $\omega_{\mu\nu}$ is defined as

$$\omega_{\mu\nu} \doteq \hat{\nabla}_{[\nu} u_{\mu]} \,, \tag{2.45}$$

i.e. it is the skew-symmetric part of the orthogonally projected covariant derivative of the velocity vector field[12].

In the case of *pure vorticity* ($\Theta = 0$ and $\sigma_{\mu\nu} = 0$), the rate of change of relative distance becomes

$$\frac{\dot{\delta l}}{\delta l} = 0 \,, \tag{2.46}$$

so all the relative distances are unchanged, while now the rate of change of direction is

$$\dot{e}^{\langle\mu\rangle} = -\omega^{\mu}{}_{\nu} e^{\nu} \,. \tag{2.47}$$

By definition, a rotation preserves all distances, so these relations show that the change is a pure rotation. So the vorticity tensor $\omega_{\mu\nu}$ determines a rigid rotation of a fluid sphere of particles with respect to a local inertial frame (see figure 2.3).

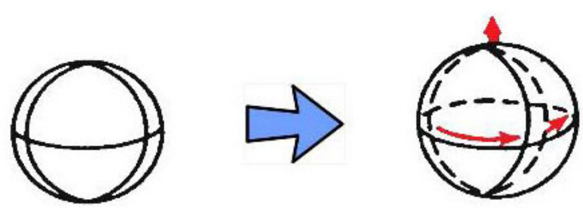

Figure 2.3: The vorticity tensor $\omega_{\mu\nu}$ determines a rigid rotation of a fluid sphere of particles with respect to a local inertial frame.

The vorticity tensor $\omega_{\mu\nu}$ obeys the following properties:

$$\omega_{\mu\nu} = \omega_{[\mu\nu]} \,, \qquad \omega_{\mu\nu} u^{\nu} = 0 \,, \qquad \omega^{\mu}{}_{\mu} = 0 \,. \tag{2.48}$$

[12]See footnote 7 (chapter 2) for different definitions.

It follows from (2.45) that the tensor $\omega_{\mu\nu}$ has only three independent components, then we can use without loss of generality the *vorticity vector*, defined as

$$\omega^{\mu} \doteq \frac{1}{2}\eta^{\mu\nu\rho}\omega_{\nu\rho}\,, \tag{2.49}$$

from which we obtain the vorticity tensor by

$$\omega_{\mu\nu} = \eta_{\mu\nu\rho}\omega^{\rho}\,. \tag{2.50}$$

From this equation, using the second expression in (2.20), we can see that ω^{μ} is orthogonal to the velocity u^{ν}, i.e.

$$\omega_{\mu}u^{\mu} = 0\,, \tag{2.51}$$

and it is an eigenvector of $\omega_{\mu\nu}$ with eigenvalue zero, i.e.

$$\omega^{\mu}\omega_{\mu\nu} = 0\,. \tag{2.52}$$

This shows that the direction of the vorticity vector ω^{μ} is the axis of rotation of matter[13].

We can also define a scalar quantity ω, the *vorticity scalar*, as[14]

$$\omega \doteq \left(\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}\right)^{\frac{1}{2}} = \left(\omega_{\mu}\omega^{\mu}\right)^{\frac{1}{2}}\,. \tag{2.53}$$

In comoving coordinates, the vorticity tensor becomes

$$\omega_{0\mu} = 0\,, \qquad \omega_{ij} = \partial_{[j}u_{i]} + u_{[j}\partial_0 u_{i]}\,, \tag{2.54}$$

where $u_i = g_{0i}$.

If the vorticity vanishes, $h_{\mu\nu}$ is the (induced) effective metric tensor for the surfaces of simultaneity for the fundamental observer.

[13]For more details about vorticity see [8, chapter 3].

[14]Note that $\omega_{\mu\nu} = 0 \Leftrightarrow \omega_{\mu} = 0 \Leftrightarrow \omega = 0$, i.e. the conditions of *vanishing vorticity tensor*, *vanishing vorticity vector* and *vanishing vorticity scalar* are equivalent.

2.4.3 Shear

The shear tensor is defined as the trace-free symmetric part of the orthogonally projected covariant derivative of the velocity vector field:

$$\sigma_{\mu\nu} \doteq \hat{\nabla}_{\langle\nu} u_{\mu\rangle} = \hat{\nabla}_{(\nu} u_{\mu)} - \frac{1}{3}\Theta h_{\mu\nu} \,. \tag{2.55}$$

In the case of *pure shear* ($\Theta = 0$ and $\omega_{\mu\nu} = 0$), the rate of change or relative distance becomes

$$\frac{\dot{\delta l}}{\delta l} = \sigma_{\mu\nu} e^{\mu} e^{\nu} \,, \tag{2.56}$$

while for the rate of change of direction we have

$$\dot{e}^{\langle\mu\rangle} = \left[\sigma^{\mu}{}_{\nu} - \left(\sigma_{\alpha\beta} e^{\alpha} e^{\beta}\right) h^{\mu}{}_{\nu}\right] e^{\nu} \,. \tag{2.57}$$

Since the shear tensor $\sigma_{\mu\nu}$ is symmetric, we can always choose an orthonormal basis of shear eigenvectors in order to have $\sigma_{\mu\nu} = \mathrm{diag}\,(0,\, \sigma_1,\, \sigma_2,\, \sigma_3)$, where $\sigma_1 + \sigma_2 + \sigma_3 = 0$. Then if there is expansion in the 1-direction ($\sigma_1 > 0$), there must be contraction in at least one other direction (say $\sigma_2 < 0$). If in this case we consider a sphere of fluid particles (in a cosmological model it could be a sphere of galaxies around the observer) at time t, at time $t + \delta t$ the distances to particles in the principal j-axis direction will have change by

$$dl = \sigma_j \delta l \delta t \tag{2.58}$$

and their direction (from eqn. (2.57)) unchanged. Thus the fluid particles form an ellipsoid, expanded in the 1-direction but contracted in the 2-direction, with the same volume as before. Each fluid particle (or galaxy) lying in a shear eigendirection will be in the same direction as before; all the others will appear to have moved, but the average change of direction will be zero (see figure 2.4). Hence we have a pure distortion, without rotation or change of volume of the sphere.

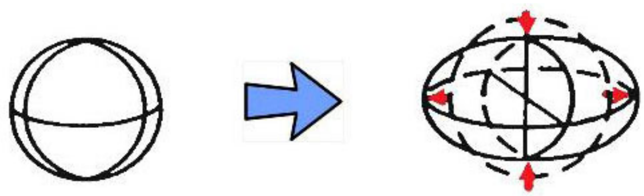

Figure 2.4: The shear tensor $\sigma_{\mu\nu}$ describes a pure distortion, without rotation or change of volume of the sphere.

From the definition of shear (2.55), we can easily obtain the following properties:

$$\sigma_{\mu\nu} = \sigma_{(\mu\nu)}, \qquad \sigma_{\mu\nu}u^{\nu} = 0, \qquad \sigma^{\mu}{}_{\mu} = 0\,. \tag{2.59}$$

As for the vorticity tensor, we can define a scalar quantity σ, the *shear scalar*, as[15]

$$\sigma \doteq \left(\frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}\right)^{\frac{1}{2}}\,. \tag{2.60}$$

In comoving coordinates the shear tensor can be written as[16]

$$\sigma^{\mu}{}_{0} = 0\,, \qquad \sigma^{i}{}_{j} = \frac{1}{2}\delta^{i}{}_{0}\partial_{0}u_{j} - \frac{1}{3}\Gamma^{\rho}{}_{\rho 0}h^{i}{}_{j} + \Gamma^{i}{}_{00}u_{j} + \Gamma^{i}{}_{j0}\,. \tag{2.61}$$

2.5 Electric and magnetic Weyl tensors

In this section we give the decomposition of the Weyl tensor, writing it also in comoving coordinates.

[15]Note that $\sigma = 0 \Leftrightarrow \sigma_{\mu\nu} = 0$.

[16]Where (2.22) and (2.25) have been used.

The Weyl tensor can be split relative to the four-velocity u^μ in two parts: the *electric* and *magnetic Weyl tensors*. The former is defined as

$$E_{\alpha\beta} \doteq C_{\alpha\gamma\beta\delta} u^\gamma u^\delta \,, \tag{2.62}$$

and has the properties

$$E^\alpha{}_\alpha = 0 \,, \qquad E_{\alpha\beta} = E_{(\alpha\beta)} \,, \qquad E_{\alpha\beta} u^\beta = 0 \,, \tag{2.63}$$

while the magnetic Weyl tensor is

$$H_{\alpha\beta} \doteq \frac{1}{2} \eta_{\alpha\gamma\delta} C^{\gamma\delta}{}_{\beta\nu} u^\nu \,, \tag{2.64}$$

for which

$$H^\alpha{}_\alpha = 0 \,, \qquad H_{\alpha\beta} = H_{(\alpha\beta)} \,, \qquad H_{\alpha\beta} u^\beta = 0 \,. \tag{2.65}$$

Both $E_{\alpha\beta}$ and $H_{\alpha\beta}$ are, by construction, projected orthogonally to the four-velocity u^μ and they are what an observer with four-velocity u^μ measures. So if we consider an observer moving with a different four-velocity, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is the same, but the values of $E_{\alpha\beta}$ and $H_{\alpha\beta}$ are different.

In comoving coordinates the electric and magnetic Weyl tensors are

$$\begin{aligned} E_{\alpha\beta} &= C_{\alpha 0 \beta 0} \,, && (2.66a) \\ H_{\alpha\beta} &= \frac{1}{2} \eta_{\alpha\gamma\delta} C^{\gamma\delta}{}_{\beta 0} = \frac{1}{2} \eta_{\alpha\gamma\delta} g^{\gamma\xi} g^{\delta\psi} C_{\xi\psi\beta 0} \,. && (2.66b) \end{aligned}$$

Finally we give the expression of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ in terms of $E_{\alpha\beta}$ and $H_{\alpha\beta}$[17]:

$$C_{\alpha\beta}{}^{\gamma\delta} = 4 \left(u_{[\alpha} u^{[\gamma} + h_{[\alpha}{}^{[\gamma} \right) E_{\beta]}{}^{\delta]} + 2 \eta_{\alpha\beta\kappa} u^{[\gamma} H^{\delta]\kappa} + 2 \eta^{\gamma\delta\kappa} u_{[\alpha} H_{\beta]\kappa} \,. \tag{2.67}$$

[17]See [1, p. 87]

2.6 Dynamics

2.6.1 Einstein field equation

In the theory of general relativity, the spacetime is specified once a metric tensor $g_{\mu\nu}(x^\rho)$ is given. The behavior of the matter is described by the energy-momentum tensor, whereas the Einstein field equation describes the interaction between geometry and matter, i.e how matter determines the geometry, which in turn determines the motion of matter.

The Einstein field equation is[18]:

$$G_{\mu\nu} \doteq R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu}\,, \tag{2.68}$$

where $G_{\mu\nu}$ is the *Einstein tensor*, $R_{\mu\nu}$ is the *Ricci tensor*, R is the *Ricci scalar*, $T_{\mu\nu}$ the *energy-momentum tensor* and Λ the *cosmological constant*[19].

Equation (2.68) is a set of coupled non-linear second order partial differential equations for the components of the metric $g_{\mu\nu}$.

Each tensor quantity in (2.68) possesses ten independent components[20], so in principle we have ten independent equations. Actually, we have to consider four differential *twice contracted Bianchi identities*

$$\nabla^\mu G_{\mu\nu} = 0\,. \tag{2.69}$$

In this way the number of independent equations can be reduced to six by the choice of the coordinates[21].

[18] The cosmological constant Λ can be added to the left-hand side of the Einstein equation and thus we consider it as a geometric term, but we can also add an identical term to the matter, on the right-hand side of Einstein equation. This second choice is the vacuum energy case.
For vanishing cosmological constant the Einstein equation assumes the original form $G_{\mu\nu} = T_{\mu\nu}$.

[19] We have included the cosmological constant Λ for the sake of generality, but throughout this work we consider only dust (see subsection 2.6.2.1) solutions of Einstein equation with $\Lambda = 0$.

[20] In the n-dimensional case the independent components of the metric, Einstein tensor and Ricci tensor are $\frac{1}{2}n(n+1)$.

[21] See [2, p. 259, 260].

Now, by taking the covariant derivative of the Einstein equation (2.68), since $\nabla^\mu g_{\mu\nu} = 0$ (∇^μ is the derivative operator naturally associated with the metric $g_{\mu\nu}$) and $\nabla^\mu \Lambda = 0$ (i.e. Λ is defined as constant in space and time) we obtain[22]

$$\nabla^\mu T_{\mu\nu} = 0\,. \tag{2.70}$$

In the case $T_{\mu\nu} = 0$, the Einstein equation (2.68) reduces to the *vacuum field equation*

$$R - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,. \tag{2.71}$$

Contracting (2.71) with the metric we have

$$R = 4\Lambda\,, \qquad \text{or equivalently} \qquad R_{\mu\nu} = \Lambda g_{\mu\nu}\,. \tag{2.72}$$

Examples of vacuum solutions are the *Minkowski spacetime* (used in special relativity), the *Schwarzschild solution* (describing static black holes) and the *Kerr solution* (for rotating black holes).

2.6.2 Energy-momentum tensor

The energy-momentum tensor $T_{\mu\nu}$ enters in the Einstein field equation (2.68) as the source term and it describes the properties of the matter in spacetime.

The tensor $T_{\mu\nu}$ can be decomposed relative to u^μ in the following form:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + 2 q_{(\mu} u_{\nu)} + \pi_{\mu\nu}\,, \tag{2.73}$$

where ρ is the *total energy density* of the matter relative to u^μ, defined as

$$\rho \doteq T_{\mu\nu} u^\mu u^\nu\,, \tag{2.74}$$

[22]Note that (2.70) means $\partial^\mu T_{\mu\nu} = g^{\mu\rho} \Gamma^\gamma{}_{\rho\mu} T_{\gamma\nu} + \Gamma^\gamma{}_{\rho\nu} T^\rho{}_\gamma$, where we have used the decomposition of the covariant derivative in terms of Christoffel symbols $\nabla_\rho T_{\mu\nu} = \partial_\rho T_{\mu\nu} - \Gamma^\gamma{}_{\rho\mu} T_{\gamma\nu} - \Gamma^\gamma{}_{\rho\nu} T_{\mu\gamma}$.

p is the *isotropic pressure*

$$p \doteq \frac{1}{3} T_{\mu\nu} h^{\mu\nu} \,, \tag{2.75}$$

$\pi_{\mu\nu}$ is the *trace-free anisotropic pressure*, defined as

$$\pi_{\mu\nu} \doteq T_{\gamma\rho} h^{\gamma}{}_{\langle\mu} h^{\rho}{}_{\nu\rangle} \,, \tag{2.76}$$

and, finally, q^{μ} is the *relativistic momentum density* (which represents also the *energy flux* relative to u^{μ})

$$q^{\mu} \doteq -T_{\nu\rho} u^{\nu} h^{\rho\mu} \,. \tag{2.77}$$

Both the relativistic momentum q^{μ} and the anisotropic pressure $\pi_{\mu\nu}$ are orthogonal to u^{μ}, i.e.

$$q_{\mu} u^{\mu} = 0 \qquad \text{and} \qquad \pi_{\mu\nu} u^{\nu} = 0 \,. \tag{2.78}$$

Above we gave the full expression for the energy-momentum tensor, but in this thesis only the case of dust (see subsection 2.6.2.1) is considered.

2.6.2.1 Particular fluids

Often we may consider a simpler form of the energy-momentum tensor $T_{\mu\nu}$, instead of the full expression (2.73), to describe a particular physical situation.
An interesting case is the *perfect fluid* one, especially its sub-case of *dust* (at which we will refer in this work).

Perfect fluid

The energy-momentum tensor $T_{\mu\nu}$ for a perfect fluid is given by[23]

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + p h_{\mu\nu} = \rho u_{\mu} u_{\nu} + p \left(g_{\mu\nu} + u_{\mu} u_{\nu} \right) \,. \tag{2.79}$$

[23] Choosing in eqn. (2.73) $q^{\mu} = 0$ and $\pi_{\mu\nu} = 0$.

This is a good approximation when the momentum density q^μ and anisotropic pressure $\pi_{\mu\nu}$ are smaller with respect to the energy density ρ and isotropic pressure p.

In this case the equation of state relates p to ρ, i.e. it has the form $p = p(\rho)$. For instance, the main component of the radiation content of the universe (the CMB) can be represented as an ideal fluid with the equation of state[24]

$$p = \frac{1}{3}\rho\,. \tag{2.80}$$

Dust

This case refers to *pressure-free matter* (also called *cold dark matter*) and it has the dynamical restrictions

$$p = q^\mu = \pi_{\mu\nu} = 0\,. \tag{2.81}$$

Now, also the pressure p (in addition to q^μ and $\pi_{\mu\nu}$) is much smaller than the energy density ρ, so the matter is represented only by its four-velocity u^μ and its energy density $\rho > 0$.

The expression (2.73) for the energy-momentum tensor $T_{\mu\nu}$ now becomes

$$T_{\mu\nu} = \rho u_\mu u_\nu\,. \tag{2.82}$$

In this case, from (2.103), we have

$$\rho \propto a^{-3}\,, \tag{2.83}$$

and[25]

$$\dot{u}^\mu = 0\,. \tag{2.84}$$

[24] See [1, p. 99].

[25] Equation (2.84) comes out from the following propagation equation, which derives from the twice-contracted Bianchi identities (2.69): $\hat{\nabla}^\mu p = -(\rho + p)\,\dot{u}^\mu - \dot{q}^\mu - \hat{\nabla}_\nu \pi^{\mu\nu} - \frac{4}{3}\Theta q^\mu - \sigma^\mu{}_\nu q^\nu - \dot{u}_\nu \pi^{\mu\nu} + \eta^{\mu\nu\kappa}\omega_\nu q_\kappa$. For more details see [8, subsection 2.4.3].

Moreover, in comoving coordinates (2.11) the energy-momentum tensor for dust assumes the form:

$$T_{\mu\nu} = \rho g_{0\mu} g_{0\nu} \,. \tag{2.85}$$

It is worth giving the expression of the Einstein equation (2.68) in the dust case using comoving coordinates (at which we refer in this work):

$$G_{\mu\nu} \doteq R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \rho g_{0\mu} g_{0\nu} \,, \tag{2.86}$$

in this way (2.68) reduces to the following system:

$$\begin{cases} G_{00} = & \rho \\ G_{0i} = & -\rho g_{0i} \\ G_{ij} = & \rho g_{0i} g_{0j} \end{cases} . \tag{2.87}$$

We discuss the exact solutions to Einstein equation in the dust case in chapters 4 and 5.

2.6.3 Riemann tensor

It is useful to give the $(3+1)$-decomposition of the Riemann tensor $R_{\alpha\beta\gamma\delta}$, that is[26]

$$R^{\alpha\beta}{}_{\gamma\delta} = R_P^{\alpha\beta}{}_{\gamma\delta} + R_E^{\alpha\beta}{}_{\gamma\delta} + R_H^{\alpha\beta}{}_{\gamma\delta} \,, \tag{2.88}$$

[26]Here P stands for perfect fluid part, E marks the part due to the electric Weyl curvature and H the one due to the magnetic Weyl curvature.

where[27]

$$
\begin{aligned}
R_{P\ \gamma\delta}^{\alpha\beta} &= \frac{2}{3}\rho\left(u^{[\alpha}u_{[\gamma}h^{\beta]}{}_{\delta]} + h^{\alpha}{}_{[\gamma}h^{\beta}{}_{\delta]}\right), && (2.89a)\\
R_{E\ \gamma\delta}^{\alpha\beta} &= 4u^{[\alpha}u_{[\gamma}E^{\beta]}{}_{\delta]} + 4h^{[\alpha}{}_{[\gamma}E^{\beta]}{}_{\delta]}, && (2.89b)\\
R_{H\ \gamma\delta}^{\alpha\beta} &= 2\eta^{\alpha\beta\kappa}u_{[\gamma}H_{\delta]\kappa} + 2\eta_{\gamma\delta\kappa}u^{[\alpha}H^{\beta]\kappa}. && (2.89c)
\end{aligned}
$$

The former term derives from the decomposition of the Ricci tensor, while the latter two terms from the splitting of the Weyl tensor.

Here we gave the expression of (2.89a)-(2.89c) in the particular case of dust with the cosmological constant Λ set to zero. For the general expressions see [8, subsection 2.3.3].

2.7 Propagation and constraint equations

There are three sets of equations to be considered, resulting from Einstein equation (2.68) and its associated integrability conditions. These three sets come out from *Ricci identities*, *Bianchi identities* contracted once and twice respectively[28].

In all the equations we set $\Lambda = p = q^{\mu} = \pi^{\mu\nu} = 0$, i.e. we consider the dust case with zero cosmological constant.

2.7.1 Ricci identities

Let us start with the first set of equations, which arise from the Ricci identities for the vector field u^{μ}. Using the definition of Riemann tensor

[27] To get this splitting we have started from (2.8), i.e. from the decomposition of the Riemann tensor in terms of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ and the Ricci tensor $R_{\alpha\beta}$, and used (2.67) and the Einstein equation (2.68) to express the Ricci tensor $R_{\alpha\beta}$ in terms of the energy-momentum tensor (equation (2.73)).

[28] Here we give only a brief description of how obtain them, for more details see [3] and [13].

(2.1), these identities are:

$$2\nabla_{[\gamma}\nabla_{\delta]}v^{\alpha} = R^{\alpha}{}_{\beta\gamma\delta}v^{\beta}\,. \tag{2.90}$$

Now, two classes of equations can be obtained: the *propagation* and *constraint* equations by projecting (2.90) by means of the projection tensors (2.21) and (2.22) respectively. Then every class can be further split in three equations by separating out the *trace*, *antisymmetric part* and s*ymmetric trace-free part*.

Propagation equations

Using the projection tensor (2.21) to project along the vector field u^{μ} and recalling (2.30), (2.68) and (2.73), we obtain three evolution equations for the quantities Θ, ω^{μ} and $\sigma^{\mu\nu}$ respectively.

1. Separating out the trace, we get the *Raychaudhuri equation*:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = -\frac{1}{2}\rho - 2\sigma^2 + 2\omega^2\,. \tag{2.91}$$

   This is the basic equation of gravitational attraction and leads to identification of ρ as the active gravitational mass density.

2. Taking the skew-symmetric part, we obtain the *vorticity propagation equation*:

$$\dot{\omega}^{\alpha} = -\frac{2}{3}\Theta\omega^{\alpha} + \sigma^{\alpha}{}_{\beta}\omega^{\beta}\,, \tag{2.92}$$

   which is independent of the matter content and the Einstein equation (2.68).

3. Taking the spatially projected symmetric trace-free part, by means of second of (2.28), and splitting the Riemann tensor as in (2.88), we get the *shear propagation equation*:

$$\dot{\sigma}^{\langle\alpha\beta\rangle} = -\frac{2}{3}\Theta\sigma^{\alpha\beta} - \sigma^{\langle\alpha}{}_{\gamma}\sigma^{\beta\rangle\gamma} - \omega^{\langle\alpha}\omega^{\beta\rangle} - E^{\alpha\beta}\,. \tag{2.93}$$

This equation shows that the Weyl tensor (which represents tidal gravitational forces) induces shear through its electric part $E_{\alpha\beta}$, which then feeds into Raychaudhuri and vorticity propagation equations, influencing the nature of the fluid flow.

Constraint equations

Using the projection tensor (2.22) to project (2.90) orthogonally to the vector field u^μ and recalling (2.30), (2.68) and (2.73), we obtain the three following constraint equations.

1. Separating out the trace, we obtain the *shear divergence constraint*:
$$\hat{\nabla}_\beta \sigma^{\alpha\beta} = \frac{2}{3}\hat{\nabla}^\alpha \Theta - \eta^{\alpha\beta\gamma}\hat{\nabla}_\beta \omega_\gamma\,. \tag{2.94}$$

2. Taking the antisymmetric part, we get the *vorticity divergence constraint*:
$$\hat{\nabla}_\alpha \omega^\alpha = 0\,. \tag{2.95}$$

3. Finally, separating out the spatially projected symmetric trace-free part, by means of second of (2.28), and splitting the Riemann tensor as in (2.88), we get the *magnetic constraint*:
$$H_{\alpha\beta} = \hat{\nabla}_{\langle\alpha}\omega_{\beta\rangle} + \eta_{\gamma\delta\langle\alpha}\hat{\nabla}^\gamma \sigma^\delta{}_{\beta\rangle}\,, \tag{2.96}$$
which characterizes the magnetic Weyl tensor and is independent of the matter content and Einstein equation (2.68).

2.7.2 Bianchi identities

The second set of equations arises from the fact that the Riemann tensor satisfies the following *Bianchi identities*:
$$R^\alpha{}_{\beta[\gamma\delta;\varepsilon]} = 0\,, \qquad \text{or} \qquad \nabla_{[\varepsilon}R_{\gamma\delta]\alpha\beta} = 0\,. \tag{2.97}$$

Now, inserting the splitting (2.8) of the Riemann tensor, using the Einstein equation (2.68) and contracting (2.97) over the indices ε and δ, the *once contracted Bianchi identities* are found to be

$$\nabla^{\delta} C_{\alpha\beta\gamma\delta} + \nabla_{[\alpha} R_{\beta]\gamma} + \frac{1}{6}\delta_{\gamma[\alpha}\nabla_{\beta]} R = 0\,. \tag{2.98}$$

We can then project it along the world lines originated by u^{μ} and on the orthogonal surfaces, to find two further propagation equations and two further constraint equations.

Propagation equations

These two equations show how gravitational radiation arises[29].

1. The *electric propagation equation* is:

$$\dot{E}_{\alpha\beta} = -\Theta E_{\alpha\beta} + 3\sigma_{\langle\alpha}{}^{\gamma u^{\mu}} E_{\beta\rangle\gamma} - \frac{1}{2}\rho\sigma_{\alpha\beta} + \eta_{\gamma\delta\langle\alpha}\left(\hat{\nabla}^{\gamma} H_{\beta\rangle}{}^{\delta} - \omega^{\gamma} H_{\beta\rangle}{}^{\delta}\right)\,. \tag{2.99}$$

2. The *magnetic propagation equation* is:

$$\dot{H}_{\alpha\beta} = -\eta_{\gamma\delta\langle\alpha}\left(\hat{\nabla}^{\gamma} E_{\beta\rangle}{}^{\delta} - \omega^{\gamma} H_{\beta\rangle}{}^{\delta}\right) - \Theta H_{\alpha\beta} + 3\sigma_{\langle\alpha}{}^{\gamma} H_{\beta\rangle\gamma}\,. \tag{2.100}$$

Constraint equations

1. The *electric constraint equation* is:

$$\hat{\nabla}_{\beta} E^{\alpha\beta} = -3\omega_{\beta} H^{\alpha\beta} + \eta^{\alpha\beta\gamma}\sigma_{\beta\delta} H_{\gamma}{}^{\delta} + \frac{1}{3}\hat{\nabla}^{\alpha}\rho\,, \tag{2.101}$$

   with source the spatial gradient of the energy density.

[29] See [3, p. 11].

2. The *magnetic constraint equation* is:

$$\hat{\nabla}_\beta H^{\alpha\beta} = -3\omega_\beta E^{\alpha\beta} - \eta^{\alpha\beta\gamma}\sigma_{\beta\delta}E_\gamma{}^\delta + \rho\omega^\alpha \,, \tag{2.102}$$

with source the fluid vorticity.

2.7.3 Twice-contracted Bianchi identities

Finally, the last propagation equation arise from the *twice contracted Bianchi identities* (2.69). Using the Einstein equation we obtain the vanishing of the covariant derivative of the energy-momentum tensor, i.e. (2.70). Inserting the expression of $T_{\mu\nu}$ (2.73), using (2.30) and projecting parallel to the four-velocity u^μ by means of (2.21) we have

$$\dot{\rho} = -\Theta\rho \,. \tag{2.103}$$

Chapter 3

Average quantities and backreaction in the dust case

In this chapter we explain what backreaction is.

After a brief introduction in order to describe the physical idea that underlies the concept of backreaction, we restrict ourselves to a universe filled with irrotational dust. Then, we describe an averaging procedure in order to introduce the Buchert equations, from which we define the backreaction variable.

Some references for this chapter are [4], [8], and [14].

3.1 An introduction to backreaction

Observations of the cosmic microwave background (CMB) and large scale structures have verified that the early universe is close to exact homogeneity and isotropy and it can be described by a FLRW model with linear perturbations. But then, at late times, the homogeneity and isotropy are broken because of the growth of non-linear structures.

However the universe appears to be homogeneous and isotropic when averaged over sufficiently large scales, i.e. the universe is *statistically* homogeneous and isotropic. So, if we take a box anywhere in the universe larger than the homogeneity scale, the mean quantities inside it do not depends on its location, orientation or size. In this way we can study the average quantities, instead of the local ones.

Physically we should first plug the inhomogeneous quantities into the Einstein equation and then take the average. In fact, because the Einstein equation is non-linear, these two procedures are not equivalent, i.e. the operations of time evolution and averaging do not commute.

The fact that the average evolution of a clumpy space is not the same of a smooth space is referred to as *backreation*.

The aim of this work is to list all the exact solutions of the Einstein equation (2.68) with the cosmological constant Λ and the vorticity $\omega_{\mu\nu}$ set to zero. Then in this chapter we consider only the case of irrotational dust. For a description of the average procedure in the general case of matter with non-zero vorticity see [8, section 4.3].

3.2 Irrotational dust

Let us start describing the universe considering the matter as dust[1]. In this case the energy-momentum tensor $T_{\mu\nu}$ has the form

$$T_{\mu\nu} = \rho u_\mu u_\nu, \tag{3.1}$$

i.e. we are assuming that the energy density of matter dominates everywhere over the pressure, the anisotropic stress and the energy flux. In this way (we have chosen the pressure to be zero) the fluid elements do not interact each other, the motion is then geodesic and the four-velocity u^μ is a tangent vector to timelike geodesics, i.e. $\dot{u}^\mu = 0$.

[1]See subsection 2.6.2.1.

From Frobenius' theorem[2] we know that if and only if the vorticity is zero, the tangent spaces orthogonal to u^μ form spatial hypersurfaces and provide a foliation that fills the spacetime exactly once. If the vorticity is non-zero, the hypersurfaces of constant proper time are no longer orthogonal to the fluid flow.

In this thesis the case $\omega_{\mu\nu} = 0$ have been studied. Vorticity contributes positively to acceleration $\dot{\Theta}$ and setting it to zero gives a lower bound to acceleration, which in the irrotational case is always non-positive (see (2.91)).

3.3 Defining the average

Before defining the backreaction variable, we have to define a procedure of averaging.

Taking the average of quantities in general relativity is an involved issue. The metric is a dynamical variable and it enters non-linearly in the Einstein equation (2.68). Moreover, if the vorticity $\omega_{\mu\nu}$ vanishes then we have a three-dimensional space to average on, but in the general case of non-zero vorticity none of these (local) three-dimensional spaces is preferred.

In this subsection we follow [8, subsection 4.2.1] and [19]. So the *spatial average $\langle f \rangle$ of a scalar quantity f* is defined as

$$\langle f \rangle (t) \doteq \frac{\int_t f\eta}{\int_t \eta}\,, \tag{3.2}$$

i.e. it is the integral of f over the hypersurface of constant proper time t orthogonal to u^μ , divided by the volume of the hypersurface.
In (3.2) η is the volume form of the spatial hypersurface considered and,

[2]For an exhaustive discussion on Frobenius' theorem see [2, p. 434-436] and [8, chapter 3].

recalling subsection 2.2.3, it is written as

$$\eta = \sqrt{|{}^{(3)}g|}dx^1 \wedge dx^2 \wedge dx^3 \,, \tag{3.3}$$

where ${}^{(3)}g$ is the determinant of the metric ${}^{(3)}g_{\mu\nu}$ on the hypersurface of constant proper time t.

In the case of irrotational dust the metric ${}^{(3)}g_{\mu\nu}$ coincides with the restriction of the projector $h_{\mu\nu}$, defined in (2.22), to the three-dimensional hypersurface of averaging.

Now, defining the scale factor $a\,(t)$ as the volume of the hypersurface of constant proper time to the power $\frac{1}{3}$, i.e.

$$a\,(t) \doteq \left(\frac{\int_t \eta}{\int_{t_0} \eta}\right)^{\frac{1}{3}} \,, \tag{3.4}$$

where $a\,(t)$ has been normalized to unity at time t_0 (which we take to be today), the *average expansion rate* $\langle\Theta\rangle$ turns out to be

$$\langle\Theta\rangle\,(t) \doteq \frac{\int_t \Theta\eta}{\int_t \eta} = \frac{\partial_t \int_t \eta}{\int_t \eta} = 3\frac{\dot{a}}{a}. \tag{3.5}$$

The *Hubble parameter* in now defined as[3]

$$H\,(t) \doteq \frac{\dot{a}\,(t)}{a\,(t)}\,. \tag{3.6}$$

The last important thing to say is that, as stated in section 3.1, the time evolution and the average procedure do not commute. Then we have the following *commutation rule*:

$$\partial_t\big\langle f\big\rangle = \big\langle\dot{f}\big\rangle + \big\langle f\Theta\big\rangle - \big\langle f\big\rangle\big\langle\Theta\big\rangle\,. \tag{3.7}$$

[3]See the relation (2.42).

To prove this equation let us take the derivative with respect to the time t of the definition (3.2) of spatial average of the quantity f:

$$\partial_t \langle f \rangle = \partial_t \left(\frac{\int_t f\eta}{\int_t \eta} \right) = \frac{\partial_t \left(\int_t f\eta \right)}{\int_t \eta} - \langle f \rangle \frac{\partial_t \left(\int_t \eta \right)}{\int_t \eta} \,. \tag{3.8}$$

Now, since from eqn. (3.5) we have

$$\partial_t \left(\int_t \eta \right) = \langle \Theta \rangle (t) \int_t \eta \,, \tag{3.9}$$

and

$$\partial_t \left(\int_t f\eta \right) = \int_t (\partial_t f)\, \eta + \int_t f\Theta\eta \,, \tag{3.10}$$

using definition (3.2) again, we get (3.7) from (3.8).

3.4 The Buchert equations

In section 2.7, a set of propagation and constraint equations have been obtained from Einstein equation (2.68) in the dust case. It is worth writing the corresponding equations for averaged quantities (we are interested in the overall geometry).

3.4.1 The scalar equations

Only scalar quantities can be straightforwardly averaged in a curved spacetime, so let us consider the scalar part of the Einstein equation (2.68).

The only equations we are interested in are the Raychaudhuri equation (2.91), the continuity equation (2.103) and the Hamiltonian constraint, i.e. respectively[4]

$$
\begin{aligned}
\dot{\Theta} + \frac{1}{3}\Theta^2 &= -\frac{1}{2}\rho - 2\sigma^2 \,, &(3.11)\\
\dot{\rho} + \Theta\rho &= 0 \,, &(3.12)\\
\frac{1}{3}\Theta^2 &= \rho - \frac{1}{2}{}^{(3)}R + \sigma^2 \,, &(3.13)
\end{aligned}
$$

where ${}^{(3)}R$ is the Ricci scalar on the three-dimensional space orthogonal to u^μ.

Equations (3.11) and (3.13) are local generalizations of the Friedmann equations[5], while equation (3.12) shows that mass is conserved (because the energy density is proportional to the inverse of the volume).

3.4.2 Buchert equations and backreaction

Let us now take the average of the equations (3.11)-(3.13), using the procedure outlined in subsection 3.3. Recalling the commutation rule (3.7) and the relation (3.6), we obtain the *Buchert equations*:

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}\langle\rho\rangle + \mathcal{Q} \qquad \text{(averaged Raychaudhuri eqn.)}\,, \quad (3.14)$$

$$\partial_t\langle\rho\rangle + 3\frac{\dot{a}}{a}\langle\rho\rangle = 0 \qquad \text{(averaged continuity eqn.)}\,, \quad (3.15)$$

$$3\frac{\dot{a}^2}{a^2} = \langle\rho\rangle - \frac{1}{2}\langle{}^{(3)}R\rangle - \frac{1}{2}\mathcal{Q} \qquad \text{(averaged Hamiltonian constr.)}\,. \quad (3.16)$$

In these equations, $\mathcal{Q}$ is the backreaction variable, defined as

$$\mathcal{Q} \doteq \frac{2}{3}\left(\langle\Theta^2\rangle - \langle\Theta\rangle^2\right) - 2\langle\sigma^2\rangle \,, \quad (3.17)$$

[4]All these three equations have been written in the case of irrotational dust, i.e. setting vorticity to zero in (2.91) and (2.103).

[5]See [8, appendix A] and [9].

and it contains the effects of inhomogeneities and anisotropies.

The Buchert equations differ from the Friedmann equations[6] in two ways. First of all, here the scale factor is the total volume of a region (as follow from its definition (3.4)), so it is not part of the metric and it does not describe the local behavior of the space like in the FLRW case. Second, they are different from a mathematical point of view, in the sense that they include the backreaction term $\mathcal{Q}$, which does not appear in in the Friedmann equations. This new term expresses the non-commutativity of time evolution and averaging and it allows acceleration and a non-trivial evolution of the average spatial curvature.

So, the Buchert equations are the generalization of the Friedmann equations to an inhomogeneous dust universe.

As we can see from (3.17), the variable $\mathcal{Q}$ has two parts: the variance of the expansion rate and the average of the shear scalar.

The average of the shear scalar is also present in the local equations. Since it is always negative, it acts to decelerate the expansion, unless in the case of homogeneous and isotropic spacetime that it vanishes. In contrast, the variance of the expansion rate has no local counterpart. It also vanishes if the expansion is homogeneous, but if not, being a variance, this term is always positive, so it accelerates the expansion.

So, from (3.14) we can see that, if the variance is sufficiently large compared to the shear and the energy density, the average expansion rate accelerates even though the Raychaudhuri equation (3.11) shows that the local expansion rate decelerates everywhere. This is precisely the way backreaction provides acceleration, which in the homogeneous and isotropic case requires the presence of either dark energy or the cosmological constant.

[6]For the Friedmann equations see [8, appendix A.2] or [9].

3.4.3 The integrability condition

The averaged Raychaudhuri equation (3.14) and the averaged Hamiltonian constraint (3.16) form an underdetermined system: we have in fact three unknown variables, $a\,(t)$, $\left\langle {}^{(3)}R \right\rangle (t)$ and $\mathcal{Q}\,(t)$.

We need then another condition, which can be obtained taking the time derivative of the averaged Hamiltonian constraint (3.16) and plug into the result both equation (3.14) and (3.16). In this way we get the following relation between the average spatial Ricci scalar $\left\langle {}^{(3)}R \right\rangle$ and the backreaction $\mathcal{Q}$:

$$\partial_t \left\langle {}^{(3)}R \right\rangle + 2\frac{\dot{a}}{a}\left\langle {}^{(3)}R \right\rangle = -\partial_t \mathcal{Q} - 6\frac{\dot{a}}{a}\mathcal{Q}\,. \tag{3.18}$$

Equation (3.18) is a necessary *integrability condition* between equations (3.14) and (3.16). So, in general, the evolution of backreaction $\mathcal{Q}$ influence the evolution of the average spatial Ricci scalar $\left\langle {}^{(3)}R \right\rangle$ and vice-versa.

There are three interesting particular solutions of the variables $\mathcal{Q}$ and $\left\langle {}^{(3)}R \right\rangle$, expressed as function of $a\,(t)$:

1. considering a *spatially flat on average* portion of the universe, $\left\langle {}^{(3)}R \right\rangle = 0$, and solving equation (3.18) we have:

$$\mathcal{Q} = \mathcal{Q}\,(t_0)\, a^{-6}\,; \tag{3.19}$$

2. if the backreaction vanishes, i.e. $\mathcal{Q} = 0$, then

$$\left\langle {}^{(3)}R \right\rangle = \left\langle {}^{(3)}R \right\rangle (t_0)\, a^{-2}\,, \tag{3.20}$$

   where $a\,(t)$ is given by the Friedmann equations. Here we are describing a system which is FLRW on average, i.e. the effects of inhomogeneities compensate each other and backreaction is zero;

3. the two prior solutions give a solution of (3.18) in the case of non-vanishing backreaction $\mathcal{Q}$ and non-vanishing average spatial Ricci scalar $\left\langle {}^{(3)}R \right\rangle$:

$$\mathcal{Q} = \mathcal{Q}\left(t_0\right) a^{-6} \qquad \text{and} \qquad \left\langle {}^{(3)}R \right\rangle = \left\langle {}^{(3)}R \right\rangle \left(t_0\right) a^{-2} . \quad (3.21)$$

Here we have independent evolution of the backreaction variable $\mathcal{Q}$ and the average spatial Ricci scalar $\left\langle {}^{(3)}R \right\rangle$ (both sides of equation (3.18) vanish).

Chapter 4

Szekeres models

The Szekeres metric is an exact inhomogeneous and anisotropic dust solution of the Einstein equation (2.68) and it has no Killing vector fields (and then no symmetries). We write this metric in comoving coordinates.

The Lemaître-Tolman-Bondi (LTB) model is contained in the Szekeres metric as the spherically symmetric special case.

We first give the solutions of the Einstein equation for this metric and then its physical interpretation. After that, we calculate the dynamical quantities for the Szekeres metric.

Finally, after a brief discussion of the LTB model, we compute the average quantities and the backreaction variable (3.17).

The Szekeres metric has been used in [48] to construct a Swiss Cheese dust model with Szekeres holes. In that paper the authors have proven that, under certain conditions, the average expansion rate is close to the FLRW model, but if one of the assumptions is violated then the first statistically homogeneous and isotropic solution in which inhomogeneity has a significant effect on the average expansion rate (i.e. backreaction is large) has been built.

In all this chapter we follow [14], [16], [20] and [23].

4.1 Szekeres metric

4.1.1 Einstein equation's solutions

The Szekeres metric is

$$ds^2 = -dt^2 + X^2\,(t,\,r,\,p,\,q)\,dr^2 + A^2\,(t,\,r,\,p,\,q)\left(dp^2 + dq^2\right)\,, \quad (4.1)$$

where

$$X^2\,(t,\,r,\,p,\,q) = \frac{\left(R' - R\frac{E'}{E}\right)^2}{\epsilon + f} \qquad \text{and} \qquad A^2\,(t,\,r,\,p,\,q) = \frac{R^2}{E^2}\,. \quad (4.2)$$

In (4.2) $' \equiv \frac{\partial}{\partial r}$, $\epsilon = 0,\, \pm 1$, $f = f\,(r) \geq -\epsilon$ is an arbitrary function of r, $R = R\,(t,\,r)$ and the function E is given by

$$E\,(r,\,p,\,q) = \frac{S}{2}\left\{\left(\frac{p-P}{S}\right)^2 + \left(\frac{q-Q}{S}\right)^2 + \epsilon\right\}\,, \quad (4.3)$$

where $S = S\,(r)$, $P = P\,(r)$, $Q = Q\,(r)$ are arbitrary functions and ϵ is the same of (4.2). When $\epsilon \geq 0$, the range of values of the coordinates p and q is $-\infty < p,\, q < +\infty$, for the $\epsilon = -1$ case see [20] and [23].

The factor ϵ determines whether the p-q two-surfaces are spherical ($\epsilon = +1$), pseudospherical ($\epsilon = -1$), or planar ($\epsilon = 0$). So it determines how the constant r two-surfaces foliate the three-dimensional spatial section of constant t.

The function E (4.3) determines how the coordinates $(p,\, q)$ map onto the unit two-sphere (plane, pseudosphere) at each value of r. At each r these two-surfaces are multiplied by the areal "radius" $R = R\,(t,\,r)$ that evolves with time. Thus the r-p-q three-surfaces are constructed out of a sequence of two-dimensional spheres (pseudospheres, planes) that are not concentric, since the metric component g_{11} depends on p and q, but also on r and t.

The two relations (4.2) can be obtained by means of the Einstein equation for dust (2.87).

Now, let us start from the diagonal components of the Einstein equations (2.87):

$$\begin{cases} G_{00} = & \rho \\ G_{ii} = & 0 \end{cases} . \tag{4.4}$$

Inserting in (4.4) the explicit form for the Einstein tensor we obtain the following four equations:

$$\begin{aligned} \rho \; = \; & \frac{2\dot{A}\dot{X}}{AX} + \frac{\left(\partial_p A\right)^2}{A^4} + \frac{\left(\partial_q A\right)^2}{A^4} - \frac{\partial_p^2 X}{A^2 X} - \frac{\partial_q^2 X}{A^2 X} - \frac{2A''}{AX^2} \\ & + \frac{2A'X'}{AX^3} - \frac{\partial_p^2 A}{A^3} - \frac{\partial_q^2 A}{A^3} + \frac{\dot{A}^2}{A^2} - \frac{\left(A'\right)^2}{A^2 X^2} , \end{aligned} \tag{4.5a}$$

$$\begin{aligned} 0 \; = \; & -\frac{\left(\partial_p A\right)^2 X^2}{A^4} - \frac{\left(\partial_q A\right)^2 X^2}{A^4} - \frac{2\ddot{A}X^2}{A} + \frac{\left(\partial_p^2 A\right) X^2}{A^3} \\ & + \frac{\left(\partial_q^2 A\right) X^2}{A^3} - \frac{\dot{A}^2 X^2}{A^2} + \frac{\left(A'\right)^2}{A^2} , \end{aligned} \tag{4.5b}$$

$$\begin{aligned} 0 \; = \; & -\frac{A\dot{A}\dot{X}}{X} + \frac{\left(\partial_p A\right)\left(\partial_p X\right)}{AX} - \frac{\left(\partial_q A\right)\left(\partial_q X\right)}{AX} - A\ddot{A} \\ & + \frac{AA''}{X^2} - \frac{AA'X'}{X^3} + \frac{\partial_q^2 X}{X} - \frac{A^2 \ddot{X}}{X} , \end{aligned} \tag{4.5c}$$

$$\begin{aligned} 0 \; = \; & -\frac{A\dot{A}\dot{X}}{X} - \frac{\left(\partial_p A\right)\left(\partial_p X\right)}{AX} + \frac{\left(\partial_q A\right)\left(\partial_q X\right)}{AX} - A\ddot{A} \\ & + \frac{AA''}{X^2} - \frac{AA'X'}{X^3} + \frac{\partial_p^2 X}{X} - \frac{A^2 \ddot{X}}{X} , \end{aligned} \tag{4.5d}$$

where $\dot{} = \frac{\partial}{\partial t}$.

From (4.5b), using (4.2) and (4.3) and calculating explicitly all the derivatives, we get

$$2R\ddot{R} + \dot{R}^2 = f\,. \tag{4.6}$$

By the integration of (4.6), we see that the function R satisfies the Friedmann equation for dust:

$$\dot{R}^2 = \frac{2M}{R} + f\,, \tag{4.7}$$

where $M = M\,(r)$ is another arbitrary function of the coordinate r.

From (4.7) it follows that the acceleration of R is negative

$$\ddot{R} = -\frac{M}{R^2}\,. \tag{4.8}$$

Equation (4.7) shows that the evolution of R depends on the value of f. It can be:

- hyperbolic, $f > 0$:

$$R = \frac{M}{f}\left(\cosh\eta - 1\right)\,, \tag{4.9}$$

$$\sinh\eta - \eta = \frac{f^{\frac{3}{2}}\sigma\,(t-a)}{M}\,; \tag{4.10}$$

- parabolic, $f = 0$:

$$R = \left[\frac{9M\,(t-a)^2}{2}\right]^{\frac{1}{3}}\,; \tag{4.11}$$

- elliptic, $f < 0$:

$$R = \frac{M}{(-f)}\left(1 - \cos\eta\right)\,, \tag{4.12}$$

$$\eta - \sin\eta = \frac{(-f)^{\frac{3}{2}}\,\sigma\,(t-a)}{M}\,. \tag{4.13}$$

In the expressions above we have introduced the last arbitrary function $a = a(r)$, giving the local time of the big bang or big crunch $R = 0$, and $\sigma = \pm 1$ allows time reversal.

The six arbitrary functions f, M, a, S, P and Q give us five functions to control the physical inhomogeneity, plus a coordinate freedom to rescale r. The behavior of $R(t, r)$ is identical to that in the LTB model[1].

Finally, from (4.5a), using also (4.5c) and (4.5d), doing again the same procedure that brought to (4.6) and plugging (4.7) and (4.8) into the result, we obtain the following expression for the density:

$$\rho = \frac{2\left(M' - 3\frac{ME'}{E}\right)}{R^2\left(R' - \frac{RE'}{E}\right)}. \tag{4.14}$$

Singularities

We can see that the bang or the crunch occur when $R = 0$, so when $t = a$ or $t = \frac{2\pi M}{(-f)^{3/2}} + a$, and the density (4.14) is here divergent.

From (4.14) we can also see that there are singularities at $R' = \frac{RE'}{E}$ and $M' = 3\frac{ME'}{E}$, when shell crossings happen, i.e. when surfaces ("shells"[2]) of different values of r intersect.

Special cases

As already stated, the LTB model is a subcase of the Szekeres one, in particular it is the spherically symmetric special case $\epsilon = +1$, $E' = 0$.

In the vacuum case we have $\left(M' - 3\frac{ME'}{E}\right) = 0$, which implies $E' = M' = 0 \Rightarrow S' = P' = Q' = 0$. For $M \neq 0$, this gives pseudospherical and planar equivalents of the Schwarzschild metric.

---

[1] For a description of the LTB model see section 4.3.

[2] We call the comoving surfaces of constant r "shells" and paths that follow constant p and q "radial". However, the shells are quite different from spheres.

Finally, the null limit is obtained by taking, after a suitable transformation, $f \to \infty$. In this limit the "dust" particles move at light speed and the metric becomes a pure radiation Robinson-Trautman metric of Petrov type D[3].

4.1.2 Physical restrictions

Here we describe the physical restrictions of the Szekeres metric (4.1).

First of all, the metric should remain Lorentzian, i.e. $(-,+,+,+)$, so we must have $\epsilon + f \geq 0$. In particular $\epsilon + f > 0$ and $R' - \frac{RE'}{E} \neq 0$, while $\epsilon + f = 0$ where $R' = \frac{RE'}{E}$.

Obviously, pseudo-spherical foliations require $f \geq 1$ and then are only possible for regions with hyperbolic evolution ($f > 0$). In the same way, planar foliations are allowed only for regions with parabolic or hyperbolic evolution ($f \geq 0$), while spherical foliations are possible for all $f \geq -1$.

Second, the metric should be non-degenerate and non-singular, except at the bang or crunch. For a well behaved r coordinate we do need to specify

$$0 < \frac{\left(R' - R\frac{E'}{E}\right)^2}{\epsilon + f} < \infty \,. \tag{4.15}$$

Failure to satisfy this requirement may only be due to a bad choice of coordinates, so there should exist a choice of the coordinate r for which it holds.

The density should be non-negative and this adds the following restriction:

$$0 \leq \frac{M' - 3\frac{ME'}{E}}{R' - \frac{RE'}{E}} < \infty \,. \tag{4.16}$$

[3]For a discussion on the Petrov classification see [15, p. 48], while for the Robinson-Trautman pure radiation field see [15, p. 435].

Then we choose $R \geq 0$, $M \geq 0$ and $S > 0$ (but the sign of S, and hence of E, can be flipped without changing the metric signature).

Finally, the various arbitrary functions should have sufficient continuity (C^1 and piecewise C^3) except possibly at a spherical origin.

4.1.3 Physical interpretation

The role of R

The factor R^2 multiplies the unit sphere or pseudo-sphere both in the metric (4.1) and in the area integral $\mathcal{A} = R^2 \int \frac{1}{E^2} dpdq$, and therefore determines the magnitude of the curvature of the constant $(t,\, r)$ surfaces. We can see it as an "areal factor" or a "curvature scale". However, when $\epsilon < 0$ it is not at all like a spherical radius.

The role of M

If we look at (4.7) we see that M looks like a mass in the gravitational energy term, in particular, for $\epsilon = +1$, $M\,(r)$ it is simply the gravitational mass contained within a comoving "radius" r. However, this last interpretation is geometrically and physically correct only in the quasispherical model, where the surfaces of constant r are non-concentric spheres enclosing a finite amount of matter. But for $\epsilon \leq 0$, since the constant t and r surfaces are not closed, R is not the spherical radius and M is not a total gravitational mass.

The role of f

In (4.7) $f\,(r)$ represents twice the total energy per unit mass of the particles in the shells of matter at constant r.

Moreover, it also determines the geometry of the spatial sections at constant t. In the quasispherical case ($\epsilon = +1$) this three-space becomes Euclidean when $f = 0$. In the quasi-pseudospherical case ($\epsilon = -1$), instead, when $f = 0$ it becomes flat but pseudo-Euclidean (the signature is $(-, +, +)$). In the quasiplanar case the value $f = 0$ is not possible, so we do not have any flat three-dimensional subspace[4].

The role of E

Here we give only a brief explanation (without any proof) of the role of E. For a complete description see [20, sections II.E and III] or [23, section 3].

In the spherical and pseudospherical cases, the factor $\frac{E'}{E}$ determines the dipole nature of the shells at constant r (when $\epsilon = -1$ we have the pseudospherical equivalent of a dipole). When $\epsilon = 0$ (planar case) the effect of $\frac{E'}{E}$ is to tilt adjacent shells relative to each other, with the only zero tilt case ($E' = 0$) free of shell crossings.

Moreover, the shell separation is regulated by the factor $\frac{E'}{E}$.

Finally, we can see from (4.14) that $\frac{E'}{E}$ affects also the density distribution on each shell[5].

4.2 Metric properties

Now, following chapter 2, we calculate all the dynamical quantities for the Szekeres metric.

We use comoving coordinates, i.e. the four-velocity of the matter particles is $u^\mu = \delta^\mu{}_0$.

[4]For more details on the quasiplanar case see [20, section V].

[5]For more details about the influence of $\frac{E'}{E}$ on the density see [20, section VIII.D].

4.2.1 Acceleration vector

The expression for the acceleration vector a^μ in comoving coordinates is (2.16):

$$a^\mu \equiv \dot{u}^\mu = \Gamma^\mu{}_{00} \, .$$

We need then the following Christoffel symbols: $\Gamma^0{}_{00}$, $\Gamma^1{}_{00}$, $\Gamma^2{}_{00}$ and $\Gamma^3{}_{00}$.

Using the general expression

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) \, , \tag{4.17}$$

and the metric (4.1)

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{\left(R' - \frac{RE'}{E}\right)^2}{\epsilon + f} & 0 & 0 \\ 0 & 0 & \left(\frac{R}{E}\right)^2 & 0 \\ 0 & 0 & 0 & \left(\frac{R}{E}\right)^2 \end{pmatrix} \, , \tag{4.18}$$

with its inverse

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{\epsilon + f}{\left(R' - \frac{RE'}{E}\right)^2} & 0 & 0 \\ 0 & 0 & \left(\frac{E}{R}\right)^2 & 0 \\ 0 & 0 & 0 & \left(\frac{E}{R}\right)^2 \end{pmatrix} \, , \tag{4.19}$$

we get

$$\Gamma^0{}_{00} = 0 \, , \tag{4.20a}$$
$$\Gamma^1{}_{00} = 0 \, , \tag{4.20b}$$
$$\Gamma^2{}_{00} = 0 \, , \tag{4.20c}$$
$$\Gamma^3{}_{00} = 0 \, . \tag{4.20d}$$

So the acceleration vector a^μ vanishes for all μ (as we expected, because the Szekeres metric is a dust solution of the Einstein equation):

$$a^\mu = 0 \qquad \forall \mu\,, \tag{4.21}$$

and the motion of the matter particles is geodesic.

4.2.2 Expansion rate

For the expansion rate Θ in comoving coordinates we use (2.44), which we rewrite here:

$$\Theta = {\Gamma^\mu}_{\mu 0}\,.$$

We see then that the only Christoffel symbols we need to calculate the expansion rate Θ are ${\Gamma^0}_{00}$, ${\Gamma^1}_{10}$, ${\Gamma^2}_{20}$ and ${\Gamma^3}_{30}$.

Using the general expression (4.17) and the metric (4.18) with its inverse (4.19), these symbols are:

$$\begin{aligned}
{\Gamma^0}_{00} &= 0\,, && (4.22a)\\
{\Gamma^1}_{10} &= \frac{\dot{R}' - \frac{\dot{R}E'}{E}}{R' - \frac{RE'}{E}}\,, && (4.22b)\\
{\Gamma^2}_{20} &= \frac{\dot{R}}{R}\,, && (4.22c)\\
{\Gamma^3}_{30} &= \frac{\dot{R}}{R}\,. && (4.22d)
\end{aligned}$$

Then the expansion rate Θ is:

$$\Theta = \frac{\dot{R}' - 3\frac{\dot{R}E'}{E} + 2\frac{\dot{R}R'}{R}}{R' - \frac{RE'}{E}}\,. \tag{4.23}$$

4.2.3 Vorticity

For convenience, here we write again the expression of the vorticity tensor $\omega_{\mu\nu}$ in comoving coordinates (2.54):

$$\omega_{0\mu} = 0\,, \qquad \omega_{ij} = \partial_{[j}u_{i]} + u_{[j}\partial_0 u_{i]}\,.$$

Since $u_i = g_{0i}$ and from (4.18) $g_{0i} = 0$, we see that the vorticity $\omega_{\mu\nu}$ always vanishes:

$$\omega_{\mu\nu} = 0 \qquad \forall\,\mu,\,\nu\,. \tag{4.24}$$

So we are dealing with a geodesic (equation (4.21)) and rotation free flow.

4.2.4 Shear

The shear tensor $\sigma^\mu{}_\nu$ in comoving coordinates is (2.61):

$$\sigma^\mu{}_0 = 0\,, \qquad \sigma^i{}_j = \frac{1}{2}\delta^i{}_0\partial_0 u_j - \frac{1}{3}\Gamma^\rho{}_{\rho 0}h^i{}_j + \Gamma^i{}_{00}u_j + \Gamma^i{}_{j0}\,.$$

Recalling that $u_j = g_{0j}$, using (4.18) and its inverse (4.19), all the Christoffel symbols calculated above and the fact that $\Gamma^i{}_{j0} = 0$ for $i \neq j$ (from (4.17)), we obtain the following diagonal form for the shear $\sigma^\mu{}_\nu$:

$$\sigma^\mu{}_\nu = \left[\frac{\dot{R}' - \frac{\dot{R}R'}{R}}{3\left(R' - \frac{RE'}{E}\right)}\right] \mathrm{diag}\left(0,\ 2,\ -1,\ -1\right)\,. \tag{4.25}$$

As we can see from the definition of the backreaction variable (3.17), it is useful to give the square of the shear σ^2, defined as in (2.60). We have:

$$
\begin{aligned}
\sigma^2 &\doteq \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} \\
&= \frac{1}{2}\left(\sigma_{00}\sigma^{00} + \sigma_{11}\sigma^{11} + \sigma_{22}\sigma^{22} + \sigma_{33}\sigma^{33}\right) \\
&= \frac{1}{3}\left(\frac{\dot{R}' - \frac{\dot{R}R'}{R}}{R' - \frac{RE'}{E}}\right)^2 . \qquad (4.26)
\end{aligned}
$$

4.2.5 Electric and magnetic Weyl tensors

Finally, let us consider the expressions of the electric and magnetic Weyl tensors in comoving coordinates (2.66a) and (2.66b):

$$
\begin{aligned}
E_{\alpha\beta} &= C_{\alpha 0\beta 0} , \\
H_{\alpha\beta} &= \frac{1}{2}\eta_{\alpha\gamma\delta}C^{\gamma\delta}{}_{\beta 0} = \frac{1}{2}\eta_{\alpha\gamma\delta}g^{\gamma\xi}g^{\delta\psi}C_{\xi\psi\beta 0} .
\end{aligned}
$$

From these relations we see that we are interested only in the components $C_{\alpha 0\beta 0}$ and $C_{\xi\psi\beta 0}$ of the Weyl tensor. After determining them, the only ones that differ from zero are:

$$C_{1010} = -\frac{2M\left(R' - \frac{RE'}{E}\right)\left(R' - \frac{M'R}{3M}\right)}{(\epsilon + f)\, R^3} , \qquad (4.27a)$$

$$C_{2020} = \frac{M\left(R' - \frac{M'R}{3M}\right)}{E^2 R\left(R' - \frac{RE'}{E}\right)} , \qquad (4.27b)$$

$$C_{3030} = \frac{M\left(R' - \frac{M'R}{3M}\right)}{E^2 R\left(R' - \frac{RE'}{E}\right)} . \qquad (4.27c)$$

So, the electric Weyl tensor is[6]

$$E^{\alpha}{}_{\beta} = \left[\frac{M\left(R' - \frac{M'R}{3M}\right)}{R^3\left(R' - \frac{RE'}{E}\right)}\right] \mathrm{diag}\left(0,\,-2,\,1,\,1\right)\,, \tag{4.28}$$

whereas the magnetic Weyl tensor always vanishes:

$$H_{\alpha\beta} = 0 \qquad \forall \alpha,\, \beta\,. \tag{4.29}$$

4.3 The Lemaître-Tolman-Bondi model

Before starting to calculate the average of the quantities estimated in the previous section, let us describe the Lemaître-Tolman-Bondi (LTB) model[7].

As stated in subsection 4.1.1, the LTB model is a subcase of the Szekeres metric with $\epsilon = +1$ and $E' = 0$.

Following [33], let us consider a spherically symmetric dust universe with radial inhomogeneities as seen from our location at the center. By choosing comoving coordinates (the spatial origin $x^i = 0$ is the symmetry center), the line element assumes the following form:

$$ds^2 = -dt^2 + X^2\left(t,\,r\right)dr^2 + R^2\left(t,\,r\right)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,. \tag{4.30}$$

Now we want to determine the functions $X\left(t,\,r\right)$ and $R\left(t,\,r\right)$.

[6]Here, for convenience, the electric Weyl tensor has been written raising the first index α: $E^{\alpha}{}_{\beta} = g^{\alpha\xi}C_{\xi 0\beta 0}$.

[7]For a complete discussion on the LTB model see [33] and [34].

The non-zero Einstein tensor's components are:

$$G_{01} = \frac{2\dot{R}'}{R} - \frac{2\dot{X}R'}{RX}\,, \tag{4.31a}$$

$$G_{00} = -\frac{(R')^2}{R^2X^2} + \frac{2\dot{R}\dot{X}}{RX} + \frac{1}{R^2} - \frac{2R''}{RX^2} + \frac{2X'R'}{RX^3} + \frac{\dot{R}^2}{R^2}\,, \tag{4.31b}$$

$$G_{11} = X^2\left[\frac{2\ddot{R}}{R} + \frac{1}{R^2} + \frac{\dot{R}^2}{R^2} - \frac{(R')^2}{R^2X^2}\right], \tag{4.31c}$$

$$G_{22} = -R^2\left[-\frac{R''}{RX^2} + \frac{\ddot{R}}{R} + \frac{R'X'}{RX^3} + \frac{\dot{R}\dot{X}}{RX} + \frac{\ddot{X}}{X}\right], \tag{4.31d}$$

$$G_{33} = G_{22}\sin^2\theta\,. \tag{4.31e}$$

The Einstein equation for dust, from (2.87), then leads to the following system of equations:

$$\begin{cases} G_{00} = & \rho \\ G_{11} = & 0 \\ G_{22} = & 0 \\ G_{33} = & 0 \\ G_{01} = & 0 \end{cases}. \tag{4.32}$$

From the last equation in (4.32) we get

$$\dot{R}' - \frac{\dot{X}R'}{X} = 0\,, \tag{4.33}$$

and from its integration we have

$$X = C\left(r\right)R'\,, \tag{4.34}$$

where the function $C\left(r\right)$ depends only on the coordinate r.

By defining

$$C(r) \doteq \frac{1}{\sqrt{1-k(r)}}\,, \tag{4.35}$$

where $k(r) < 1$, we can rewrite the LTB metric (4.30) in its usual form

$$ds^2 = -dt^2 + \frac{(R')^2}{1-k(r)}dr^2 + R^2(t,\, r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,, \tag{4.36}$$

where $k(r)$ is a function associated with the curvature of $t = const.$ hypersurfaces[8].

The Friedmann equation for R is obtained from the second equation in (4.32). We have

$$R\ddot{R} + \frac{1}{2}\dot{R}^2 = -\frac{1}{2}k\,, \tag{4.37}$$

and from its integration we get

$$\dot{R}^2 = \frac{2M}{R} - k\,, \tag{4.38}$$

where $M = M(r)$. We see that the result is the same of the Szekeres metric (4.7).

Then, also the acceleration of R is the same. In fact we have

$$\ddot{R} = -\frac{M}{R^2}\,,$$

which is (4.8).

To calculate the density we start from the first and third equations in (4.32). They give

$$\begin{cases} -\frac{(R')^2}{R^2X^2} + \frac{2\dot{R}\dot{X}}{RX} + \frac{1}{R^2} - \frac{2R''}{RX^2} + \frac{2X'R'}{RX^3} + \frac{\dot{R}^2}{R^2} = \rho \\ -\frac{R''}{RX^2} + \frac{\ddot{R}}{R} + \frac{R'X'}{RX^3} + \frac{\dot{R}\dot{X}}{RX} + \frac{\ddot{X}}{X} = 0 \end{cases}\,, \tag{4.39}$$

[8] In the limit $R(t,\, r) \longrightarrow a(t)\, r$ and $k(r) \longrightarrow kr^2$ we obtain the FLRW model.

and, substituting the second equation in the first one, the expression for the density becomes

$$\rho = \frac{2M'}{R^2 R'}\,, \tag{4.40}$$

which is (4.14) with $E' = 0$.

Now we calculate the properties of the LTB metric.

Acceleration vector

As in the Szekeres case, by following subsection 4.2.1, the acceleration vector a^μ always vanishes:

$$a^\mu = 0 \qquad \forall \mu\,, \tag{4.41}$$

and the motion is geodesic.

Expansion rate

As in subsection 4.2.2, the expansion rate Θ in comoving coordinates is (2.44):

$$\Theta = \Gamma^\mu{}_{\mu 0}\,.$$

Then we see that the only Christoffel symbols we need to calculate the expansion rate Θ are $\Gamma^0{}_{00}$, $\Gamma^1{}_{10}$, $\Gamma^2{}_{20}$ and $\Gamma^3{}_{30}$.

Using (4.17) and the metric (4.36)

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{(R')^2}{1-k} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2 \sin^2\theta \end{pmatrix}, \tag{4.42}$$

and its inverse

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-k}{(R')^2} & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2 \sin^2\theta} \end{pmatrix}, \tag{4.43}$$

we get:

$$\Gamma^0{}_{00} = 0\,, \tag{4.44a}$$

$$\Gamma^1{}_{10} = \frac{\dot{R}'}{R'}\,, \tag{4.44b}$$

$$\Gamma^2{}_{20} = \frac{\dot{R}}{R}\,, \tag{4.44c}$$

$$\Gamma^3{}_{30} = \frac{\dot{R}}{R}\,. \tag{4.44d}$$

Then the expansion rate Θ is:

$$\Theta = \frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}\,, \tag{4.45}$$

that is (4.23) with $E' = 0$.

Vorticity

Like in subsection 4.2.3, we can see that the vorticity $\omega_{\mu\nu}$ always vanishes:

$$\omega_{\mu\nu} = 0 \qquad \forall\ \mu,\ \nu\,. \tag{4.46}$$

So we are dealing with a rotation free flow.

Shear

To calculate the shear ${\sigma^\mu}_\nu$, we follow subsection 4.2.4.

The shear tensor ${\sigma^\mu}_\nu$ in comoving coordinates is (2.61):

$$ {\sigma^\mu}_0 = 0\,, \qquad {\sigma^i}_j = \frac{1}{2}{\delta^i}_0\partial_0 u_j - \frac{1}{3}{\Gamma^\rho}_{\rho 0}{h^i}_j + {\Gamma^i}_{00}u_j + {\Gamma^i}_{j0}\,. $$

Since from (4.36) we have $u_j = g_{0j} = 0$, using all the Christoffel symbols calculated above and the fact that ${\Gamma^i}_{j0} = 0$ for $i \neq j$ (see (4.17)), we obtain the following diagonal form for the shear ${\sigma^\mu}_\nu$:

$$ {\sigma^\mu}_\nu = \frac{1}{3}\left(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R}\right)\mathrm{diag}\,(0,\,2,\,-1,\,-1)\,, \tag{4.47} $$

i.e. (4.25) with $E' = 0$.

From (2.60), the square of the shear is

$$ \begin{aligned} \sigma^2 &\doteq \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} \\ &= \frac{1}{3}\left(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R}\right)^2\,. \end{aligned} \tag{4.48} $$

Spatial Ricci scalar

From the Hamiltonian constraint (3.13), we can calculate the spatial Ricci scalar ${}^{(3)}R$. We have:

$$ {}^{(3)}R = 2\left(\rho + \sigma^2 - \frac{1}{3}\Theta^2\right)\,. \tag{4.49} $$

Now, using (4.40), (4.45), (4.48) and rewriting the density (4.40) by means of the Friedmann equations (4.38), we get:

$$ {}^{(3)}R = 2\frac{(Rk)'}{R^2R'}\,. \tag{4.50} $$

Electric and magnetic Weyl tensors

Finally, the expressions of the electric and magnetic Weyl tensors in comoving coordinates are (2.66a) and (2.66b):

$$\begin{aligned} E_{\alpha\beta} &= C_{\alpha 0 \beta 0}\,, \\ H_{\alpha\beta} &= \frac{1}{2}\eta_{\alpha\gamma\delta}C^{\gamma\delta}{}_{\beta 0} = \frac{1}{2}\eta_{\alpha\gamma\delta}g^{\gamma\xi}g^{\delta\psi}C_{\xi\psi\beta 0}\,. \end{aligned}$$

Doing again the same procedure as in subsection 4.2.5, the electric Weyl tensor is (4.28) with $E' = 0$, i.e.

$$E^{\alpha}{}_{\beta} = \frac{M}{R^3 R'}\left(R' - \frac{M' R}{3M}\right)\mathrm{diag}\left(0,\ -2,\ 1,\ 1\right)\,, \tag{4.51}$$

whereas the magnetic Weyl tensor always vanishes:

$$H_{\alpha\beta} = 0 \qquad \forall \alpha,\, \beta\,. \tag{4.52}$$

4.4 Averaging in the quasispherical Szekeres model

The quasispherical Szekeres model is a non-symmetrical generalization of the spherically symmetric LTB model.

In this section we see that the volume averaging within the quasispherical Szekeres model leads to the same solutions as those obtained within the LTB model.

Here we consider only the quasispherical case because the averaging procedure in the quasihyperbolic and quasiplane cases requires a special treatment. In fact, an area of a surface of constant t and r in the quasihyperbolic and quasiplane models in infinite. Moreover in these two cases there is no origin (in the quasihyperbolic model r cannot be zero and in the quasiplane case r can only asymptotically approach the origin[9]).

Now, following [14] and chapter 3, we calculate the average quantities. The volume is calculated around the observer located at the origin.

First of all, we see that the volume in the Szekeres model is the same as in the LTB model. To prove this, we rewrite here the Szekeres metric (4.1) in the quasispherical case:

$$ds^2 = -dt^2 + X^2\left(t,\, r,\, p,\, q\right) dr^2 + A^2\left(t,\, r,\, p,\, q\right)\left(dp^2 + dq^2\right)\ ,$$

where now

$$X^2\left(t,\, r,\, p,\, q\right) = \frac{\left(R' - R\frac{E'}{E}\right)^2}{1+f} \qquad \text{and} \qquad A^2\left(t,\, r,\, p,\, q\right) = \frac{R^2}{E^2}\,. \tag{4.53}$$

The volume is then

$$V_{\mathcal{D}} \doteq \int_0^{r_{\mathcal{D}}} dr \iint dpdq \sqrt{\left|{}^{(3)}g\right|}\,, \tag{4.54}$$

where

$${}^{(3)}g \doteq \det \begin{pmatrix} X^2\left(t,\, r,\, p,\, q\right) & 0 & 0 \\ 0 & A^2\left(t,\, r,\, p,\, q\right) & 0 \\ 0 & 0 & A^2\left(t,\, r,\, p,\, q\right) \end{pmatrix}. \tag{4.55}$$

So, we get

$$\begin{aligned} V_{\mathcal{D}} &= \int_0^{r_{\mathcal{D}}} dr \iint dpdq X A^2 \\ &= \int_0^{r_{\mathcal{D}}} dr \iint dpdq \frac{R^2}{\sqrt{1+f}} \left(R' - \frac{RE'}{E}\right) \frac{1}{E^2} \\ &= \int_0^{r_{\mathcal{D}}} dr \left[\frac{R^2 R'}{\sqrt{1+f}} \iint dpdq \frac{1}{E^2} + \frac{1}{2} \frac{R^3}{\sqrt{1+f}} \frac{\partial}{\partial r} \left(\iint \frac{dpdq}{E^2}\right)\right] . \end{aligned} \tag{4.56}$$

[9]For more details see [20].

Since $\frac{dpdq}{E^2}$ is the metric of a unit sphere, we have

$$\iint \frac{dpdq}{E^2} = 4\pi \, . \tag{4.57}$$

Thus the volume becomes

$$V_{\mathcal{D}} = 4\pi \int_0^{r_{\mathcal{D}}} dr \frac{R^2 R'}{\sqrt{1+f}} \, . \tag{4.58}$$

The same result is obtained if initially we set E' to zero, that is we have the same volume as in the LTB model.

This also happens to the density. Using (4.14), the average density is:

$$\begin{aligned} \langle \rho \rangle_{\mathcal{D}} &\doteq \frac{1}{V_{\mathcal{D}}} \int_0^{r_{\mathcal{D}}} dr \iint dpdq X A^2 \rho \\ &= \frac{2}{V_{\mathcal{D}}} \int_0^{r_{\mathcal{D}}} dr \iint dpdq \frac{1}{\sqrt{1+f}} \frac{1}{E^2} \left(M' - 3\frac{ME'}{E} \right) \\ &= \frac{2}{V_{\mathcal{D}}} \int_0^{r_{\mathcal{D}}} dr \left[\frac{M'}{\sqrt{1+f}} \iint \frac{dpdq}{E^2} + \frac{3M}{\sqrt{1+f}} \frac{\partial}{\partial r} \left(\iint \frac{dpdq}{E^2} \right) \right] \, , \end{aligned} \tag{4.59}$$

and recalling (4.57) we get

$$\langle \rho \rangle_{\mathcal{D}} = \frac{8\pi}{V_{\mathcal{D}}} \int_0^{r_{\mathcal{D}}} dr \frac{M'}{\sqrt{1+f}} \, . \tag{4.60}$$

This result is the same as in the LTB model, i.e. setting from the beginning $E' = 0$.

The average of the expansion rate is, from (4.23):

$$
\begin{aligned}
\langle\Theta\rangle_{\mathcal{D}} &\doteq \frac{1}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint dpdqXA^2\Theta \\
&= \frac{1}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint dpdqXA^2 \frac{\dot{R}' - 3\frac{\dot{R}E'}{E} + 2\frac{\dot{R}R'}{R}}{R' - \frac{RE'}{E}} \\
&= \frac{1}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint dpdq \frac{R^2}{\sqrt{1+f}}\frac{1}{E^2}\left(\dot{R}' - 3\frac{\dot{R}E'}{E} + 2\frac{\dot{R}R'}{R}\right) \\
&= \frac{1}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \frac{R^2}{\sqrt{1+f}}\left[\iint \frac{dpdq}{E^2}\left(\dot{R}' + 2\frac{\dot{R}R'}{R}\right) + \frac{3}{2}\dot{R}\frac{\partial}{\partial r}\left(\iint \frac{dpdq}{E^2}\right)\right] \\
&= \frac{4\pi}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \frac{R^2 R'}{\sqrt{1+f}}\left(\frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}\right) . \qquad (4.61)
\end{aligned}
$$

As above, E' does not contribute to the final result, so we have the same average of the expansion rate as in the LTB model (see (4.45)).

Now we can give the backreaction variable, defined as in (3.17):

$$
\mathcal{Q}_{\mathcal{D}} \doteq \frac{2}{3}\left(\langle\Theta^2\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}^2\right) - 2\langle\sigma^2\rangle_{\mathcal{D}} .
$$

First we should calculate the following term:

$$
\begin{aligned}
\frac{2}{3}\langle\Theta^2\rangle_{\mathcal{D}} - 2\langle\sigma^2\rangle_{\mathcal{D}} &\doteq 2\left[\frac{1}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint dpdqXA^2\left(\frac{1}{3}\Theta^2 - \sigma^2\right)\right] \\
&= \frac{2}{3V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint \frac{dpdq}{E^2}\frac{R^2}{\sqrt{1+f}\left(R' - \frac{RE'}{E}\right)} \\
&\quad \left[\left(\dot{R}' - 3\frac{\dot{R}E'}{E} + 2\frac{\dot{R}R'}{R}\right)^2 - \left(\dot{R}' - \frac{\dot{R}R'}{R}\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{3V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint \frac{dpdq}{E^2}\frac{R^2}{\sqrt{1+f}\left(R' - \frac{RE'}{E}\right)} \\
&\quad \left[3\frac{\dot{R}^2\left(R'\right)^2}{R^2} + 6\frac{\dot{R}R'\dot{R}'}{R} + 9\frac{\dot{R}^2\left(E'\right)^2}{E^2}\right. \\
&\quad \left. -6\frac{\dot{R}\dot{R}'E'}{E} - 12\frac{\dot{R}^2R'E'}{R}\right] \\
&= \frac{2}{3V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint \frac{dpdq}{E^2}\frac{R^2}{\sqrt{1+f}\left(R' - \frac{RE'}{E}\right)} \\
&\quad \left[3\frac{\dot{R}^2R'}{R^2} + 6\frac{\dot{R}\dot{R}'}{R} - 9\frac{\dot{R}^2E'}{RE}\right]. \qquad (4.62)
\end{aligned}
$$

Using (4.57) we get:

$$
\frac{2}{3}\left\langle\Theta^2\right\rangle_{\mathcal{D}} - 2\left\langle\sigma^2\right\rangle_{\mathcal{D}} = \frac{8\pi}{3V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr\frac{R^2R'}{\sqrt{1+f}}\left(3\frac{\dot{R}^2}{R^2} + 6\frac{\dot{R}\dot{R}'}{RR'}\right), \qquad (4.63)
$$

thus the backreaction variable becomes:

$$
\begin{aligned}
\mathcal{Q}_{\mathcal{D}} &= \frac{8\pi}{3V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr\frac{R^2R'}{\sqrt{1+f}}\left(3\frac{\dot{R}^2}{R^2} + 6\frac{\dot{R}\dot{R}'}{RR'}\right) \\
&\quad -\frac{2}{3}\left[\frac{4\pi}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr\frac{R^2R'}{\sqrt{1+f}}\left(\frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}\right)\right]^2. \qquad (4.64)
\end{aligned}
$$

This result is the same if we set initially $E' = 0$, so the backreaction does not change from the LTB model.

Finally, we can calculate the spatial Ricci scalar ${}^{(3)}R$ and then give its average.

With the same procedure used in the LTB model, starting from the Hamiltonian constraint (3.13), the expression for the spatial Ricci scalar

${}^{(3)}R$ is (4.49), which we rewrite here:

$$ {}^{(3)}R = 2\left(\rho + \sigma^2 - \frac{1}{3}\Theta^2\right) . $$

Now, inserting (4.14), (4.23), (4.26) and using the Friedmann equation (4.7) in the density expression (4.14), we get

$$ {}^{(3)}R = -\frac{2f}{R^2}\left(\frac{\frac{Rf'}{f} - 2\frac{RE'}{E}}{R' - \frac{RE'}{E}} + 1\right) , \tag{4.65} $$

which reduces to the LTB expression (4.50) when $E' = 0$ and $f = -k$.

Averaging the relation (4.65) yields

$$ \begin{aligned} \left\langle {}^{(3)}R \right\rangle_{\mathcal{D}} &\doteq \frac{1}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint dpdq X A^{2(3)}R \\ &= \frac{1}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint \frac{dpdq}{E^2}\frac{R' - \frac{RE'}{E}}{\sqrt{1+f}} \\ &\quad \left[-2f\left(\frac{\frac{Rf'}{f} - 2\frac{RE'}{E}}{R' - \frac{RE'}{E}} + 1\right)\right] \\ &= \frac{1}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \iint \frac{dpdq}{E^2}\frac{(-2f)}{\sqrt{1+f}} \\ &\quad \left(\frac{Rf'}{f} - 2\frac{RE'}{E} + R' - \frac{RE'}{E}\right) , \end{aligned} \tag{4.66} $$

and inserting (4.57) we have

$$ \left\langle {}^{(3)}R \right\rangle_{\mathcal{D}} = -\frac{8\pi}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} dr \frac{(Rf)'}{\sqrt{1+f}} . \tag{4.67} $$

This expression, as above, is the same if we set E' to zero from the beginning, so we have the same average spatial Ricci scalar as in the LTB model.

So, in this section we have proved that the Szekeres model is a generalization of the LTB model, in fact all our solutions are the same if initially we had set E' to zero.

Thus all the results found when studying the average within the LTB model also apply to the Szekeres case. For instance, we can conclude that $\mathcal{Q}_{\mathcal{D}} = 0$ when $f = 0$ (as in the parabolic LTB model, i.e. when $k = 0$). A proof of this fact can be found in [35, subsection 3.2].

Moreover, in [35], [49] and [50] has been proven that in the LTB model the inhomogeneities can induce acceleration.

Chapter 5

Non-tilted Bianchi models

Despite the success of the FLRW models, the structure that we observe today, means that our universe is neither homogeneous nor isotropic, at least on certain scales. To follow the late time evolution of the universe on these scales we need models with more degrees of freedom than the FLRW ones. For this, Bianchi models have long been studied.

Then, in this chapter we are interested in Bianchi models, which describe homogeneous but anisotropic ($q = 0$, $s = 3 \Rightarrow r = 3$) universes. So we are dealing with a three-dimensional group of isometries[1] G_3.

Here we follow [25, section 3], [24, section II] and [18, chapter 9].

First we give a modern classification of Bianchi models. Then we specialize in non-tilted (i.e. with the flow-lines of the fluid normal to the hypersurfaces of homogeneity) Bianchi models: after a general discussion of the spatially homogeneous model in the synchronous system, we study all the different types of dust metrics, with their properties.

Tilted Bianchi models go beyond the aim of this thesis, but a complete description of the metrics that characterize these universes can be found in [28]-[32].

[1] See section A.2 for a classification of cosmological models by means of their symmetries.

5.1 Classification of Bianchi models

In this section we give the modern classification of Bianchi models (for the original classification see appendix B and [27]).

The scheme for classifying the equivalence classes of three-dimensional Lie algebras uses the irreducible parts of the structure constant tensor under linear transformations (rather than the more complicated derived group approach of Lie and Bianchi).

Following [25, section 3], we decompose the structure constants $C^a{}_{bc}$ (see (A.4)) into a symmetric contravariant tensor n^{ab} and a covariant vector a_b:

$$C^a{}_{bc} = \varepsilon_{dbc} n^{ad} + \delta^a{}_c a_b - \delta^a{}_b a_c \,, \tag{5.1}$$

where ε_{abc} is the three-dimensional antisymmetric tensor and n^{ab} and a_b are defined as

$$a_b \doteq \frac{1}{2} C^a{}_{ba} \,, \tag{5.2}$$

$$n^{ab} \doteq \frac{1}{2} C^{(a}{}_{cd} \varepsilon^{b)cd} \,. \tag{5.3}$$

The structure constants $C^a{}_{bc}$ expressed as in (5.1) clearly satisfy the skew-symmetry property (see (A.5)). The Jacobi identity (A.6) is satisfied only if the vector a_b has zero contraction with the tensor n^{ab}, that is

$$C^a{}_{e[b} C^e{}_{cd]} = 0 \qquad \Leftrightarrow \qquad n^{ab} a_b = 0 \,. \tag{5.4}$$

We can choose a convenient basis (the tetrad basis, chosen to be invariant under the group of isometries[2]) to diagonalize n^{ab} to obtain $n^{ab} = \mathrm{diag}\,(n_1,\, n_2,\, n_3)$ and to set $a_b = (a,\, 0,\, 0)$. Then the Jacobi identity (5.4) becomes

$$n_1 a = 0 \,. \tag{5.5}$$

[2]For a description of how to construct an invariant basis see [18, section 6.3].

At this point we can classify the structure constants in two classes:

- class A: $a = 0$;
- class B: $a \neq 0$.

Then both the classes can be classified further by the sign of the eigenvalues of n^{ab} (i.e. the signs of n_1, n_2 and n_3). For this classification see table 5.1 [25].

Group class	Group type	n^{ab} eigenvalues	a
	I	0, 0, 0	0
	II	+, 0, 0	0
A	VI_0	0, +, −	0
	VII_0	0, +, +	0
	$VIII$	−, +, +	0
	IX	+, +, +	0
	V	0, 0, 0	+
	IV	0, 0, +	+
B	VI_h	0, +, −	+
	III	0, +, −	$n_2 n_3$
	VII_h	0, +, +	+

Table 5.1: Classification of homogeneous cosmological models into ten equivalence classes.

The parameter h in class B is defined by the scalar constant of proportionality in the following relation

$$a_b a_c = \frac{h}{2}\varepsilon_{bik}\varepsilon_{cjl}n^{ij}n^{kl}\,. \tag{5.6}$$

In the case of diagonal n^{ab}, the factor h has a simple form: $h = \frac{a^2}{n_2 n_3}$.
In addition to the classification given above we can distinguish two further Bianchi models:

- *orthogonal models*: here the fluid flow lines are orthogonal to the surfaces of homogeneity. So, the fluid four-velocity u^μ is parallel to the normal vector n^μ to the surfaces. In this case the matter variables will be just the fluid density and pressure;

- *tilted models*: the fluid flow lines are no longer orthogonal to the surfaces of homogeneity, thus the four-velocity u^μ is not parallel to the normal vector n^μ. Here, in addition to the fluid pressure and density, we need the peculiar velocity of the fluid relative to the normal vectors.

In this work, as stated at the beginning of this chapter, we are dealing only with non-tilted models.

5.2 The general spatially homogeneous model in the synchronous system

In this section we want to calculate the Ricci tensor $R_{\mu\nu}$ in the synchronous system, from which the Einstein equation (2.68) can be derived.

We follow the procedure used in [18, chapter 9].

We know that, in a spatially homogeneous model described by a manifold M, through every point it passes an invariant or homogeneous, three-dimensional hypersurface S. This hypersurface is generated by the three-dimensional isometry group G of the model.

A one-parameter family of these hypersurfaces fill M and the direction of the axis of the parameter t may be chosen quite freely. Once this choice has been made, we can find the one-forms[3] ω^i.

A useful choice, which defines the *synchronous system*, for the direction of t is the timelike direction, perpendicular to each hypersurface S. In this case, the normal to the hypersurface is given by $n_\mu = \nabla_\mu t$,

[3]For more details see [18, chapter 6].

such that $g^{\mu\nu}n_\mu n_\nu = -1$. The existence of such a timelike normal vector assumes that the $S\left(t\right)$ are spacelike.

By parametrizing with the proper time t, the four-dimensional metric is written as

$$ds^2 = -dt^2 + g_{ij}\omega^i\omega^j \,, \tag{5.7}$$

where the three-dimensional metric g_{ij} depends only on t, that is $g_{ij} = g_{ij}\left(t\right)$.

The one-forms ω^i satisfy[4]

$$d\omega^i = \frac{1}{2}C^i{}_{st}\omega^s \wedge \omega^t \,, \tag{5.8}$$

where $C^i{}_{st}$ are the structure constants of the isometry group G.

The group structure of the manifold M implies the existence of a vector field basis dual to $\left\{-dt,\, \omega^i\right\}$ such that

$$\left[Y_0,\, Y_i\right] = 0\,, \quad \left[Y_i,\, Y_j\right] = -C^s{}_{ij}Y_s\,, \tag{5.9}$$

and

$$Y_0 \cdot Y_0 = -1\,, \qquad Y_0 \cdot Y_i = 0\,, \qquad Y_i \cdot Y_j = g_{ij}\left(t\right)\,. \tag{5.10}$$

As long as the homogeneous hypersurfaces $S\left(t\right)$ remain spacelike, the synchronous basis is unique. This because the hypersurfaces are picked out by the group action unambiguously. Then the vector Y_0 is the unique normal to these surfaces[5].

Now we want to rewrite the metric (5.7) in the *orthonormal synchronous basis*, defined by

$$\sigma^0 = dt \qquad \text{and} \qquad \sigma^i = b_{is}\left(t\right)\omega^s\,, \tag{5.11}$$

[4] See [18, relation (6.19)].

[5] If the hypersurfaces $S\left(t\right)$ change from spacelike to timelike, the synchronous system breaks down and we must use another basis.

where $b_{is}b_{sj} = g_{ij}$ and $b_{ij} = b_{ji}$. The matrix $B = (b_{ij})$ is the symmetric square root of $G = (g_{ij})$.

With this choice, the metric (5.7) becomes diagonal:

$$ds^2 = -\left(\sigma^0\right)^2 + \left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \left(\sigma^3\right)^2 . \tag{5.12}$$

Now we define the scalar $(\det B)^{\frac{1}{3}}$ in the following way:

$$(\det B)^{\frac{1}{3}} \doteq e^{-\Omega(t)} , \tag{5.13}$$

where $\Omega(t)$ is a scalar.

We write also

$$e^{\Omega} B \doteq \left({e^{\beta(t)}}_{ij}\right) , \tag{5.14}$$

where $(\beta_{ij}(t))$ is a 3×3 symmetric, traceless matrix[6].

Therefore we have

$$B = \left(e^{-\Omega} {e^{\beta}}_{ij}\right) . \tag{5.15}$$

We have then split the matrix B in its volume and distortion parts, where the scalar Ω represents the volumetric expansion.

In order to calculate the Ricci tensor, we need to compute the affine connection forms. To do this we need the curls of the one-forms σ^μ:

$$\begin{aligned} d\sigma^i &= \left(\frac{db_{is}}{dt}\right) dt \wedge \omega^s + b_{is} d\omega^s \\ &= \left(-\dot{\Omega} e^{-\Omega} {e^{\beta}}_{is} + e^{-\Omega}\left(\dot{{e^{\beta}}}_{is}\right)\right) dt \wedge \omega^s + e^{-\Omega} {e^{\beta}}_{is} d\omega^s \\ & \quad \left(-\dot{\Omega} e^{-\Omega} {e^{\beta}}_{is} + e^{-\Omega}\left(\dot{{e^{\beta}}}_{is}\right)\right) dt \wedge \omega^s + \frac{1}{2} e^{-\Omega} {e^{\beta}}_{is} {C^s}_{tu} \omega^t \wedge \omega^u \end{aligned}$$

[6] e^β means matrix exponentiation, i.e. $e^\beta = \sum_{r=0}^{\infty} \frac{1}{r!}\beta^r$.
We have: $\det e^\beta = e^{\mathrm{Tr}\beta} = 1$.

$$
\begin{aligned}
&= \left(-\dot{\Omega}\delta_{iu} + \left(\dot{e^{\beta}}_{it}\right)e^{-\beta}{}_{tu}\right) e^{-\Omega}e^{\beta}{}_{us}dt \wedge \omega^s \\
&\quad + \frac{1}{2}e^{-\Omega}e^{\beta}{}_{is}C^s{}_{tu}\omega^t \wedge \omega^u \,,
\end{aligned} \tag{5.16}
$$

where (5.8) has been used. Moreover, recalling the property of the exterior derivative $d \circ d = 0$, we have $d\sigma^0 = 0$.

Now, using the expression of ω^i in terms of σ^i (from (5.11)), i.e.

$$
\omega^i = e^{\Omega}e^{-\beta}{}_{is}\sigma^s \,, \tag{5.17}
$$

we can rewrite (5.16) as

$$
\begin{aligned}
d\sigma^i &= \left(-\dot{\Omega}\delta_{iu} + \left(\dot{e^{\beta}}_{it}\right)e^{-\beta}{}_{tu}\right) e^{\beta}{}_{us}e^{-\beta}{}_{sj}dt \wedge \sigma^j \\
&\quad + \frac{1}{2}e^{\beta}{}_{is}C^s{}_{tu}e^{-\beta}{}_{tj}e^{\Omega}e^{-\beta}{}_{uk}\sigma^j \wedge \sigma^k \\
&= \left(-\dot{\Omega}\delta_{ij} + \left(\dot{e^{\beta}}_{it}\right)e^{-\beta}{}_{tj}\right) dt \wedge \sigma^j + \frac{1}{2}e^{\Omega}e^{\beta}{}_{is}C^s{}_{tu}e^{-\beta}{}_{tj}e^{-\beta}{}_{uk}\sigma^j \wedge \sigma^k \\
&= k_{ij}dt \wedge \sigma^j + \frac{1}{2}d^i{}_{jk}\sigma^j \wedge \sigma^k \,,
\end{aligned} \tag{5.18}
$$

where we have defined

$$
k_{ij} \doteq -\dot{\Omega}\delta_{ij} + \left(\dot{e^{\beta}}_{it}\right)e^{-\beta}{}_{tj} \,, \tag{5.19}
$$

$$
d^i{}_{jk} \doteq e^{\Omega}e^{\beta}{}_{is}C^s{}_{tu}e^{-\beta}{}_{tj}e^{-\beta}{}_{uk} \,. \tag{5.20}
$$

The expression for k_{ij} simplifies if $\dot{\beta}$ commutes with β. In fact in this case we have $\left(\dot{e^{\beta}}_{it}\right)e^{-\beta}{}_{tj} = \dot{\beta}_{ij}$.

We can also see that $d^i{}_{jk}$ has the same symmetry as $C^i{}_{jk}$ and satisfies the two properties

$$
\begin{aligned}
d^i{}_{jk} &= -d^i{}_{kj} && \text{(skew-symmetry)}\,, && (5.21)\\
d^a{}_{s[i}d^s{}_{jk]} &= 0 && \text{(Jacobi identity)}\,. && (5.22)
\end{aligned}
$$

We now calculate the connection one-forms (see [18, section 2.5])

$$\sigma^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\sigma^\rho\,. \tag{5.23}$$

From (5.23) and the fact that the covariant derivative of the metric vanishes, we have[7]:

$$\sigma_{\mu\nu} + \sigma_{\nu\mu} = 0\,. \tag{5.24}$$

This equation implies[8]:

$$
\begin{aligned}
\sigma^0{}_0 &= 0\,, && (5.25a)\\
\sigma^i{}_i &= 0\,, && (5.25b)\\
\sigma^0{}_i &= \sigma^i{}_0\,, && (5.25c)\\
\sigma^i{}_j &= -\sigma^j{}_i\,. && (5.25d)
\end{aligned}
$$

Using the first Cartan equation (see [18, section 2.5])

$$d\sigma^\rho = -\sigma^\rho{}_\mu \wedge \sigma^\mu = \Gamma^\rho{}_{\mu\nu}\sigma^\mu \wedge \sigma^\nu\,, \tag{5.26}$$

we have

$$\Gamma^0{}_{\mu\nu}\sigma^\mu \wedge \sigma^\nu = 0\,, \tag{5.27}$$

and

$$k_{ij}\sigma^0 \wedge \sigma^j + \frac{1}{2}d^i{}_{jk}\sigma^j \wedge \sigma^k = \Gamma^i{}_{\mu\nu}\sigma^\mu \wedge \sigma^\nu\,. \tag{5.28}$$

---

[7] In fact, from (5.12), which can be written as $\eta = \eta_{ik}\sigma^i \wedge \sigma^k$,and $\nabla_j \eta_{ik} = 0$, it comes out that $d\eta_{ik} = \eta_{ik,j}\sigma^j = \left(\Gamma^h{}_{kj}\eta_{ih} + \Gamma^h{}_{ij}\eta_{hk}\right)\sigma^j$. Then, using (5.23) we have: $d\eta_{ik} = \sigma_{ik} + \sigma_{ki} = 0$.

[8] Note that to lower or raise the indices we should use the metric $\eta_{\mu\nu}$, i.e. (5.12).

Using the last two equations and the symmetry (5.24) we have

$$\begin{aligned} k_{ij} &= \Gamma^i{}_{0j} - \Gamma^i{}_{j0}\,, && (5.29a) \\ d^i{}_{jk} &= \Gamma^i{}_{jk} - \Gamma^i{}_{kj}\,, && (5.29b) \\ 0 &= \Gamma^0{}_{\mu\nu} - \Gamma^0{}_{\nu\mu}\,. && (5.29c) \end{aligned}$$

The solutions of equations (5.29a)-(5.29c) are

$$\begin{aligned} \Gamma^0{}_{i0} &= 0\,, && (5.30a) \\ \Gamma^0{}_{ij} &= \Gamma^0{}_{ji} = \frac{1}{2}\left(k_{ij} + k_{ji}\right) \doteq l_{ij}\,, && (5.30b) \\ \Gamma^i{}_{j0} &= \frac{1}{2}\left(k_{ji} - k_{ij}\right) \doteq m_{ji} = -m_{ij}\,, && (5.30c) \\ \Gamma^i{}_{jk} &= \frac{1}{2}\left(d^i{}_{jk} - d^k{}_{ij} - d^j{}_{ik}\right)\,, && (5.30d) \end{aligned}$$

whence

$$\begin{aligned} \sigma^0{}_i &= l_{ij}\sigma^j\,, && (5.31a) \\ \sigma^i{}_j &= -m_{ij}\sigma^0 + \frac{1}{2}\left(d^i{}_{jk} - d^k{}_{ij} - d^j{}_{ik}\right)\sigma^k\,. && (5.31b) \end{aligned}$$

Now we calculate the components of the Ricci tensor in the $\{\sigma^\mu\}$ basis.

For the (00)-component, starting from the definition of Ricci tensor (2.3) and using the property (2.2a) of the Riemann tensor, we have

$$R_{00} = R^\mu{}_{0\mu 0} = -R^0{}_{i0i}\,, \tag{5.32}$$

where the index i is summed over 1, 2, 3 and we have used the metric $\eta_{\mu\nu}$ to raise and lower the indices.

From the definition of the curvature two-form

$$\Omega^\mu{}_\nu = R^\mu{}_{\nu\xi\rho}\sigma^\xi \wedge \sigma^\rho\,, \tag{5.33}$$

and the second Cartan equation

$$\Omega^\mu{}_\nu = d\sigma^\mu{}_\nu + \sigma^\mu{}_\rho \wedge \sigma^\rho{}_\nu \,, \tag{5.34}$$

we get

$$\begin{aligned} \Omega^0{}_i &= R^0{}_{i\mu\nu}\sigma^\mu \wedge \sigma^\nu \,, && (5.35) \\ \Omega^0{}_i &= d\sigma^0{}_i + \sigma^0{}_\rho \wedge \sigma^\rho{}_i \,. && (5.36) \end{aligned}$$

By taking the $(0i)$-component of the last two equations we find

$$R^0{}_{i0i} = \dot{l}_{ii} + l_{ij}k_{ji} - m_{ji}l_{ij} \,, \tag{5.37}$$

then

$$R_{00} = -\dot{l}_{ii} - l_{ij}k_{ji} \,. \tag{5.38}$$

A similar calculation brings to

$$R_{0i} = R^\mu{}_{0\mu i} = l_{jk}d^k{}_{ji} + l_{ki}d^j{}_{jk} \,. \tag{5.39}$$

Finally we need the spatial components of the Ricci tensor $R_{\mu\nu}$. A tedious calculation shows that

$$\begin{aligned} R_{ij} &= \dot{l}_{ij} + l_{ij}l_{kk} + l_{ik}m_{kj} + l_{jk}m_{ki} + \frac{1}{2}d^h{}_{hk}\left(d^i{}_{kj} - d^j{}_{ki}\right) \\ &\quad -\frac{1}{2}d^k{}_{ih}\left(d^k{}_{jh} + d^h{}_{jk}\right) + \frac{1}{4}d^i{}_{kh}d^j{}_{kh} \,, \end{aligned} \tag{5.40}$$

where we have used the fact that the Jacobi identity (5.22) implies $d^h{}_{hk}d^k{}_{ij} = 0$.

At this point it is easy to find the Einstein equation (2.68) for dust.

In the next section we will use the comoving coordinates, then the Einstein equation (2.68) can be written as in (2.87).

5.3 Bianchi dust metrics and their properties

In this section we give the known Bianchi dust solutions in comoving coordinates, i.e. writing the four-velocity as in (2.11). For the general discussion of Bianchi perfect fluid solutions see [17, section 9.3].

Bianchi models describe spatially homogeneous universes, then we do not need to define the averaging procedure (chapter 3) and the backreaction variable (3.17) assumes the following simple form:

$$\mathcal{Q} = -2\sigma^2 \,. \tag{5.41}$$

Since σ^2 is always positive, in Bianchi models the backreaction variable is always negative, then it acts to decelerate the expansion. We can also see that if the shear tensor $\sigma^\mu{}_\nu$ vanishes, we reduce to the isotropic case (FLRW model).

Now we study the different types of Bianchi metrics for dust.

We have dust solutions of the Einstein equation only for the following (non-tilted) Bianchi types: I, II, III, V and VI_h. For the remaining cases (see appendix B) we do not have dust solutions.

Since we deal with dust solutions of the Einstein equation (2.68), the acceleration vector always vanishes (as we have seen in chapter 4 for the Szekeres metric) and the motion is always geodesic. Then, we omit the calculation for the different types of Bianchi metrics.

Moreover, since non-tilted Bianchi models are irrotational by construction, also the vorticity always vanishes. Then, we write the calculation only for the type I and omit it for all the other types.

Finally, we give the expressions of electric and magnetic Weyl tensors only for types I and III. The other cases can be easily found following the calculations done for these two types.

5.3.1 Type I

The general solution of Bianchi type I for dust is

$$ds^2 = -dt^2 + t^{2p_1} A^{2q_1} dx^2 + t^{2p_2} A^{2q_2} dy^2 + t^{2p_3} A^{2q_3} dz^2 \,, \tag{5.42}$$

where the function $A = A\,(t)$ is defined as

$$A\,(t) \doteq \alpha + m^2 t \,, \tag{5.43}$$

with α, $m = constant$.

The constants p_1, p_2 and p_3 satisfy the Kasner constraints

$$p_1 + p_2 + p_3 = 1 \,, \qquad p_1^2 + p_2^2 + p_3^2 = 1 \,, \tag{5.44}$$

and the following property

$$q_i = \frac{2}{3} - p_i \qquad (i = 1,\, 2,\, 3) \,. \tag{5.45}$$

Now we want to calculate the properties of the metric (5.42).

It turns out that the Einstein tensor $G_{\mu\nu}$, defined in (2.68), is diagonal. By plugging it into the Einstein equation for dust in comoving coordinates (2.87) we get the following set of equations:

$$\begin{aligned} \rho \;=\; & \frac{1}{A^2 t^2} \left[m^4 t^2 \left(q_1 q_2 + q_1 q_3 + q_2 q_3 \right) + A t m^2 \left(q_1 p_2 + q_2 p_1 + q_1 p_3 \right. \right. \\ & \left. \left. + q_3 p_1 + q_2 p_3 + q_3 p_2 \right) + A^2 \left(p_1 p_2 + p_1 p_3 + p_2 p_3 \right) \right] \,, \end{aligned} \tag{5.46a}$$

$$\begin{aligned} 0 \;=\; & m^4 t^2 \left(q_2 q_3 + q_2^2 - q_2 + q_3^2 - q_3 \right) + A t m^2 \left(2 q_2 p_2 + 2 q_3 p_3 \right. \\ & \left. + q_2 p_3 + q_3 p_2 \right) + A^2 \left(p_2^2 - p_2 + p_3^2 - p_3 + p_2 p_3 \right) \,, \end{aligned} \tag{5.46b}$$

$$
\begin{aligned}
0 \;=\;& m^4t^2\left(q_1q_3+q_1^2-q_1+q_3^2-q_3\right)+Atm^2\,(2q_1p_1+2q_3p_3\\
& q_1p_3+q_3p_1)+A^2\left(p_1^2-p_1+p_3^2-p_3+p_1p_3\right)\,, \qquad (5.46c)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& m^4t^2\left(q_1q_2+q_1^2-q_1+q_2^2-q_2\right)+Atm^2\,(2q_1p_1+2q_2p_2\\
& q_1p_2+q_2p_1)+A^2\left(p_1^2-p_1+p_2^2-p_2+p_1p_2\right)\,. \qquad (5.46d)
\end{aligned}
$$

We insert in (5.46a), the sum of (5.46b)-(5.46d) and recalling (5.44) and (5.45), after some steps, we get the following expression for the density:

$$\rho=\frac{4m^2}{3At}\,. \tag{5.47}$$

If $m=0$, we obtain the Kasner vacuum solutions (see [15, subsection 13.3.2] and [17, subsection 6.2.2]).

Moreover, as $t\to 0^+$, the line element assume the Kasner form with exponents p_i (see [17, subsection 9.1.1]) whereas, as $t\to+\infty$, the line element approaches the flat FLRW metric (see [17, subsection 6.3.1]).

Now, following chapter 2 we calculate the dynamical quantities of the metric (5.42).

Expansion rate

The expansion rate Θ in comoving coordinates is expressed in (2.44), that is:

$$\Theta={\Gamma^{\mu}}_{\mu 0}\,.$$

Using (4.17) and the metric (5.42) with its inverse

$$g^{\mu\nu}=\begin{pmatrix}-1&0&0&0\\0&\frac{1}{t^{2p_1}A^{2q_1}}&0&0\\0&0&\frac{1}{t^{2p_2}A^{2q_2}}&0\\0&0&0&\frac{1}{t^{2p_3}A^{2q_3}}\end{pmatrix}, \tag{5.48}$$

we have

$$\Gamma^0{}_{00} = 0\,, \tag{5.49a}$$

$$\Gamma^i{}_{i0} = \frac{p_i}{t} + \left(\frac{2}{3} - p_i\right)\frac{m^2}{\alpha + m^2 t}\,. \tag{5.49b}$$

Then, recalling (5.44), the expansion rate Θ is:

$$\Theta = \frac{\alpha + 2m^2 t}{t\,(\alpha + m^2 t)}\,. \tag{5.50}$$

Vorticity

The expression for the vorticity tensor $\omega_{\mu\nu}$ in comoving coordinates is given in (2.54), which we write again here:

$$\omega_{0\mu} = 0\,, \qquad \omega_{ij} = \partial_{[j}\,u_{i]} + u_{[j}\,\partial_0 u_{i]}\,.$$

Since $u_i = g_{0i}$ and from (5.42) $g_{0i} = 0$, we see that the vorticity $\omega_{\mu\nu}$ always vanishes:

$$\omega_{\mu\nu} = 0 \qquad \forall\,\mu,\,\nu\,. \tag{5.51}$$

So we are dealing with a rotation free flow.

Shear

The shear tensor $\sigma^\mu{}_\nu$ in comoving coordinates is (2.61), i.e.

$$\sigma^\mu{}_0 = 0\,, \qquad \sigma^i{}_j = \frac{1}{2}\delta^i{}_0\partial_0 u_j - \frac{1}{3}\Gamma^\rho{}_{\rho 0}h^i{}_j + \Gamma^i{}_{00}u_j + \Gamma^i{}_{j0}\,.$$

Recalling that $u_j = g_{0j}$, using (5.42) and its inverse (5.48), all the Christoffel symbols calculated above and the fact that $\Gamma^i{}_{00} = 0$ for all i

and ${\Gamma^i}_{j0} = 0$ for $i \neq j$ (from (4.17)), we obtain the following diagonal form for the shear ${\sigma^\mu}_\nu$:

$$ {\sigma^\mu}_\nu = \left[\frac{\alpha}{3t\left(\alpha + m^2 t\right)}\right] \mathrm{diag}\left(0,\ 3p_1 - 1,\ 3p_2 - 1,\ 3p_3 - 1\right) . \tag{5.52} $$

Since the determinant of the metric (5.42) is

$$ g \doteq \det\left(g_{\mu\nu}\right) = -t^2\left(\alpha + m^2 t\right)^2 , \tag{5.53} $$

the shear ${\sigma^\mu}_\nu$ can be written as

$$ {\sigma^\mu}_\nu = \left[\frac{\alpha}{3\sqrt{-g}}\right] \mathrm{diag}\left(0,\ 3p_1 - 1,\ 3p_2 - 1,\ 3p_3 - 1\right) . \tag{5.54} $$

So, the constant α is a measure of the shear, whereas the p_i characterize its dependence upon direction. The particular case $\alpha = 0$ leads to an (isotropic) FLRW universe.

The square of the shear σ^2 is

$$ \begin{aligned} \sigma^2 &\doteq \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} \\ &= \frac{\alpha^2}{3t^2\left(\alpha + m^2 t\right)} \\ &= -\frac{\alpha^2}{3g} , \end{aligned} \tag{5.55} $$

and the backreaction (5.41) becomes

$$ \begin{aligned} \mathcal{Q} &= -\frac{2\alpha^2}{3t^2\left(\alpha + m^2 t\right)} \\ &= \frac{2\alpha^2}{3g} . \end{aligned} \tag{5.56} $$

Electric and magnetic Weyl tensors

The electric Weyl tensor $E^{\alpha}{}_{\beta}$ in comoving coordinates is given in (2.66a), which we rewrite here[9]

$$E^{\alpha}{}_{\beta} = g^{\alpha\xi} C_{\xi 0 \beta 0}$$

The Weyl tensor's components we need here are:

$$\begin{aligned}
C_{0000} &= 0\,, \qquad (5.57a)\\
C_{0101} &= -\frac{t^{2p_1}A^{2q_1}}{6A^2t^2}\Big(-q_1q_2\dot{A}^2t^2 - q_1q_3\dot{A}^2t^2 + 2q_2q_3\dot{A}^2t^2 + 4p_1q_1A\dot{A}t\\
&\quad -2p_2q_2A\dot{A}t + 2q_1^2\dot{A}^2t^2 - 2q_1\dot{A}^2t^2 - q_2^2\dot{A}^2t^2 + q_2\dot{A}^2t^2 - 2p_1A^2\\
&\quad +2p_1^2A^2 + p_2A^2 - p_2^2A^2 + p_3A^2 - p_3^2A^2 + q_3^2\dot{A}^2t^2 - q_3\dot{A}^2t^2\\
&\quad -2p_3q_3A\dot{A}t - p_2q_1A\dot{A}t - p_1q_2A\dot{A}t - p_3q_1A\dot{A}t - p_1q_3A\dot{A}t\\
&\quad +2p_3q_2A\dot{A}t + 2p_2q_3A\dot{A}t - p_1p_2A^2 - p_1p_3A^2 + 2p_2p_3A^2\Big)\,, \qquad (5.57b)\\
C_{0202} &= -\frac{t^{2p_2}A^{2q_2}}{6A^2t^2}\Big(q_1q_2\dot{A}^2t^2 - 2q_1q_3\dot{A}^2t^2 + q_2q_3\dot{A}^2t^2 + 2p_1q_1A\dot{A}t\\
&\quad -4p_2q_2A\dot{A}t + q_1^2\dot{A}^2t^2 - q_1\dot{A}^2t^2 - 2q_2^2\dot{A}^2t^2 + 2q_2\dot{A}^2t^2 - p_1A^2\\
&\quad +p_1^2A^2 + 2p_2A^2 - 2p_2^2A^2 - p_3A^2 + p_3^2A^2 + q_3^2\dot{A}^2t^2 - q_3\dot{A}^2t^2\\
&\quad +2p_3q_3A\dot{A}t + p_2q_1A\dot{A}t + p_1q_2A\dot{A}t - 2p_3q_1A\dot{A}t - 2p_1q_3A\dot{A}t\\
&\quad +p_3q_2A\dot{A}t + p_2q_3A\dot{A}t + p_1p_2A^2 - 2p_1p_3A^2 + p_2p_3A^2\Big)\,,
\end{aligned}$$

[9] For convenience, here, we have written the electric Weyl tensor (2.66a) raising the index α.

$$
\begin{aligned}
C_{0303} &= -\frac{t^{2p_3}A^{2q_3}}{6A^2t^2}\Big(-2q_1q_2\dot{A}^2t^2 + q_1q_3\dot{A}^2t^2 + q_2q_3\dot{A}^2t^2 + 2p_1q_1A\dot{A}t \\
&\quad +2p_2q_2A\dot{A}t + q_1^2\dot{A}^2t^2 - q_1\dot{A}^2t^2 + q_2^2\dot{A}^2t^2 - q_2\dot{A}^2t^2 - p_1A^2 \\
&\quad +p_1^2A^2 - p_2A^2 + p_2^2A^2 + 2p_3A^2 - 2p_3^2A^2 - 2q_3^2\dot{A}^2t^2 + 2q_3\dot{A}^2t^2 \\
&\quad -4p_3q_3A\dot{A}t - 2p_2q_1A\dot{A}t - 2p_1q_2A\dot{A}t + p_3q_1A\dot{A}t + p_1q_3A\dot{A}t \\
&\quad +p_3q_2A\dot{A}t + p_2q_3A\dot{A}t - 2p_1p_2A^2 + p_1p_3A^2 + p_2p_3A^2\Big)\ , \qquad (5.57c) \\
C_{0i0j} &= 0 \quad \text{for } i \neq j\,. \qquad (5.57d)
\end{aligned}
$$

Then, using the Einstein equations (5.46b)-(5.46d) and the properties (5.44) and (5.45), the electric Weyl tensor $E^{\alpha}{}_{\beta}$ assumes the following diagonal form:

$$
\begin{aligned}
E^{i}{}_{i} &= g^{ii}C_{i0i0} \\
&= \frac{1}{3}\left(\frac{\dot{A}}{A}\right)^2\left(p_i - 3p_i^2 + \frac{2}{3}\right) + \frac{2\dot{A}}{3At}\left(3p_i^2 - 2p_i - \frac{1}{3}\right) \\
&\quad +\frac{1}{t^2}\left(p_i - p_i^2\right)\,. \qquad (5.58)
\end{aligned}
$$

For magnetic Weyl tensor $H_{\alpha\beta}$ in comoving coordinates we have the expression (2.66b), i.e.

$$
H_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\gamma\delta}C^{\gamma\delta}{}_{\beta 0} = \frac{1}{2}\eta_{\alpha\gamma\delta}g^{\gamma\xi}g^{\delta\psi}C_{\xi\psi\beta 0}\,.
$$

Since the non-zero components of the Weyl tensor are (5.57b)-(5.57c) and C_{1212}, C_{1313}, C_{2323}, we get

$$
H_{\alpha\beta} = 0 \qquad \forall\,\alpha,\,\beta\,. \qquad (5.59)
$$

5.3.2 Type II

The Bianchi type II metric for dust is

$$ds^2 = -dt^2 + t^{2p_1} A^{2p_1} (dx + bzdy)^2 + t^{2p_2} A^{2p_3} dy^2 + t^{2p_3} A^{2p_2} dz^2 \,, \quad (5.60)$$

where the function $A = A(t)$ is defined as in (5.43) and the constant b is given by

$$b \doteq \frac{1}{4} m^2 \,, \quad (5.61)$$

with $m = constant$.

The constants p_1, p_2 and p_3 satisfy the Kasner constraints (5.44) and are given by

$$p_1 = \frac{1}{3} (1 - 2\cos\psi) \,, \quad (5.62)$$

$$p_{2,3} = \frac{1}{3} \left(1 + \cos\psi \pm \sqrt{3} \sin\psi\right) \,, \quad (5.63)$$

with $\cos\psi = \frac{1}{8}$.

With the same procedure used in subsection 5.3.1, we find the density to be

$$\rho = \frac{5m^2}{4At} \,. \quad (5.64)$$

As for the type I solution, if $m = 0$ we obtain the Kasner vacuum solution.

Now we compute the dynamical quantities for the metric (5.60).

Expansion rate

The expansion rate Θ in comoving coordinates is given by (2.44).

As in subsection 5.3.1, using (4.17) and the metric (5.60), we have

$$\Gamma^1{}_{10} = p_1\left(\frac{1}{t}+\frac{\dot{A}}{A}\right), \tag{5.65a}$$

$$\Gamma^2{}_{20} = \frac{1}{3b^2z^2t^{2p_1}A^{2p_1}-t^{2p_2}A^{2p_3}}\Bigg[3b^2z^2p_1t^{2p_1}A^{2p_1} \left(\frac{1}{t}+\frac{\dot{A}}{A}\right) - t^{2p_2}A^{2p_3}\left(\frac{p_2}{t}+p_3\frac{\dot{A}}{A}\right)\Bigg], \tag{5.65b}$$

$$\Gamma^3{}_{30} = \frac{p_3}{t}+p_2\frac{\dot{A}}{A}. \tag{5.65c}$$

Then, since $\Gamma^0{}_{00}=0$, the expansion rate Θ becomes:

$$\Theta = \left(p_1+p_3\right)\frac{1}{t}+\left(p_1+p_2\right)\frac{\dot{A}}{A} + \frac{1}{3b^2z^2t^{2p_1}A^{2p_1}-t^{2p_2}A^{2p_3}}\Bigg[3b^2z^2p_1t^{2p_1}A^{2p_1} \left(\frac{1}{t}+\frac{\dot{A}}{A}\right) - t^{2p_2}A^{2p_3}\left(\frac{p_2}{t}+p_3\frac{\dot{A}}{A}\right)\Bigg]. \tag{5.66}$$

Shear

The shear tensor $\sigma^\mu{}_\nu$ in comoving coordinates is (2.61).

We need also the following Christoffel symbols:

$$\Gamma^1{}_{20} = -\frac{2bzt^{2p_2}A^{2p_3}\left(p_1A+p_1\dot{A}t-p_2A-p_3\dot{A}t\right)}{At\left(3b^2z^2t^{2p_1}A^{2p_1}-t^{2p_2}A^{2p_3}\right)}, \tag{5.67a}$$

$$\Gamma^{1}{}_{30} = 0\,, \tag{5.67b}$$
$$\Gamma^{2}{}_{10} = 0\,, \tag{5.67c}$$
$$\Gamma^{2}{}_{30} = 0\,, \tag{5.67d}$$
$$\Gamma^{3}{}_{10} = 0\,, \tag{5.67e}$$
$$\Gamma^{3}{}_{20} = 0\,. \tag{5.67f}$$

Recalling, from the metric (5.60), that $u_j = g_{0j} = 0$ and all the Christoffel symbols calculated above, since $\Gamma^{i}{}_{00} = 0$ for all i, the non-zero components of the shear tensor $\sigma^{\mu}{}_{\nu}$ are:

$$\sigma^{1}{}_{1} = \frac{2}{3}p_1\left(\frac{1}{t}+\frac{\dot{A}}{A}\right) - \frac{1}{3}\left(\frac{p_3}{t}+p_2\frac{\dot{A}}{A}\right) - \frac{1}{9b^2z^2t^{2p_1}A^{2p_1}-3t^{2p_2}A^{2p_3}}$$
$$\left[3b^2z^2p_1t^{2p_1}A^{2p_1}\left(\frac{1}{t}+\frac{\dot{A}}{A}\right) - t^{2p_2}A^{2p_3}\left(\frac{p_2}{t}+p_3\frac{\dot{A}}{A}\right)\right]\,, \tag{5.68a}$$

$$\sigma^{2}{}_{2} = -\frac{1}{3}\left[\left(p_1+p_3\right)\frac{1}{t}+\left(p_1+p_2\right)\frac{\dot{A}}{A}\right] + \frac{2}{9b^2z^2t^{2p_1}A^{2p_1}-3t^{2p_2}A^{2p_3}}$$
$$\left[3b^2z^2p_1t^{2p_1}A^{2p_1}\left(\frac{1}{t}+\frac{\dot{A}}{A}\right) - t^{2p_2}A^{2p_3}\left(\frac{p_2}{t}+p_3\frac{\dot{A}}{A}\right)\right]\,, \tag{5.68b}$$

$$\sigma^{3}{}_{3} = \frac{2}{3}\left(\frac{p_3}{t}+p_2\frac{\dot{A}}{A}\right) - \frac{1}{3}p_1\left(\frac{1}{t}+\frac{\dot{A}}{A}\right) - \frac{1}{9b^2z^2t^{2p_1}A^{2p_1}-3t^{2p_2}A^{2p_3}}$$
$$\left[3b^2z^2p_1t^{2p_1}A^{2p_1}\left(\frac{1}{t}+\frac{\dot{A}}{A}\right) - t^{2p_2}A^{2p_3}\left(\frac{p_2}{t}+p_3\frac{\dot{A}}{A}\right)\right]\,, \tag{5.68c}$$

$$\sigma^{1}{}_{2} = -\frac{bzt^{2p_2}A^{2p_3}}{3b^2z^2t^{2p_1}A^{2p_1}-t^{2p_2}A^{2p_3}}\left[\left(p_1-p_2\right)\frac{1}{t}+\left(p_1-p_3\right)\frac{\dot{A}}{A}\right]\,.$$

From these expressions we can easily calculate the square of the shear, that is $\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$.

5.3.3 Type III

The Bianchi type III line element for dust is

$$ds^2 = \frac{A^4}{\lambda^2}\left(-dt^2 + dx^2 + e^{4x}dy^2\right) + \frac{B^2}{\lambda^2 A^2}dz^2\,, \tag{5.69}$$

where $\lambda = constant$.

The functions $A = A\,(t)$ and $B = B\,(t)$ are given in three subcases by

1. $A = \cosh t\,,\quad B = \alpha \sinh t + \beta^2\left(t\sinh t - \cosh t\right)\;;$
2. $A = \sinh t\,,\quad B = \alpha \cosh t + \beta^2\left(t\cosh t - \sinh t\right)\;;$
3. $A = e^t\,,\qquad B = e^t\left(\alpha + \beta^2 t\right)\;;$

where α and β are constants.

The Einstein tensor $G_{\mu\nu}$, defined in (2.68), is diagonal and its components are given by

$$G_{00} = \frac{4}{AB}\left(\dot{A}\dot{B} - AB\right)\,, \tag{5.70a}$$

$$G_{11} = -\frac{1}{AB}\left(\ddot{A}B - 2\dot{A}\dot{B} + A\ddot{B}\right)\,, \tag{5.70b}$$

$$G_{22} = e^{4x}G_{11}\,, \tag{5.70c}$$

$$G_{33} = \frac{4B^2}{A^7}\left(-\ddot{A} + A\right)\,. \tag{5.70d}$$

Before writing the Einstein equation for dust, we should note that the (00)-component of the metric (5.69) is $g_{00} = -\frac{A^4}{\lambda^2}$, so in comoving coordinates, the four-velocity is given by

$$u^\mu = \frac{\lambda}{A^2}\delta^\mu{}_0\,. \tag{5.71}$$

Then, from the Einstein equation (2.68) for dust, we get the following set of equations:

$$\frac{4\lambda^2}{A^5B}\left(\dot{A}\dot{B} - AB\right) = \rho\,, \tag{5.72a}$$

$$\ddot{A}B - 2\dot{A}\dot{B} + A\ddot{B} = 0\,, \tag{5.72b}$$

$$-\ddot{A} + A = 0\,. \tag{5.72c}$$

By plugging the second and third equations in the first one we obtain

$$\rho = \frac{4\lambda^2}{A^3B}\left(\frac{\ddot{B} - B}{2A}\right)\,. \tag{5.73}$$

In all the three subcases listed above we have

$$\left(\frac{\ddot{B} - B}{2A}\right) = \beta^2\,, \tag{5.74}$$

so the final expression for the density is

$$\rho = \frac{4\lambda^2\beta^2}{A^3B}\,, \tag{5.75}$$

and if $\beta = 0$, we obtain the vacuum limits.

Now we calculate the properties of the metric (5.69).

Expansion rate

To calculate the expansion rate Θ we can use both method (2.43) and (2.44).

Here we choose the first, so we have[10]

$$\Theta = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}u^\mu\right)\,.$$

[10] See footnote 10, chapter 2.

Recalling (5.71), the expression of the expansion rate Θ can be rewritten as

$$\Theta = \frac{\lambda}{A^2}\left(\frac{\dot{g}}{2g} - \frac{2\dot{A}}{A}\right) , \tag{5.76}$$

where $g \doteq \det\left(g_{\mu\nu}\right)$ and $\dot{g} = \frac{\partial g}{\partial t}$.

From the metric (5.69), we see that

$$g = -\frac{e^{4x}A^{10}B^2}{\lambda^8} , \tag{5.77}$$

$$\dot{g} = -\frac{2e^{4x}A^9 B}{\lambda^8}\left(5\dot{A}B + A\dot{B}\right) , \tag{5.78}$$

and then

$$\Theta = \frac{\lambda}{A^2}\left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) . \tag{5.79}$$

Shear

The shear tensor $\sigma^\mu{}_\nu$ in comoving coordinates, with the four velocity given by (5.71), is

$$\sigma^\mu{}_0 = 0 , \qquad \sigma^i{}_j = -\frac{1}{3}\Theta h^i{}_j + \lambda^2 \Gamma^i{}_{00} u_j + \frac{\lambda}{A^2}\Gamma^i{}_{j0} , \tag{5.80}$$

where we have dropped the term containing the vorticity.

Recalling that $u_j = g_{0j} = 0$ (see the metric (5.69)), (5.79), since $\Gamma^0{}_{00} = 2\frac{\dot{A}}{A}$, $\Gamma^i{}_{00} = 0$ for all i and $\Gamma^i{}_{j0} = 0$ for $i \neq j$ (from (4.17)), we obtain the following diagonal form for the shear $\sigma^\mu{}_\nu$:

$$\sigma^\mu{}_\nu = \left[\frac{\lambda}{3A^2}\right] \mathrm{diag}\left[0, \frac{3\dot{A}}{A} - \frac{\dot{B}}{B}, \frac{3\dot{A}}{A} - \frac{\dot{B}}{B}, 2\left(\frac{3\dot{A}}{A} - \frac{\dot{B}}{B}\right)\right] . \tag{5.81}$$

Then the backreaction variable (5.41) is

$$\mathcal{Q} = -\frac{2\lambda^2}{A^4}\left[3\left(\frac{\dot{A}}{A}\right)^2 + \frac{1}{3}\left(\frac{\dot{B}}{B}\right)^2 - 2\frac{\dot{A}\dot{B}}{AB}\right] . \tag{5.82}$$

Electric and magnetic Weyl tensors

The electric Weyl tensor $E^{\alpha}{}_{\beta}$ in comoving coordinates, with four-velocity (5.71), is given by

$$E^{\alpha}{}_{\beta} = \frac{\lambda^2}{A^4} g^{\alpha\xi} C_{\xi 0 \beta 0} . \tag{5.83}$$

The Weyl tensor's components we need here are:

$$\begin{aligned}
C_{0000} &= 0 , && (5.84a)\\
C_{0101} &= -\frac{A^2}{6\lambda^2 B}\Big(-12\dot{A}^2 B + 3A\ddot{A}B + 4A^2 B \\
&\quad +6A\dot{A}\dot{B} - A^2\ddot{B}\Big) , && (5.84b)\\
C_{0202} &= -\frac{e^{4x}A^2}{6\lambda^2 B}\Big(-12\dot{A}^2 B + 3A\ddot{A}B + 4A^2 B \\
&\quad +6A\dot{A}\dot{B} - A^2\ddot{B}\Big) , && (5.84c)\\
C_{0303} &= \frac{B}{3\lambda A^4}\Big(-12\dot{A}^2 B + 3A\ddot{A}B + 4A^2 B \\
&\quad +6A\dot{A}\dot{B} - A^2\ddot{B}\Big) , && (5.84d)\\
C_{0i0j} &= 0 \quad \text{for } i \neq j . && (5.84e)
\end{aligned}$$

So, recalling (5.72b) and (5.72c), the non-zero electric Weyl tensor's components $E^{\alpha}{}_{\beta}$ are:

$$
\begin{aligned}
E^1{}_1 &= \frac{\lambda^2}{A^2}\left[2\left(\frac{\dot{A}}{A}\right)^2 - \frac{\ddot{B}}{3B} - \frac{5}{3}\right], && (5.85a)\\
E^2{}_2 &= E^1{}_1\,, && (5.85b)\\
E^3{}_3 &= -\frac{2}{A^2}E^1{}_1\,. && (5.85c)
\end{aligned}
$$

For the magnetic Weyl tensor $H_{\alpha\beta}$ in comoving coordinates we have

$$
H_{\alpha\beta} = \frac{\lambda}{2A^2}\eta_{\alpha\gamma\delta}C^{\gamma\delta}{}_{\beta 0}\,, \tag{5.86}
$$

and, as i subsection 5.3.1, it always vanishes.

In fact the non-zero components of the Weyl tensor are (5.84b)-(5.84d) and C_{1212}, C_{1313}, C_{2323}, then

$$
H_{\alpha\beta} = 0 \qquad \forall\, \alpha,\, \beta\,. \tag{5.87}
$$

5.3.4 Type V

The Bianchi type V metric for dust is

$$
ds^2 = -N^2 dt^2 + t^2\left[dx^2 + e^{2rx}\left(e^{2h(t)}dy^2 + e^{-2h(t)}dz^2\right)\right], \tag{5.88}
$$

where the function $N = N(t)$ is defined as

$$
N^2 = \left(\frac{m^2}{t} + \frac{3s^2}{t^4} + r^2\right)^{-1}, \tag{5.89}
$$

and

$$
\dot{h}(t) = \frac{3sN}{t^3}\,, \tag{5.90}
$$

and m, s and r are constants.

Following subsection 5.3.3, from the Einstein equation for dust, we get the following expression for the density:

$$\rho = \frac{3m}{t^3} \,. \tag{5.91}$$

In a way totally analogous to subsection 5.3.3 we calculate the dynamical quantities of the metric (5.88).

Expansion rate

By proceeding as in subsection 5.3.3, we start from

$$\Theta = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} u^\mu \right) \,.$$

Since the four-velocity u^μ in comoving coordinates is

$$u^\mu = \frac{1}{N} \delta^\mu{}_0 \,, \tag{5.92}$$

then the expansion rate Θ becomes

$$\Theta = \frac{\dot{g}}{2Ng} - \frac{\dot{N}}{N^2} \,, \tag{5.93}$$

where $\dot{g} = \frac{\partial g}{\partial t}$ and

$$\begin{aligned} g &\doteq \det\left(g_{\mu\nu}\right) \\ &= -N^2 t^6 e^{4rx} \,. \end{aligned} \tag{5.94}$$

Then we have

$$\Theta = \frac{3}{Nt} \,. \tag{5.95}$$

Shear

Proceeding as in subsection 5.3.3, the shear tensor $\sigma^{\mu}{}_{\nu}$ in comoving coordinates, with the four velocity given by (5.92), is

$$\sigma^{\mu}{}_{0} = 0\,, \qquad \sigma^{i}{}_{j} = -\frac{1}{3}\Theta h^{i}{}_{j} + \frac{1}{N^2}\Gamma^{\frac{\dot{N}}{N},i}_{00} u_j + \frac{1}{N}\Gamma^{i}{}_{j0}\,. \tag{5.96}$$

From the metric (5.88) we can see that $u_j = g_{0j} = 0$ and, recalling (5.95), since $\Gamma^{0}{}_{00} = \frac{\dot{N}}{N}$, $\Gamma^{i}{}_{00} = 0$ for all i and $\Gamma^{i}{}_{j0} = 0$ for $i \neq j$ (from (4.17)), we obtain:

$$\sigma^{\mu}{}_{\nu} = \left[\frac{\dot{h}}{N}\right] \mathrm{diag}\left[0,\ 0,\ 1,\ -1\right]\,. \tag{5.97}$$

Then the backreaction variable (5.41) is

$$\mathcal{Q} = -\frac{4\dot{h}^2}{N^2}\,. \tag{5.98}$$

5.3.5 Type VI_h

Four classes of dust solutions of Bianchi type VI_h have been found.

The line-element for each class has the following form:

$$\lambda^2 ds^2 = -A^{2a_0}B^{2b_0}dt^2 + A^{2a_1}B^{2b_1}dx^2 + A^{2a_2}B^{2b_2}e^{2c_2x}dy^2 + A^{2a_3}B^{2b_3}e^{2c_3x}dz^2\,, \tag{5.99}$$

where $A = A(t)$, $B = B(t)$ and $\lambda = constant$. The two constants c_2 and c_3 determine the group type. The two sets of constant exponents a_μ, b_μ ($\mu = 0,\ 1,\ 2,\ 3$) are related to the Kasner exponents, which satisfy the Kasner constraints (5.44) and are defined as

$$p_1 = \frac{1}{3}(1-2k)\,, \quad p_{2,3} = \frac{1}{3}\left(1 + k \mp \sqrt{3\left(1-k^2\right)}\right)\,, \tag{5.100}$$

$$q_1 = \frac{1}{3}(1+2k)\,, \quad q_{2,3} = \frac{1}{3}\left(1 - k \pm \sqrt{3\left(1-k^2\right)}\right)\,, \tag{5.101}$$

where the constant k is related to the group parameter h according to

$$k = \frac{1}{\sqrt{1-3h}}\,. \tag{5.102}$$

The costants p_i and q_i $(i = 1,\, 2,\, 3)$ satisfy also (5.45).

Now we give a list of the different classes of Bianchi type VI_h metrics with their properties.

First class: $k = \frac{1}{4}$**.** The functions A and B are

$$A = t\,, \tag{5.103}$$

$$B = \alpha + m^2 t + \frac{3}{5}\beta^2 t^{\frac{5}{3}}\,, \tag{5.104}$$

where α, β and m are arbitrary constants. The other constants are given by

$$a_0 = b_0 = 0\,, \qquad a_i = p_i\,, \qquad b_i = q_i\,, \tag{5.105}$$

and

$$c_{2,3} = \frac{\beta}{2\,(1-k)}\left(\sqrt{1-k^2} \pm \sqrt{3}k\right)\,. \tag{5.106}$$

With the same procedure used in the previous subsections, the density is found to be

$$\rho = \frac{4\lambda^2 m^2}{3}\,(AB)^{-2q_1}\,. \tag{5.107}$$

Second class: $k = \frac{5}{8}$**.** There are three different forms for the functions A and B, that are:

1. $A = \cosh t\,, \quad B = \left[\alpha + \beta^2 \int \frac{\cosh^r \bar{t}}{\sinh^2 \bar{t}} d\bar{t}\right] \sinh t\,;$

2. $A = \sinh t\,, \quad B = \left[\alpha + \beta^2 \int \frac{\sinh^r \bar{t}}{\cosh^2 \bar{t}} d\bar{t}\right] \cosh t\,;$

3. $A = e^t\,, \quad B = \begin{cases} e^t \left[\alpha + \frac{\beta^2}{r-2}\right] e^{(r-2)t}\,, & r \neq 2\,, \\ \\ e^t \left(\alpha + \beta^2 t\right)\,, & r = 2\,, \end{cases}$

where α and β are arbitrary constants and $r = \frac{2k}{1-k}$.

All the other constants are given by (5.105) and (5.106).

The density is

$$\rho = \frac{16\lambda^2}{3}\,. \tag{5.108}$$

Third class: $k = \frac{1}{2}$. The functions A and B are defined as in the second class. The constants are

$$a_0 = a_1\,, \quad b_0 = b_1\,, \quad a_i = 3q_i\,, \quad b_i = p_i\,, \tag{5.109}$$

and

$$c_{2,3} = \frac{1}{\sqrt{1-k^2}}\left(\sqrt{1-k^2} \pm \sqrt{3}k\right)\,. \tag{5.110}$$

Then the density becomes

$$\rho = \frac{4\lambda^2\beta^2}{A^3 B}\,. \tag{5.111}$$

Fourth class: $k = -\frac{1}{2}$. In this case the solution is obtained from the case $k = \frac{1}{2}$ by replacing k with $-k$.

Now we list all the dynamical quantities for the different classes. The procedure that we have used is the same as in subsections 5.3.1-5.3.4.

Expansion rate

For the first two classes the expression of the expansion rate Θ is the same. The procedure is the same used in subsection 5.3.3 and it gives

$$\Theta = \lambda \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) , \tag{5.112}$$

where now the four-velocity is

$$u^\mu = \lambda \delta^0{}_\mu . \tag{5.113}$$

For the last two classes the four-velocity is given by

$$u^\mu = \frac{1}{A^{3q_1} B^{p_1}} \delta^0{}_\mu , \tag{5.114}$$

and the expansion rate Θ becomes

$$\Theta = \frac{1}{A^{3q_1} B^{p_1}} \left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) . \tag{5.115}$$

Shear

To compute the shear tensor $\sigma^\mu{}_\nu$ we use always the same method as in subsection 5.3.3.

Then, for the first two classes, with $k = \frac{1}{4}$ and $k = \frac{5}{8}$, we get

$$\sigma^\mu{}_\nu = \left[\lambda \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \right] \mathrm{diag} \left[0,\, p_1 - \frac{1}{3},\, p_2 - \frac{1}{3},\, p_3 - \frac{1}{3} \right] , \tag{5.116}$$

and the backreaction variable (5.41) is

$$\mathcal{Q} = -\frac{2}{3} \lambda^2 \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right)^2 . \tag{5.117}$$

In the same way, for the classes with $k = \pm\frac{1}{2}$, the shear tensor $\sigma^{\mu}{}_{\nu}$ assumes the following form

$$\sigma^{\mu}{}_{\nu} = \left[\frac{1}{A^{3q_1}B^{p_1}}\left(\frac{\dot{B}}{B} - 3\frac{\dot{A}}{A}\right)\right] \mathrm{diag}\left[0,\, p_1 - \frac{1}{3},\, p_2 - \frac{1}{3},\, p_3 - \frac{1}{3}\right] , \tag{5.118}$$

and the backreaction (5.41) is

$$\mathcal{Q} = -\frac{2}{3A^{6q_1}B^{2p_1}}\left(\frac{\dot{B}}{B} - 3\frac{\dot{A}}{A}\right)^2 . \tag{5.119}$$

Conclusions

In this work dust solutions of the Einstein field equation have been studied.
We first have introduced the covariant formalism, which is the scaffold on which this work has been built. Then we have described in detail the averaging procedure in the framework of Buchert's approach (considering only the dust case). We have obtained the Buchert equations, from which we have defined the backreaction variable.
Backreaction is a valid alternative to modified gravity or dark energy to give an explanation to the late time expansion of the universe. This idea emerges from the fact that the late time universe is far from exact homogeneity and isotropy due to the formation of non-linear structures, i.e. galaxies, clusters of galaxies, voids, etc. and this can have effect on the expansion of the universe, or rather it could explain the current expansion of the universe without introducing exotic matter with negative pressure, i.e. dark energy.
Finally we have studied all the properties of Szekeres and Bianchi metrics, which describe, respectively, inhomogeneous and anisotropic universes and spatially homogeneous but anisotropic universes.

Appendix A

Symmetries

Cosmological models can be classified by their symmetries.

Symmetries of a space are transformations of the space into itself that leave the metric tensor and all physical and geometrical properties invariant.

We first give some mathematical preliminaries and then a classification of cosmological models.

A.1 Mathematical introduction

Following [25, appendix A], let us start with the definition of *diffeomorphism*:

Definition 1 (Diffeomorphism) *A C^∞ map φ is called diffeomorphism if it is one-to-one and onto and its inverse φ^{-1} is C^∞.*

Let $(M,\, g)$ be a (pseudo-)Riemannian manifold with the metric tensor $g_{\mu\nu}$. A diffeomorphism $\varphi : M \to M$ is an *isometry* if it preserves the metric, i.e. if

$$\varphi^* g_{\varphi(p)} = g_p \,, \tag{A.1}$$

where $p \in M$ and φ^* is the pull-back[1] of φ.

The isometries of a space of dimension n form a group[2], in fact the identity map, the composition of two isometries and the inverse of an isometry are all isometries too.

Now, if ϕ_t is a one-parameter group of isometries[3] (i.e. $\phi_t^* g_{\mu\nu} = g_{\mu\nu}$), the vector field ξ^μ which generates ϕ_t is called a *Killing vector field.* The necessary and sufficient condition for ϕ_t to be a group of isometries is

$$\mathcal{L}_\xi g_{\mu\nu} = 0\,, \tag{A.2}$$

where we have introduced the Lie derivative along the vector field ξ^μ: $\mathcal{L}_\xi$[4].

The necessary and sufficient condition for ξ^μ to be a Killing field is to satisfy the *Killing equation*[5]:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\,. \tag{A.3}$$

The maximum number of symmetries is related to the dimension of the manifold M and, if $\dim M = m$, is $\frac{1}{2}m\,(m+1)$. Spaces which admit $\frac{1}{2}m\,(m+1)$ Killing vector fields are called *maximally symmetric spaces.*

Let ξ_μ and ξ_ν be two Killing vector fields, then:

---

[1] Let M and N be manifolds (not necessarily of the same dimension) and let $\phi : M \to N$ be a C^∞ map. Consider a function $f \in C^\infty(N)$, then we define the pull-back of ϕ as:

$$\phi^* f \doteq f \circ \phi \in C^\infty(M)\,.$$

For more details see [2, appendix C.1].

[2] A group $(G,\,\cdot)$ is a set of elements G with an operation $\cdot : G \times G \to G$ that combines any two elements $g,\, g' \in G$ to form a third element $g \cdot g' \in G$. The operation $\cdot$ satisfies four conditions: closure, associativity, identity and invertibility.

[3] A *one-parameter group of diffeomorphisms* ϕ_t is a C^∞ map from $\mathbb{R} \times M$ to M such that, for fixed $t \in \mathbb{R}$, $\phi_t : M \to M$ is a diffeomorphism and for all $t,\, s \in \mathbb{R}$, we have $\phi_t \circ \phi_s = \phi_{t+s}$. See also [2, subsection 2.2].

[4] See [2, appendix C.2].

[5] For more details see [2, appendices C.2 and C.3], whereas for a complete discussion on Killing equation see [10, subsection 8.2].

- a linear combination $\alpha\xi_\mu + \beta\xi_\nu$ is a Killing vector field (α, $\beta \in \mathbb{R}$);
- the commutator $[\xi_\mu, \xi_\nu]$ is a Killing vector field[6].

Thus all Killing vector fields form a Lie algebra[7] of the symmetric operations on the manifold M, with structure constants $C^\lambda{}_{\mu\nu}$:

$$[\xi_\mu, \xi_\nu] = C^\lambda{}_{\mu\nu}\xi_\lambda \,, \tag{A.4}$$

where λ, μ, $\nu = 1, 2, \ldots, r$ and $r \leq \frac{1}{2}m\,(m+1)$.

Structure constants satisfy the following two properties:

- skew-symmetry:
$$C^\lambda{}_{\mu\nu} = -C^\lambda{}_{\nu\mu}\,; \tag{A.5}$$
- Jacobi identity:
$$C^\lambda{}_{\rho[\mu}C^\rho{}_{\nu\sigma]} = 0\,. \tag{A.6}$$

The transformations generated by the Lie algebra form a Lie group[8] of the same dimension.

Now we study the action of a Lie group G on a manifold M.

Definition 2 (Action) *Let G be a Lie group and M a manifold. The* action *of G on M is a C^∞ map $\sigma : G \times M \to M$ which satisfies the following conditions:*

- *$\sigma\,(e,\, p) = p \quad \forall\, p \in M$;*

---

[6]If ξ and η are two Killing vector fields, then $\mathcal{L}_{[\xi,\,\eta]}g_{\mu\nu} = \mathcal{L}_\xi\mathcal{L}_\eta g_{\mu\nu} - \mathcal{L}_\eta\mathcal{L}_\xi g_{\mu\nu} = 0$, where we have used the property of the Lie derivative $\mathcal{L}_{[\xi,\,\eta]} = \mathcal{L}_\xi\mathcal{L}_\eta - \mathcal{L}_\eta\mathcal{L}_\xi$.

[7]A *Lie algebra* $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ (called the *bracket*) such that, for all x, y, $z \in \mathfrak{g}$, $[x,\, y] = -[y,\, x]$ and $[[x,\, y]\,,\, z] + [[y,\, z]\,,\, x] + [[z,\, x]\,,\, y] = 0$ (Jacobi identity).

[8]A *Lie group* G is a differentiable manifold which is endowed with a group structure such that the map $G \times G \to G$ defined by $(\sigma,\, \tau) \mapsto \sigma\tau^{-1}$ is C^∞.

- $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p) \quad \forall\, g_1, g_2 \in G \quad and \quad p \in M.$

The action σ is said to be *transitive* if, for any $p_1, p_2 \in M$, there exists an element $g \in G$ such that $\sigma(g, p_1) = p_2$. Then the map σ can move any point of M into any point of M.

Given a point $p \in M$, the action of G on p takes p to various points in M. The *orbit* $\mathcal{O}_p$ of p is the set of all points into which p can be moved by the action of the isometries of a space, i.e.

$$\mathcal{O}_p = \{\sigma(g, p) \,|\, g \in G\} \, . \tag{A.7}$$

Any orbit $\mathcal{O}_p$ is clearly a subset of M and obviously the action of G on any orbit $\mathcal{O}_p$ is transitive. If the action of G on M is transitive, the orbit of any point $p \in M$ is M itself. So, the maximum dimension of orbits is $\dim M$, i.e $s \leq m$, where $s = \dim \mathcal{O}_p$ and $m = \dim M$.

A subgroup[9] H of a Lie group G that acts on a manifold M is called *isotropy group* if it leaves the point $p \in M$ fixed:

$$H = \{g \in G \,|\, \sigma(g, p) = p\} \, . \tag{A.8}$$

We have

$$\dim H = q \, , \qquad \text{where} \qquad q \leq \frac{1}{2} m (m-1) \, . \tag{A.9}$$

Interestingly, the cosets of the isotropy group correspond to the elements in the orbit:

$$\mathcal{O}_p \sim {}^{G}\!/_{H} \, . \tag{A.10}$$

It holds the following theorem:

[9] (H, φ) is a Lie subgroup of the Lie group G if:

- H is a Lie group;
- (H, φ) is a submanifold of G;
- $\varphi \in \mathrm{Hom}(H, G)$.

Theorem 1 *For any subgroup H of a Lie group G, the coset space ${}^{G}\!/_{H}$ admits a differentiable structure and becomes a manifold called homogeneous space. The dimension of this coset space is given by:*

$$\dim\left({}^{G}\!/_{H}\right) = \dim G - \dim H \,. \tag{A.11}$$

The dimension r of the group of symmetries of the space (group of isometries) is then:

$$r = s + q \,, \tag{A.12}$$

and $0 \leq r \leq m + \frac{1}{2}m\,(m-1) = \frac{1}{2}m\,(m+1)\,.$

The relation (A.12) can be viewed as

$$\begin{aligned}\dim\left(group\ of\ isometries\right) &= \dim\left(group\ of\ translational\ symmetries\right) \\ &\quad + \dim\left(group\ of\ rotational\ symmetries\right)\,.\end{aligned}$$

If $q = 0$ then $r = s$, which means that the dimension of the group of isometries is just enough to move each point in an orbit into any other point. This is called *simply transitive* group. There is no continuous isotropy group in this case.

A.2 Classification of cosmological models

Cosmological models can be classified by their symmetries.

We saw in the previous section that the dimension of the group of symmetries of the space is given by the sum of the dimensions of groups of translational and rotational symmetries, i.e. (A.12). Then the value of q determines the *isotropy properties* of the model, whereas the value of s determines the *homogeneity properties*.

In a four-dimensional cosmological model, r can have different values, obtained by a variety of q and s. The possibilities for the dimension of orbits are $s = 0, 1, 2, 3, 4$. Whereas for the isotropy of the spatial dimensions, q can be 0, 1 or 3, but not 2. In fact we consider non-empty perfect fluid models in which $(p+\rho) > 0$ and hence there will be uniquely defined notions of the average velocity of the matter and corresponding preferred world lines. The four-velocity u^{μ} is given by (2.9) and, since it is unique, it is invariant. So, allowed rotations are those which act orthogonally to u^{μ} and the isotropy group has to be a subgroup of these allowed rotations. Since there is no two-dimensional subgroup of $O\,(3)$, the case $q=2$ is then excluded (see [3, section 5.2] and [25, section 2]).

All over the space r must stay the same. Then, for isotropy the possibilities are:

1. $q=3$: *isotropic.* The Weyl tensor and all kinematical quantities, except the expansion rate Θ, vanish. All observations (at every point) are isotropic, this is the FLRW family of spacetime geometries;

2. $q=1$: *local rotational symmetry* (LRS). The Weyl tensor is of algebraic Petrov type D or O[10] and kinematical quantities are rotationally symmetric about a preferred spatial direction. All observations at every general point are rotationally symmetric about this direction.;

3. $q=0$: *anisotropic.* There are no rotational symmetries. Observations in each direction are different from each other.

For homogeneity we have:

1. $s=4$: *spacetime homogeneous* models. These models are unchanging in space and time, so the density ρ is constant and from (2.103) we see that $\Theta=0$, i.e. they cannot expand. Their only relevance in cosmology is as a non-expanding asymptotic state of an expanding model;

[10]For Petrov types see [15, chapter 4].

2. $s = 3$: *spatially homogeneous* universes. In this case we have the major models of theoretical cosmology because they express the idea of the cosmological principle: all the points of space at the same time are equivalent to each other;

3. $s \leq 2$: *spatially inhomogeneous* universes.

Using the above classes, we can make every cosmological model with a given symmetry (see table A.1 [25])[11].

For example, the family of FLRW spaces, that model the standard cosmology, are isotropic and spatially homogeneous universes ($q = 3$, $s = 3 \Rightarrow r = 6$).

The LTB family of models correspond to spatially inhomogeneous universes with LRS ($q = 1$, $s = 2 \Rightarrow r = 3$).

Another interesting case is the spatially homogeneous but anisotropic family ($q = 0$, $s = 3 \Rightarrow r = 3$), which is the family of Bianchi universes. In this case we have a simply transitive group of isometries G_3 and hence no continuous isotropy group.

[11] For more details on the classification of q and s given above see [3, section 5.2].

	$q = 0$	$q = 1$	$q = 3$
$s = 0$	Szekeres-Szafron Stephani-Barnes Oleson type N		
$s = 1$	general metric form independent of one coordinate		
$s = 2$	generic metric form known, spatially self-similar, abelian G_2 on $2D$ spacelike surfaces, non-abelian G_2	LTB family	none (cannot happen)
$s = 3$	Bianchi: orthogonal, tilted	Kantowski-Sachs LRS Bianchi	FLRW family
$s = 4$	Oszvath/Kerr	Gödel	Einstein static

Table A.1: Classification of cosmological models by isotropy and homogeneity.

Appendix B

The original classification by Bianchi of the various type of G_3

In his paper [27], L. Bianchi begins with a finite-dimensional continuous Lie group G_r generated by r infinitesimal transformations $\xi_1, \xi_2, \ldots, \xi_r$ of a m-dimensional Riemannian manifold M. Then, the problem of determining which spaces possess a continuous group of motions reduces to the classification of all possible forms of metrics which possess a Lie group $G_r = \{\xi_1, \ldots, \xi_m\}$ which transforms the metric into itself.

Since in chapter 5 we study Bianchi models, for which the dimension of the group of motions is $r = 3$ (see table A.1), here we give the original classification of the different types of transitive G_3. For a complete discussion of this topic see [27].

- Type I: the metric has the form

$$ds^2 = dx_1^2 + \alpha dx_2^2 + 2\beta dx_2 dx_3 + \gamma dx_3^2 \,. \tag{B.1}$$

Here α, β and γ are constants and the space is of zero curvature.

The composition rule is:

$$[\xi_1, \xi_2]\, f = [\xi_1, \xi_3]\, f = [\xi_2, \xi_3]\, f = 0\,, \tag{B.2}$$

where f is a test-function.

- Type II: the metric form is

$$ds^2 = dx_1^2 + dx_2^2 + 2x_1 dx_2 dx_3 + \left(x_1^2 + 1\right) dx_3^2\,, \tag{B.3}$$

and it holds the following rule:

$$[\xi_1, \xi_2]\, f = [\xi_1, \xi_3]\, f = 0 \qquad \text{and} \qquad [\xi_2, \xi_3]\, f = \xi_1 f\,. \tag{B.4}$$

- Type III: the line element is

$$ds^2 = dx_1^2 + e^{2x_1} dx_2^2 + 2ne^{2x_1} dx_2 dx_3 + dx_3^2\,, \tag{B.5}$$

where n is a constant.
We have:

$$[\xi_1, \xi_2]\, f = 0\,, \qquad [\xi_1, \xi_3]\, f = \xi_1 f\,, \qquad [\xi_2, \xi_3]\, f = 0\,. \tag{B.6}$$

- Type IV: the metric has the form

$$ds^2 = dx_1^2 + e^{2x_1} \left[dx_2^2 + 2x_1 dx_2 dx_3 + \left(x_1^2 + n^2\right) dx_3^2\right]\,, \tag{B.7}$$

with $n = const.$ and

$$[\xi_1, \xi_2]\, f = 0\,, \quad [\xi_1, \xi_3]\, f = \xi_1 f\,, \quad [\xi_2, \xi_3]\, f = \xi_1 f + \xi_2 f\,. \tag{B.8}$$

- Type V: the line element is

$$ds^2 = dx_1^2 + e^{2hx_1} \left(dx_2^2 + dx_3^2\right)\,, \tag{B.9}$$

where $h = const.$ and the composition is

$$[\xi_1, \xi_2]\, f = 0\,, \quad [\xi_1, \xi_3]\, f = \xi_1 f\,, \quad [\xi_2, \xi_3]\, f = \xi_2 f\,. \tag{B.10}$$

- Type VI: for this group the metric is

$$ds^2 = dx_1^2 + e^{2x_1}dx_2^2 + 2ne^{2(h+1)x_1}dx_2dx_3 + e^{2hx_1}dx_3^2 \,. \tag{B.11}$$

We have

$$[\xi_1, \xi_2]\, f = 0\,, \quad [\xi_1, \xi_3]\, f = \xi_1 f\,, \quad [\xi_2, \xi_3]\, f = h\xi_2 f\,, \tag{B.12}$$

with $h \neq 0$, 1.

- Type VII_1: here the line element has the form

$$ds^2 = dx_1^2 + (n + \cos x_1)\, dx_2^2 + 2\sin x_1 dx_2 dx_3 + (n - \cos x_1)\, dx_3^2\,, \tag{B.13}$$

where $n > 1$.
The composition rule is

$$[\xi_1, \xi_2]\, f = 0\,, \quad [\xi_1, \xi_3]\, f = \xi_2 f\,, \quad [\xi_2, \xi_3]\, f = -\xi_1 f\,. \tag{B.14}$$

- Type VII_2: the metric form is

$$\begin{aligned} ds^2 \;&=\; dx_1^2 + e^{-hx_1}\Bigg[\Big(n + \cos vx_1\Big)dx_2^2 \\ &\quad +\Big(h\cos vx_1 + v\sin vx_1 + nh\Big)dx_2dx_3 \\ &\quad +\Big(\frac{2-v^2}{2}\cos vx_1 + \frac{hv}{2}\sin vx_1 + n\Big)dx_3^2\Bigg]\,, \end{aligned} \tag{B.15}$$

with the following composition:

$$[\xi_1, \xi_2]\, f = 0\,, \quad [\xi_1, \xi_3]\, f = \xi_2 f\,, \quad [\xi_2, \xi_3]\, f = -\xi_1 f\, + h\xi_2 f, \tag{B.16}$$

where $0 < h < 2$.

- Type $VIII$: the line element is

$$ds^2 = dx_1^2 + \alpha dx_2^2 + 2\left(\beta - \alpha x_2\right) dx_2 dx_3 + \left(\alpha x_2^2 - 2\beta x_2 + \gamma\right) dx_3^2\,, \tag{B.17}$$

with α, β and γ functions of x_1.
The composition is

$$\left[\xi_1, \xi_2\right] f = \xi_1 f\,, \quad \left[\xi_1, \xi_3\right] f = 2\xi_2 f\,, \quad \left[\xi_2, \xi_3\right] f = \xi_3 f\,. \tag{B.18}$$

Bibliography

[1] G.F.R. Ellis, R. Maartens and M.A.H. MacCallum, *Relativistic Cosmology*, Cambridge University Press, Cambridge, 2012.

[2] R.M. Wald, *General Relativity*, University of Chicago Press, Chicago, 1984.

[3] G.F.R. Ellis and H. van Elst, *Cosmological Models (Cargèse lectures 1998)*, NATO Adv. Study Inst. Ser. C. Math. Phys. Sci. **541** (1999) 1-116 [arXiv:gr-qc/9812046].

[4] T. Buchert, *On Average Properties of Inhomogeneous Fluids in General Relativity I: Dust Cosmologies*, Gen. Rel. Grav. **32** (2000) 105-125 [arXiv:gr-qc/9906015v2].

[5] C. Clarkson, G.F.R. Ellis, J. Larena and O. Umeh, *Does the growth of structure affect our dynamical models of the universe? The averaging, backreaction and fitting problems in cosmology*, Rept. Prog. Phys. **74** (2011) 112901 [arXiv:1109.2314v1 [astro-ph.CO]].

[6] T. Buchert and M. Carfora, *Matter seen at many scales and the geometry of averaging in relativistic cosmology*, General Relativity, Cosmology, and Gravitational Lensing, G. Marmo, C. Rubano and P. Scudellaro (eds.), Napoli Series on Physics and Astrophysics, Bibliopolis, Naples (2002) 29-44 [arXiv:gr-qc/0101070v1].

[7] M. Carfora and K. Piotrkowska, *A Renormalization Group Approach to Relativistic Cosmology*, Phys.Rev. **D52** (1995) 4393-4424 [arXiv:gr-qc/9502021v1].

[8] S. Magni, *Backreaction and the Covariant Formalism of General Relativity*, Master's thesis, Università degli Studi di Pavia, 2011 [arXiv:1202.0430v1 [gr-qc]].

[9] S. Räsänen, *Lecture notes: Cosmology I. Chapter 3: The Friedmann-Robertson-Walker model*, url: http://theory.physics.helsinki.fi/~cosmology (lecture notes of 2012, visited on July 24, 2013).

[10] J. Plebański and A. Krasiński, *An Introduction to General Relativity and Cosmology*, Cambridge University Press, Cambridge, 2006.

[11] P. Peter and J.P. Uzan, *Primordial Cosmology*, Oxford University Press, Oxford, 2009.

[12] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman and Company, New York, 1973.

[13] S. Räsänen, *On the relation between the isotropy of the CMB and the geometry of the universe*, Phys. Rev. **D79** (2009) 123522 [arXiv:0903.3013v2 [astro-ph.CO]].

[14] K. Bolejko, *Volume averaging in the quasispherical Szekeres model*, Gen. Rel. Grav. **41** (2009) 1585-1593 [arXiv:0808.0376v2 [astro-ph]].

[15] H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, Cambridge, 2009.

[16] A. Krasiński, *Inhomogeneous Cosmological Models*, Cambridge University Press, Cambridge, 1997.

[17] J. Wainwright and G.F.R. Ellis, *Dynamical Systems in Cosmology*, Cambridge University Press, Cambridge, 1997.

[18] M.P. Ryan and L.C. Shepley, *Homogeneous Relativistic Cosmologies*, Princeton University Press, Princeton, 1975.

[19] S. Räsänen, *Light propagation in statistically homogeneous and isotropic dust universes*, JCAP **02** (2009) 011 [arXiv:0812.2872v2 [astro-ph]].

[20] C. Hellaby and A. Krasiński, *Physical and Geometrical Interpretation of the $\epsilon \leq 0$ Szekeres Models*, Phys. Rev. **D77** (2008) 023529 [arXiv:0710.2171v1 [gr-qc]].

[21] P. Szekeres, *A Class of Inhomogeneous Cosmological Models*, Commun. Math. Phys. **41** (1975) 55.

[22] A. Krasiński, *Geometry and topology of the quasi-plane Szekeres model*, Phys. Rev. **D78** (2008) 064038 [arXiv:0805.0529v4 [gr-qc]].

[23] C. Hellaby and A. Krasiński, *You Can't Get Through Szekeres Wormholes or Regularity, Topology and Causality in Quasi-Spherical Szekeres Models*, Phys. Rev. **D66** (2002) 084011 [arXiv:gr-qc/0206052v4].

[24] C.B. Collins and S.W. Hawking, *Why is the universe isotropic?*, Astrophys. J. **180** (1973) 317-334.

[25] T. Ghosh, A. Hajian and T. Souradeep, *Unveiling Hidden Patterns in CMB Anisotropy Maps*, (2006) [arXiv:astro-ph/0604279v1].

[26] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge, 1973.

[27] L. Bianchi, *On the Three-dimensional Spaces which Admit a Continuous Group of Motions*, Gen. Rel. Grav. **33**, Issue 12 (2001) 2171-2253.

[28] A. Krasiński, *Rotating Dust Solutions of Einstein's Equations with* 3*-dimensional Symmetry Groups. Part* 1*: Two Killing Fields Spanned on* u^α *and* w^α, J. Math. Phys. **39** (1998) 380-400 [arXarXiv:gr-qc/0005109v1iv:gr-qc/9705010v1].

[29] A. Krasiński, *Rotating Dust Solutions of Einstein's Equations with* 3*-dimensional Symmetry Groups. Part* 2*: One Killing Field Spanned on* u^α *and* w^α, J. Math. Phys. **39** (1998) 401-422 [arXiv:gr-qc/9705078v3].

[30] A. Krasiński, *Rotating Dust Solutions of Einstein's Equations with* 3*-dimensional Symmetry Groups. Part* 3*: All Killing Fields Linearly Independent of* u^α *and* w^α, J. Math. Phys. **39** (1998) 2148-2179 [arXiv:gr-qc/9707021v2].

[31] A. Krasiński, *Rotating Bianchi type V dust models generalizing the* $k = -1$ *Friedmann model*, J. Math. Phys. **42** (2001) 355-367 [arXiv:gr-qc/0005109v1].

[32] A. Krasiński, *Friedmann limits of rotating hypersurface-homogeneous dust models*, J. Math. Phys. **42** (2001) 3628-3664 [arXiv:gr-qc/0102040v1].

[33] K. Enqvist, *Lemaître-Tolman-Bondi model and accelerating expansion*, Gen. Rel. Grav. **40** (2008) 451-466 [arXiv:0709.2044 [astro-ph]].

[34] K. Bolejko and L. Andersson, *Apparent and average acceleration of the universe*, JCAP **0810** (2008) 003 89 [arXiv:0807.3577v2 [astro-ph]].

[35] A. Paranjape and T.P. Singh , *The Possibility of Cosmic Acceleration via Spatial Averaging in Lemaitre-Tolman-Bondi Models*, Class. Quant. Grav. **23** (2006) 6955-6969 [arXiv:astro-ph/0605195].

[36] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, New-York, 1983.

[37] C.G. Tsagas, A. Challinor and R. Maartens, *Relativistic cosmology and large-scale structure*, Phys. Rept. **465** (2008) 61-147 [arXiv:0705.4397v3 [astro-ph]].

[38] C. Sicka, T. Buchert and M. Kerscher, *Backreaction in Cosmological Models*, Proc. of the Fifth SFB-375 Ringberg workshop, Ed. Daniel Thomas, Ringberg Castle, Tegernsee, Germany, 1999 [arXiv:astro-ph/9907137v1].

[39] T. Buchert, *On average properties of inhomogeneous fluids in general relativity II: perfect fluid cosmologies*, Gen. Rel. Grav. **33** (2001) 1381-1405 [arXiv:gr-qc/0102049].

[40] A. Ishibashi, R.M. Wald, *Can the Acceleration of Our Universe Be Explained by the Effects of Inhomogeneities?*, Class. Quant. Grav. **23** (2006) 235-250 [arXiv:gr-qc/0509108].

[41] C. Bonvin, R. Durrer, M.A. Gasparini, *Fluctuations of the luminosity distance*, Phys. Rev. **D73** (2006) 023523 [arXiv:astro-ph/0511183].

[42] T. Buchert and J. Ehlers, *Averaging inhomogeneous Newtonian cosmologies*, Astron. Astrophys. **320** (1997) 1-7 [arXiv:astro-ph/9510056v3].

[43] T. Buchert and M. Carfora, *Regional averaging and scaling in relativistic cosmology*, Class. Quant. Grav. **19** (2002) 6109-6145 [arXiv:gr-qc/0210037v2].

[44] T. Buchert and M. Carfora, *Cosmological parameters are dressed*, Phys. Rev. Lett. **90** (2003) 031101 [arXiv:gr-qc/0210045v1].

[45] G.F.R. Ellis and T. Buchert, *The universe seen at different scales*, Phys. Lett. **A347** (2005) 38-46 [arXiv:gr-qc/0506106v2].

[46] S. Räsänen, *Cosmological acceleration from structure formation*, Int. J. Mod. Phys. **D15** (2006) 2141-2146 [arXiv:astro-ph/0605632].

[47] S. Räsänen, *Accelerated expansion from structure formation*, JCAP **0611** (2006) 003 [arXiv:astro-ph/0607626].

[48] M. Lavinto, S. Räsänen and S.J. Szybka, *Average expansion rate and light propagation in a cosmological Tardis spacetime* [arXiv:1308.6731v1 [astro-ph.CO]].

[49] T. Kai, H. Kozaki, K-I. Nakao, Y. Nambu and C-M. Yoo, *Can inhomogeneties accelerate the cosmic volume expansion?*, Prog. Theor. Phys. **117** (2007) 229-240 [arXiv:gr-qc/0605120].

[50] C-H. Chuang, J-A. Gu and W-Y. P. Hwang, *Inhomogeneity-Induced Cosmic Acceleration in a Dust Universe*, Class. Quant. Grav. **25** (2008) 175001 [arXiv:astro-ph/0512651].