9 Inflation: perturbations

9.1 The evolution of perturbations

9.1.1 The equations of motion

In the previous chapter, we discussed background evolution during inflation. Let us now look at how perturbations are generated during inflation and how they evolve. In order to do a consistent calculation, we would have to consider perturbations both in the inflaton field and the spacetime metric. Instead of delving into cosmological perturbation theory, we will go for a simplified treatment where we neglect perturbations in the metric. (This calculation, properly interpreted, will give the right result to leading order in the slow-roll parameters.)

We split the inflaton field into a background part that depends only on time and a perturbation that depends also on space:

$$\varphi(t, \boldsymbol{x}) = \bar{\varphi}(t) + \delta\varphi(t, \boldsymbol{x}) \ . \tag{9.1}$$

This split is not unique, as we could add a time-dependent part to the perturbation and subtract it from the background. This can be fixed by for example demanding that the spatial average of $\delta\varphi(t, \boldsymbol{x})$ vanishes. This still leaves open the question of how the hypersurface of constant t on which this averags is taken is chosen (the spacetime is no longer exactly homogeneous and isotropic, so there is no obviously preferred time slicing). This is related to the *gauge freedom* of cosmological perturbation theory, and we will not consider it further.

In chapter 8, we derived the equation of motion of the scalar field,

$$\ddot{\bar{\varphi}} + 3H\dot{\bar{\varphi}} = -V'(\bar{\varphi}) \ , \tag{9.2}$$

The equation of motion for the full field (neglecting perturbations in the metric) is similar,

$$\ddot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + 3H\dot{\varphi} = -V'(\varphi) \ , \tag{9.3}$$

where the new addition is the spatial derivatives, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Note the a^{-2} factor, which corresponds to the fact that the measure of proper length is $a(t)\mathrm{d}x^i$, not $\mathrm{d}x^i$. Using the decomposition (9.1) and expanding $V'(\varphi) = V'(\bar{\varphi}) + V''(\bar{\varphi})\delta\varphi + \mathcal{O}(\delta\varphi^2)$, we get to first order in $\delta\varphi$,

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \left(-\frac{1}{a^2}\nabla^2 + V''(\bar{\varphi})\right)\delta\varphi = 0 \ . \tag{9.4}$$

Although we have neglected metric perturbations, this expression is (in a suitable coordinate system) correct during slow-roll to leading order in the slow-roll parameters.

As the equation of motion is linear, it is easily solved with a Fourier transformation. Let us assume that the universe is spatially flat ($K = 0$). We can then write

$$\delta\varphi(t, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\,\delta\varphi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ , \tag{9.5}$$

Because the universe expands, the variable $\boldsymbol{k}$, called the *comoving momentum* or *comoving wavenumber*, is not the physical momentum, which is given by $\boldsymbol{k}/a$. With the scale factor normalised to unity today, the comoving momentum of a Fourier mode is the physical momentum it has today.

Spatial flatness is crucial here. If space was curved, plane waves would not form a complete set of basis functions, and we would instead have to use more complicated functions. (There would also be an additional scale present, given by the spatial curvature term K/a^2.)

Different Fourier modes decouple, and (9.4) reduces to

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = 0\ , \tag{9.6}$$

where we have denoted $m^2(\bar{\varphi}) \equiv V''(\bar{\varphi})$.

9.2 Fourier decomposition

As a short interjection, let us give a few results regarding Fourier transform and Fourier series. We will want to interconvert between the two. Following Liddle & Lyth [1] we have, for any function $g(t,\boldsymbol{x})$

$$\begin{aligned} g(t,\boldsymbol{x}) &= \frac{1}{(2\pi)^{3/2}}\int g(t,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d^3k \\ g(t,\boldsymbol{k}) &= \frac{1}{(2\pi)^{3/2}}\int g(t,\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}d^3x\,. \end{aligned} \tag{9.7}$$

To take the limit of infinite box size, $L^3 \to \infty$, we replace

$$\begin{aligned} \left(\frac{2\pi}{L}\right)^3\sum_{\boldsymbol{k}} &\quad\to\quad \int d^3k \\ \left(\frac{L}{2\pi}\right)^3 g_{\boldsymbol{k}}(t) &\quad\to\quad \frac{1}{(2\pi)^{3/2}}g(t,\boldsymbol{k}) \\ \left(\frac{L}{2\pi}\right)^3 \delta_{\boldsymbol{k}\boldsymbol{k}'} &\quad\to\quad \delta^3(\boldsymbol{k}-\boldsymbol{k}') \end{aligned} \tag{9.8}$$

It is usually easiest to work with the series and convert to the integral at the end (to avoid dealing with products of delta functions).

Cosmological perturbations generated by inflation are *Gaussian*, which means that different $\boldsymbol{k}$ modes are independent (except for the reality condition $g_{-\boldsymbol{k}} = g^*_{\boldsymbol{k}}$), have a Gaussian distribution, and all statistical information is encoded in the variance. The variance of a quantity whose average vanishes, $\langle g(\boldsymbol{x})\rangle = 0$ (we don't explicitly write the time-dependence here), is[1]

$$\begin{aligned} \langle g(\boldsymbol{x})^2\rangle = \sum_{\boldsymbol{k}}\langle|g_{\boldsymbol{k}}|^2\rangle &\equiv \left(\frac{2\pi}{L}\right)^3\sum_{\boldsymbol{k}}\frac{1}{4\pi k^3}\mathcal{P}_g(k) \\ \to\quad \frac{1}{4\pi}\int\frac{d^3k}{k^3}\mathcal{P}_g(k) &= \int_0^\infty\frac{dk}{k}\mathcal{P}_g(k) = \int_{-\infty}^\infty\mathcal{P}_g(k)d\ln k\ , \end{aligned} \tag{9.9}$$

[1] Note that the result has no $\boldsymbol{x}$-dependence. Even though the function $g(\boldsymbol{x})^2$ varies from place to place, its expectation value is the same everywhere.

where we have defined the *power spectrum* $\mathcal{P}_g(k)$ as

$$\mathcal{P}_g(k) \equiv \left(\frac{L}{2\pi}\right)^3 4\pi k^3 \langle |g_{\boldsymbol{k}}|^2 \rangle = \frac{L^3}{2\pi^2} k^3 \langle |g_{\boldsymbol{k}}|^2 \rangle \ . \tag{9.10}$$

The power spectrum of g gives the contribution of a logarithmic scale interval to the variance of $g(\boldsymbol{x})$. For Gaussian perturbations, the power spectrum gives a complete statistical description, and all statistical quantities can be calculated from it.

9.2.1 Solutions

Let us now return to the equation of motion (9.6) for the field perturbations and solve it. During inflation, H and m^2 change slowly. Thus, we now make an approximation where we treat them as constants. The general solution of (9.6) is then

$$\delta\varphi_{\boldsymbol{k}}(t) = a^{-3/2}\left[A_{\boldsymbol{k}} J_{-\nu}\left(\frac{k}{aH}\right) + B_{\boldsymbol{k}} J_\nu\left(\frac{k}{aH}\right)\right] , \tag{9.11}$$

where J_ν is the Bessel function of order ν, with

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \ . \tag{9.12}$$

The time dependence of the scale factor for constant H is

$$a(t) \propto e^{Ht} \ . \tag{9.13}$$

If the slow-roll approximation is valid, the inflaton has negligible mass, $m^2 \ll H^2$, since

$$\frac{m^2}{H^2} = 3M_{\rm Pl}^2 \frac{V''}{V} = 3\eta \ll 1 \, . \tag{9.14}$$

Thus we can drop m^2/H^2 in (9.12), so

$$\nu = \frac{3}{2} \, . \tag{9.15}$$

Bessel functions of half-integer order are the spherical Bessel functions which can be expressed in terms of trigonometric functions. The solution (9.11) now reduces to

$$\delta\varphi_{\boldsymbol{k}}(t) = A_{\boldsymbol{k}} w_k(t) + B_{\boldsymbol{k}} w_k^*(t) \, , \tag{9.16}$$

where the constants $A_{\boldsymbol{k}}, B_{\boldsymbol{k}}$ have been redefined to absorb some numerical constants, compared to (9.11), and

$$w_k(t) = \left(i + \frac{k}{aH}\right) \exp\left(\frac{ik}{aH}\right) \ . \tag{9.17}$$

Well before Hubble exit, $k \gg aH$, the exponent is large, and the solution oscillates rapidly. After Hubble exit, $k \ll aH$, the solution stops oscillating and approaches the constant value $i(A_{\boldsymbol{k}} - B_{\boldsymbol{k}})$. As the equation for the field perturbation is linear, we need some other information to fix the constants of integration in (9.16), i.e. the initial conditions. They are given by quantum mechanical vacuum fluctuations.

9.3 The generation of perturbations

It may sound somewhat odd to discuss the generation of perturbations. This implies that we consider the state of a system which is homogeneous and isotropic at some initial time, but where the behaviour is nevertheless different at different positions at a later time. This may seem impossible, because then we would have to a have a rule that would say where the perturbations are going to be, which would distinguish one position from another. Therefore it would seem that perturbations have to be given as an initial condition, and cannot be calculated from first principles. In a deterministic theory, this is true. However, quantum theory offers a way out of this impasse. It is is indeterministic, and there is no rule that will tell what the outcome of a quantum process will be, only probabilities of various outcomes (i.e. statistical distributions) are calculable. To discuss quantum behaviour of the inflaton field, we need to use quantum field theory in an inflating FRW universe. To warm up let us first consider quantum field theory of a scalar field in Minkowski space.

9.3.1 Vacuum fluctuations in Minkowski space

The field equation for a massive free (i.e. $V(\varphi) = \frac{1}{2}m^2\varphi^2$) real scalar field in Minkowski space is

$$\ddot{\varphi} - \nabla^2\varphi + m^2\varphi = 0\,, \tag{9.18}$$

or

$$\ddot{\varphi}_{\boldsymbol{k}} + E_k^2\varphi_{\boldsymbol{k}} = 0\,, \tag{9.19}$$

where $E_k^2 = k^2 + m^2$. We recognise (9.19) as the equation for a harmonic oscillator. Thus each Fourier component of the field behaves as an independent harmonic oscillator.

In the quantum mechanical treatment of the harmonic oscillator one introduces the creation and annihilation operators, which raise and lower the energy state of the system. It will be useful to do that here.

We have a different pair of creation and annihilation operators $\hat{a}_{\boldsymbol{k}}^\dagger$, $\hat{a}_{\boldsymbol{k}}$ for every Fourier mode $\boldsymbol{k}$. We denote the ground state of the system by $|0\rangle$, and call it the *vacuum*. Particles are quanta of the oscillations of the field. The vacuum is a state with no particles. Operating on the vacuum with the creation operator $\hat{a}_{\boldsymbol{k}}^\dagger$ adds one quantum with momentum $\boldsymbol{k}$ and energy E_k to the system, i.e. creates one particle. We denote this state with one particle with momentum $\boldsymbol{k}$ by $|1_{\boldsymbol{k}}\rangle$. Thus

$$\hat{a}_{\boldsymbol{k}}^\dagger|0\rangle = |1_{\boldsymbol{k}}\rangle\ , \tag{9.20}$$

and the state is normalised as $\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}'}\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}$. This particle has a well-defined momentum $\boldsymbol{k}$, and therefore it is completely unlocalised, as dictated by the Heisenberg uncertainty principle. The annihilation operator acting on the vacuum gives zero, i.e. not the vacuum state but the zero element of Hilbert space (the space of all quantum states),

$$\hat{a}_{\boldsymbol{k}}|0\rangle = 0\ . \tag{9.21}$$

We denote the hermitian conjugate of the vacuum state by $\langle 0|$. Thus

$$\langle 0|\hat{a}_{\boldsymbol{k}} = \langle 1_{\boldsymbol{k}}| \qquad \text{and} \qquad \langle 0|\hat{a}_{\boldsymbol{k}}^\dagger = 0\,. \tag{9.22}$$

The commutation relations of the creation and annihilation operators are

$$[\hat{a}_{\boldsymbol{k}}^{\dagger}, \hat{a}_{\boldsymbol{k}'}^{\dagger}] = [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}] = 0\,, \qquad [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^{\dagger}] = \delta_{\boldsymbol{k}\boldsymbol{k}'}\ . \tag{9.23}$$

When going from classical physics to quantum physics, classical observables are replaced by operators. We can then calculate expectation values for these observables using the operators. Here the classical observable

$$\varphi(t, \boldsymbol{x}) = \sum \varphi_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{9.24}$$

is replaced by the *field operator*

$$\hat{\varphi}(t, \boldsymbol{x}) = \sum \hat{\varphi}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{9.25}$$

where[2]

$$\hat{\varphi}_{\boldsymbol{k}}(t) = w_k(t)\hat{a}_{\boldsymbol{k}} + w_k^*(t)\hat{a}_{-\boldsymbol{k}}^{\dagger} \tag{9.26}$$

and

$$w_k(t) = L^{-3/2}\frac{1}{\sqrt{2E_k}} e^{-iE_k t} \tag{9.27}$$

is the mode function, a solution of the field equation (9.19). (The normalisation has been fixed to get the right commutation relations, (9.29).) We are using the Heisenberg picture, i.e. we have time-dependent operators and the quantum states are time-independent. Note that since the operator $\hat{\varphi}(t, \boldsymbol{x})$ is Hermitian (corresponding to a real field), $\hat{\varphi}(t, \boldsymbol{x})^{\dagger} = \hat{\varphi}(t, \boldsymbol{x})$, the corresponding Fourier components satisfy $\hat{\varphi}_{\boldsymbol{k}}(t)^{\dagger} = \hat{\varphi}_{-\boldsymbol{k}}(t)$. So the Fourier component operators are not Hermitian.

In quantum mechanics, we have two conjugate variables, position and momentum. In quantum field theory, we have the field and the corresponding canonical momentum, which is in this case just given by the time derivative of the field. Combining (9.26) and (9.27), we have

$$\dot{\hat{\varphi}}_{\boldsymbol{k}}(t) = -iE_k\left(w_k(t)\hat{a}_{\boldsymbol{k}} - w_k^*(t)\hat{a}_{-\boldsymbol{k}}^{\dagger}\right)\ . \tag{9.28}$$

We can now calculate the commutator between the field operator and the corresponding velocity operator. A straightforward calculation with the rules (9.23) gives

$$[\hat{\varphi}_{\boldsymbol{k}}(t), \dot{\hat{\varphi}}_{\boldsymbol{k}'}(t)] = iL^{-3}\delta_{\boldsymbol{k},-\boldsymbol{k}'}\ . \tag{9.29}$$

(**Exercise:** Show that demanding the canonical commutation relation (9.29) fixes the normalisation to be the one given in (9.27).)

The Hamiltonian density of the scalar field in Minkowski space is

$$\begin{aligned}\hat{\mathcal{H}} &= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\hat{\varphi}\,\partial_\nu\hat{\varphi} + V(\hat{\varphi}) \\ &= \frac{1}{2}\dot{\hat{\varphi}}^2 - \frac{1}{2}\delta^{ij}\partial_i\hat{\varphi}\partial_j\hat{\varphi} + V(\hat{\varphi})\ ,\end{aligned} \tag{9.30}$$

[2] We skip the detailed derivation of the field operator, which belongs to a course of quantum field theory. See e.g. Peskin & Schroeder, section 2.3 (note the different normalisations of operators and states, related to doing Fourier integrals rather than sums, and considerations of Lorentz invariance).

and the Hamiltonian is the spatial integral of the Hamiltonian density,

$$\hat{H} = \int d^3x \hat{\mathcal{H}} \ . \tag{9.31}$$

Since the Hamiltonian depends on the field velocity operator, it does not commute with the field operator,

$$[\hat{H}, \hat{\varphi}] \neq 0 \ . \tag{9.32}$$

As a result, the Hamiltonian and the field operator do not share a complete set of eigenstates. So, in general an eigenstate of the Hamiltonian is not an eigenstate of the field operator. Eigenstates of the Hamiltonian operator are the energy eigenstates, and the state with the smallest energy is called the vacuum state. Since the vacuum is not an eigenstate of the field operator, the eigenvalues of the field operator are not well defined, instead we have only a distribution of values. In other words, the scalar field has *vacuum fluctuations.*

The vacuum fluctuations of the field are Gaussian (we skip the proof), and are thus completely completely characterised by their variance, which we can express with the power spectrum as (note that $\langle\hat{\varphi}\rangle = 0$)

$$\langle\hat{\varphi}(\boldsymbol{x})^2\rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_\varphi(k) \ . \tag{9.33}$$

For the vacuum state $|0\rangle$, the expectation value of $|\varphi_{\boldsymbol{k}}|^2$ is

$$\begin{aligned} \langle 0|\hat{\varphi}_{\boldsymbol{k}}\hat{\varphi}^\dagger_{\boldsymbol{k}}|0\rangle \ &= \\ &|w_k|^2\langle 0|\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle + w_k^2\langle 0|\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}|0\rangle + (w_k^*)^2\langle 0|\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle + |w_k|^2\langle 0|\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}|0\rangle \\ &= \ |w_k|^2\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}}\rangle \ = \ |w_k|^2 \end{aligned} \tag{9.34}$$

since all but the first term give 0, and the states are normalised so that $\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}'}\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}$. Therefore the power spectrum is, using the definition (9.10),

$$\mathcal{P}_\varphi(k) = L^3 \frac{k^3}{2\pi^2}|w_k|^2 \ . \tag{9.35}$$

From (9.27) we have $|w_k|^2 = 1/(2L^3E_k)$, so we get the final result

$$\mathcal{P}_\varphi(k) = \frac{k^3}{4\pi^2 E_k} \ . \tag{9.36}$$

In the case of inflation, the mode functions are different because space is expanding, but the reasoning remains the same.

9.3.2 Vacuum fluctuations during inflation

During inflation the field equation for inflaton perturbations is, from (9.4),

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = 0 \ . \tag{9.37}$$

In inflation, the background field is treated classically, and only the perturbations around the mean value of the field are quantised. In fact, if we were to take into account perturbations on the metric in a coordinate-independent manner, we would see that the variables that are quantised are a linear combination of the scalar field perturbations and metric perturbations. Thus in inflation, part of the spacetime metric is quantised. Inflation may thus be called the first quantum gravity scenario which has been confronted with observations – with great success. However, just like the background scalar field, the background metric is not quantised. How to quantise the metric in general, and not just small perturbations, remains one of the most studied and most difficult questions in physics. In this course, we just treat the field perturbation during inflation the same way that we treated the field in Minkowski space. That is, the Fourier modes of the field perturbation are written as

$$\delta\hat{\varphi}_{\boldsymbol{k}}(t) = w_k(t)\hat{a}_{\boldsymbol{k}} + w_k^*(t)\hat{a}^\dagger_{-\boldsymbol{k}} \ , \tag{9.38}$$

where the mode function $w_k(t)$ satisfies the classical equation of motion (9.4), with the normalisation fixed by the canonical commutation relation,

$$[\delta\hat{\varphi}_{\boldsymbol{k}}(t), \delta\dot{\hat{\varphi}}_{\boldsymbol{k}'}(t)] = i(aL)^{-3}\delta_{\boldsymbol{k},-\boldsymbol{k}'} \ , \tag{9.39}$$

where the only difference from the Minkowski space commutator (9.29) is the change $L \to aL$ on the right-hand side.

Taking the solution of (9.4) given in section 9.2.1, under the approximations $H = \text{const.}$ and $\frac{m^2}{H^2} = 3\eta \approx 0$ and fixing the normalisation with (9.39), we get the solution

$$w_k(t) = L^{-3/2}\frac{H}{\sqrt{2k^3}}\left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right) \ , \tag{9.40}$$

where the time-dependence is $a(t) \propto e^{Ht}$.

When the scale k is well inside the Hubble radius, $k \gg aH$, $\delta\varphi_{\boldsymbol{k}}(t)$ oscillates rapidly compared to the Hubble time H^{-1}. If we consider distance and time scales much smaller than the Hubble scale, spacetime curvature does not matter and things should behave like in Minkowski space. Considering (9.40) in this limit, one finds (**exercise**) that $w_k(t)$ indeed becomes (up to a slowly varying phase), equal to the Minkowski space mode function (9.27), with the lengths scaled by a. (The prefactor in (9.40) was chosen so that the normalisations would agree.) Therefore the mode function $w_k(t)$ of (9.40) tells how the perturbation behaves as it approaches and exits the Hubble radius.

The calculation of the power spectrum of inflaton fluctuations is the same as in Minkowski space, with the same result,

$$\mathcal{P}_{\delta\varphi}(k) = L^3\frac{k^3}{2\pi^2}|w_k|^2 \ . \tag{9.41}$$

Well before Hubble exit, $k \gg aH$, and on timescales $\ll H^{-1}$, the field operator $\delta\hat{\varphi}_{\boldsymbol{k}}(t)$ agrees with the Minkowski space field operator and we have the same kind of initial $\delta\varphi$ vacuum fluctuations as in Minkowski space. However, the time evolution of the perturbations is different. Well after Hubble exit, $k \ll aH$, the mode function approaches a constant

$$w_k(t) \ \to \ L^{-3/2}\frac{iH}{\sqrt{2k^3}} \ , \tag{9.42}$$

so the vacuum fluctuations "freeze" and the power spectrum acquires the constant value

$$\mathcal{P}_{\delta\varphi}(k) \;=\; L^3\frac{k^3}{2\pi^2}|w_k|^2 \;=\; \left(\frac{H}{2\pi}\right)^2 \,. \tag{9.43}$$

We have calculated the power spectrum of the inflaton field perturbations by using the quantum mechanical expectation value of the square of the field perturbation. We now identify this with the expectation value of a probability distribution of a classical variable, i.e. we assume that the quantum mechanical fluctuations become classical. Some part of this process is understood (it can be shown that the quantum mechanical expectation values become equal to those of a classical stochastic distribution, or "squeezed"), but the emergence of (at least the appearance of) classical reality from a quantum system remains an unsolved problem. In particle physics appeal is often made to the Copenhagen interpretation according to which states become classical when they are measured, but for cosmology this is inadequate. We simply assume that we can replace an expectation value of a quantum state with the ensemble average of a classical distribution.

For our purposes, quantum mechanics generates the initial perturbations and solves the problem of how perturbations can emerge from a state which is homogeneous and isotropic. As a remnant of the indeterministic origin of the perturbations, we cannot predict the specific member of the ensemble which is realised in the universe, we can only calculate the statistical distribution of perturbations. As noted, this distribution is Gaussian, so all Fourier modes $\delta\varphi_{\boldsymbol{k}}$ are independent random variables (except for the reality condition $\delta\varphi_{-\boldsymbol{k}} = \delta\varphi^*_{\boldsymbol{k}}$) with a Gaussian probability distribution.

9.3.3 The comoving curvature perturbation

We now calculated the inflationary prediction for the power spectrum of the field perturbation. Relating that prediction to the power spectrum of the density perturbation in the late universe requires a number of extra steps. We will not discuss the details, just outline some main points. Generally, the field perturbation $\delta\varphi_{\boldsymbol{k}}$ is related to the *comoving curvature perturbation* $\mathcal{R}_{\boldsymbol{k}}$, which is a measure of how much the field curves spacetime. The advantage of using $\mathcal{R}_{\boldsymbol{k}}$ is that it is constant on super-Hubble scales even when the Hubble parameter and the field change, and thus $\delta\varphi_{\boldsymbol{k}}$ changes: we see from (9.37) that $\delta\varphi_{\boldsymbol{k}}$ is not in general constant even for super-Hubble modes. The perturbation $\mathcal{R}_{\boldsymbol{k}}$ is conserved (on super-Hubble scales) not only during inflation, but during reheating, when the inflaton decays into particles, and after. In the late universe, we can thus relate $\mathcal{R}_{\boldsymbol{k}}$ to the density perturbation of the gas formed by those particles, which eventually forms galaxies and other structures.

The result (9.43) was obtained treating H as a constant. However, H does change, albeit slowly, during inflation. To take into account evolution we use for each scale k the value of H which is representative for the evolution of that particular scale through the Hubble radius. That is, we choose the value of H at Hubble exit[3],

[3] A more precise calculation, where the evolution of $H(t)$ is taken into account gives a correction to the amplitude of $\mathcal{P}_{\mathcal{R}}(k)$ that is first order in slow-roll parameters and a correction to the spectral index n that is second order in the slow-roll parameters. Note that H is assumed to be constant only for each k mode during the time it crosses the Hubble radius. The equations of motion of the different modes are independent, so in principle H could be very different for modes that exit at very different times without violating our assumptions.

so that $aH = k$. Thus the power spectrum is

$$\mathcal{P}_{\delta\varphi}(k) \;=\; L^3 \frac{k^3}{2\pi^2}|w_k|^2 \;=\; \left(\frac{H}{2\pi}\right)^2_{aH=k} , \tag{9.44}$$

where the subscript notation signifies that the value of H for each k is to be taken at Hubble exit of that particular scale.

Since we have only one quantity which has fluctuations, the inflaton field, and the perturbations are treated in linear theory, the perturbations of any other quantity are related to the inflaton field fluctuation by linear and local equations. So the distribution of the perturbations inherits the property of homogeneity and isotropy from the symmetry of the background on which they are created and evolve. Perturbations generated by inflation are statistically homogeneous and isotropic, i.e. the power spectrum depends only on the magnitude k of $\boldsymbol{k}$, not on the direction.

In particular, for the comoving curvature perturbation we have (we skip the calculation and just give the result)

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\bar{\varphi}}}\right)^2 \mathcal{P}_{\delta\varphi}(k) = \left(\frac{H}{\dot{\bar{\varphi}}}\frac{H}{2\pi}\right)^2_{aH=k} . \tag{9.45}$$

This the main result for quantum fluctuations during inflation. The problem has been completely reduced to the evolution of the background scalar field and the background Hubble parameter. We just need to specify the inflation potential and calculate how the background evolves, and plug it in (9.45) to get complete information about the perturbations. That, in turn, is the starting point for calculating structure formation and the CMB anisotropy. Turning this around, observations of large-scale structure and the CMB can be used obtain information about quantum processes in the primordial universe. Note that the power spectrum depends only on k. Statistical homogeneity and isotropy of the perturbations, inherited from the symmetry of the background, is a strong feature of inflation. (I use the word 'feature' rather than 'prediction', because it is possible to construct models where, for example, space expands anisotropically during inflation. However, that requires untypical assumptions, such as having a short period of inflation, so that the anisotropy is not washed away, or inflation driven by a vector field instead of a scalar field.)

9.4 The primordial spectrum in slow-roll inflation

So, inflation generates primordial perturbations $\mathcal{R}_{\boldsymbol{k}}$ with the power spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\varphi}}\frac{H}{2\pi}\right)^2_{aH=k} , \tag{9.46}$$

(In this section, we drop the overbar from the background values.) Let's now get back to the inflaton potential and the presentation of the dynamics of slow-roll inflation in terms of the two slow-roll variables. Applying the slow-roll equations

$$H^2 = \frac{V}{3M_{\rm Pl}^2} \qquad \text{and} \qquad 3H\dot{\varphi} = -V' ,$$

equation (9.46) becomes

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{12\pi^2}\frac{1}{M_{\rm Pl}^6}\frac{V^3}{V'^2} = \frac{1}{24\pi^2}\frac{1}{M_{\rm Pl}^4}\frac{V}{\varepsilon} , \tag{9.47}$$

where ε is the slow-roll parameter.

According to observations of CMB and large-scale structure, the amplitude of the primordial power spectrum is [2]

$$\mathcal{P}_\mathcal{R}(k)^{1/2} \approx 4.6 \times 10^{-5} \tag{9.48}$$

on cosmological scales. This gives a constraint on inflation

$$\left(\frac{V}{\varepsilon}\right)^{1/4} \approx 24^{1/4}\sqrt{\pi}\sqrt{4.6 \times 10^{-5}} M_{\text{Pl}} \approx 0.027 M_{\text{Pl}} = 6.4 \times 10^{16} \text{ GeV} . \tag{9.49}$$

Since $\varepsilon < 1$, this implies an upper limit on the energy scale of inflation,

$$V^{1/4} < 0.027 M_{\text{Pl}} . \tag{9.50}$$

This puts a limit on the Hubble scale during inflation. From $H^2 = V/(3M_{\text{Pl}}^2)$, the constraint on V translates into $H < 6 \times 10^{14}$ GeV, or in terms of length, $H^{-1} > 3 \times 10^{-31}$ m.

Since during slow-roll inflation V and V' change slowly while a wide range of scales k exit the Hubble radius, we expect $\mathcal{P}_\mathcal{R}(k)$ to be a slowly varying function of k. We describe this small variation with the *spectral index* n of the primordial spectrum, defined as[4]

$$n(k) - 1 \equiv \frac{d \ln \mathcal{P}_\mathcal{R}}{d \ln k} . \tag{9.51}$$

If the spectral index is independent of k, we say that the spectrum is *scale-free.* In this case the primordial spectrum is a *power law*

$$\mathcal{P}_\mathcal{R}(k) = A^2 \left(\frac{k}{k_*}\right)^{n-1} , \tag{9.52}$$

where the pivot scale k_* is some chosen reference scale (for the Planck data, $k_* = 0.05$ Mpc^{-1}, and and A is the amplitude at the pivot scale.

If the power spectrum is constant,

$$\mathcal{P}_\mathcal{R}(k) = \text{const.} , \tag{9.53}$$

corresponding to $n = 1$, we say that the spectrum is *scale-invariant* (which is a special case of a scale-free spectrum). A scale-invariant spectrum is also called the *Harrison–Zel'dovich* spectrum.

If $n \neq 1$, the spectrum is called *tilted.* A tilted spectrum is called *red* if $n < 1$ (more power on large scales) and *blue* if $n > 1$ (more power on small scales). If $dn/dk \neq 0$, it is said that there is a *running spectral index.*

Using (9.47) and (9.51), we can calculate the spectral index for slow-roll inflation. Since $\mathcal{P}_\mathcal{R}(k)$ is evaluated from (9.47) when $k = aH$, we have

$$\frac{d \ln k}{dt} = \frac{d \ln(aH)}{dt} = \frac{\dot{a}}{a} + \frac{\dot{H}}{H} = (1 - \varepsilon) H ,$$

[4]The -1 is in the definition for historical reasons, related to other ways of defining the power spectrum of perturbations.

where we used the fact that in the slow-roll approximation $\dot{H} = -\varepsilon H^2$ in the last step. Thus

$$\frac{d}{d\ln k} = \frac{1}{1-\varepsilon}\frac{1}{H}\frac{d}{dt} = \frac{1}{1-\varepsilon}\frac{\dot{\varphi}}{H}\frac{d}{d\varphi} = -\frac{M_{\rm Pl}^2}{1-\varepsilon}\frac{V'}{V}\frac{d}{d\varphi} \approx -M_{\rm Pl}^2\frac{V'}{V}\frac{d}{d\varphi}\,. \tag{9.54}$$

Let us first calculate the scale dependence of the slow-roll parameters:

$$\frac{d\varepsilon}{d\ln k} = -M_{\rm Pl}^2\frac{V'}{V}\frac{d}{d\varphi}\left[\frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2\right] = M_{\rm Pl}^4\left[\left(\frac{V'}{V}\right)^4 - \left(\frac{V'}{V}\right)^2\frac{V''}{V}\right] = 4\varepsilon^2 - 2\varepsilon\eta \tag{9.55}$$

and, in a similar manner (**exercise**),

$$\frac{d\eta}{d\ln k} = \ldots = 2\varepsilon\eta - \xi\,, \tag{9.56}$$

where we have defined a third slow-roll parameter

$$\xi \equiv M_{\rm Pl}^4\frac{V'}{V^2}V'''\,. \tag{9.57}$$

The parameter ξ is typically second-order small in the sense that $\sqrt{|\xi|}$ is of the same order of magnitude as ε and η.

We can now calculate the spectral index:

$$\begin{aligned} n-1 &= \frac{1}{\mathcal{P}_\mathcal{R}}\frac{d\mathcal{P}_\mathcal{R}}{d\ln k} = \frac{\varepsilon}{V}\frac{d}{d\ln k}\left(\frac{V}{\varepsilon}\right) = \frac{1}{V}\frac{dV}{d\ln k} - \frac{1}{\varepsilon}\frac{d\varepsilon}{d\ln k} \\ &= -M_{\rm Pl}^2\frac{V'}{V}\cdot\frac{1}{V}\frac{dV}{d\varphi} - 4\varepsilon + 2\eta \;=\; -6\varepsilon + 2\eta\;. \end{aligned} \tag{9.58}$$

Slow-roll requires $\varepsilon \ll 1$ and $|\eta| \ll 1$, so the spectrum is predicted to be close to scale invariant. This agrees well with observations. Note how, as in the case of dark matter, things fall into place automatically. In order to have negative pressure, a scalar field has to roll slowly. Once the background evolution is slowly rolling, the perturbations are close to scale-invariant, without needing to add new ingredients or tune anything.

Assuming that at late times the universe is described by the ΛCDM model, the current constraint on the spectral index from CMB data by the Planck satellite and the BICEP2/Keck telescope is, assuming the presence of running and possible tensor perturbations [2]

$$n = 0.9640 \pm 0.0043\;. \tag{9.59}$$

The value is model-dependent, and with a different cosmological model (different dark energy model, the presence of cosmic strings, and so on), the preferred value of the spectral index can change slightly. However, in all but the most exotic models it remains close to scale-invariant.

From the results of the running of ε and η, we obtain the running of the spectral index:

$$\frac{dn}{d\ln k} = 16\varepsilon\eta - 24\varepsilon^2 - 2\xi\;. \tag{9.60}$$

The running is second order in slow-roll parameters, so it's expected to be even smaller than the deviation from scale invariance. The observational range is

$$\frac{dn}{d\ln k} = -0.0071 \pm 0.0068 \ , \tag{9.61}$$

Some inflation models have $|n-1|$ and $|dn/d\ln k|$ larger than this, others do not. Observations have ruled out some inflation models, while a zoo of dozens to hundreds of viable models remains [3].

CMB experiments have measured the CMB temperature anisotropy over a range $\Delta \log k \approx 8$, from the largest scales down to Mpc scales. On scales smaller than those that have been probed, the CMB anisotropy is expected to be negligible, so we expect there is nothing more to see in the CMB temperature anisotropies. However, it is possible to probe these smaller scales by observations of large-scale structure. Recall that for high energy-scale inflation, the number of e-folds until the end of inflation when the largest observable modes are generated is about 60, so we are only seeing a small part of inflation.

The above results do not allow an independent determination of the two slow-roll parameters ε and η. However, it turns out that the spectral index of *tensor perturbations* produced by inflation is independent of η (it is -2ε). So if tensor perturbations are detected (they have a definite signature on the CMB) and their spectrum is measured, we can get both ε and η. The amplitude of the tensor perturbations also depends directly on the Hubble parameter during inflation, so it will provide a measurement of the energy scale of inflation. Typically, large-field inflation models produce tensor perturbations with much larger amplitude than small-field inflation models. In the small-field case they may be too small to be detectable in the near future. It is possible to calculate the spectrum of gravity waves the same way as we did for the scalar perturbations.

Example: Consider the simple inflation model

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 \, . \tag{9.62}$$

In chapter 8 we already calculated the slow-roll parameters for this model:

$$\varepsilon = \eta = 2\frac{M_{\rm Pl}^2}{\varphi^2} \tag{9.63}$$

and we immediately see that $\xi = 0$. We thus have

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}} &= \frac{1}{96\pi^2}\frac{m^2}{M_{\rm Pl}^2}\left(\frac{\varphi}{M_{\rm Pl}}\right)^4 && (9.64)\\
n &= 1 - 6\varepsilon + 2\eta = 1 - 8\left(\frac{M_{\rm Pl}}{\varphi}\right)^2 && (9.65)\\
\frac{dn}{d\ln k} &= 16\varepsilon\eta - 24\varepsilon^2 - 2\xi = -32\left(\frac{M_{\rm Pl}}{\varphi}\right)^4 \, . && (9.66)
\end{aligned}$$

To get the numbers, we need the values of φ when the relevant cosmological scales left the Hubble radius. We know that the number of inflation e-foldings after that should be about $N = 60$, depending on the preheating history. We have

$$N(\varphi) = \frac{1}{M_{\rm Pl}^2}\int_{\varphi_{\rm end}}^{\varphi}\frac{V}{V'}d\varphi = \frac{1}{M_{\rm Pl}^2}\int\frac{\varphi}{2}d\varphi = \frac{1}{4M_{\rm Pl}^2}\left(\varphi^2 - {\varphi_{\rm end}}^2\right) \, , \tag{9.67}$$

and we estimate $\varphi_{\rm end}$ from $\varepsilon(\varphi_{\rm end}) = 2M_{\rm Pl}^2/\varphi_{\rm end}{}^2 = 1 \quad \Rightarrow \quad \varphi_{\rm end} = \sqrt{2}M_{\rm Pl}$ to get

$$\varphi^2 = \varphi_{\rm end}{}^2 + 4M_{\rm Pl}^2 N = 2M_{\rm Pl}^2 + 4M_{\rm Pl}^2 N \approx 4M_{\rm Pl}^2 N\,. \tag{9.68}$$

Thus

$$\left(\frac{M_{\rm Pl}}{\varphi}\right)^2 = \frac{1}{4N}\,, \tag{9.69}$$

so we get

$$\begin{aligned} \mathcal{P}_\mathcal{R} &= \frac{N^2}{6\pi^2}\frac{m^2}{M_{\rm Pl}^2} = \frac{600}{\pi^2}\frac{m^2}{M_{\rm Pl}^2} && (9.70) \\ n &= 1 - \frac{2}{N} = 0.97 \\ \frac{dn}{d\ln k} &= -\frac{2}{N^2} = -0.0006\,, && (9.71) \end{aligned}$$

where we have input $N = 60$. For $\mathcal{P}_\mathcal{R}$ we have, according to (9.48) $\mathcal{P}_\mathcal{R} = 2.1 \times 10^{-9}$, which gives

$$m \approx \frac{8}{N}10^{14}\ \text{GeV} \approx 1 \times 10^{13}\ \text{GeV} \approx 6 \times 10^{-6} M_{\rm Pl}\,, \tag{9.72}$$

for $N = 60$. We get $V^{1/4} = (2Nm^2M_{\rm Pl}^2)^{1/4} \approx 2 \times 10^{16}$ GeV as the energy scale for the period when the perturbations seen in the CMB were generated. Potential energy at the end of inflation is

$$V_{\rm end}^{1/4} = \left(\frac{1}{2}m^2\varphi_{\rm end}{}^2\right)^{1/4} = \sqrt{\frac{m}{M_{\rm Pl}}}M_{\rm Pl} \approx 2 \times 10^{-3} M_{\rm Pl} \approx 6 \times 10^{15}\ \text{GeV}\,. \tag{9.73}$$

Because of the high energy scale, the amplitude of tensor perturbations, as quantified by *the tensor-to-scalar* ratio r is significant, $r = 8/N \approx 0.13$. The current upper limit from combined Planck and BICEP2/Keck data is $r < 0.079$ [2]. Therefore, the simple $m^2\varphi^2$ model is ruled out, although it fitted the observations fine until the Planck data.

Exercise: It can be shown that the power spectrum of gravitational waves produced by inflation is

$$\mathcal{P}_t(k) = \frac{8}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2_{aH=k}.$$

Find the tensor-to-scalar ratio

$$r \equiv \frac{\mathcal{P}_t(k)}{\mathcal{P}_\mathcal{R}(k)}$$

and the tensor spectral index

$$n_t \equiv \frac{d\ln \mathcal{P}_t}{d\ln k}$$

in terms of the slow-roll parameters to first order.

Exercise: From the limit $r < 0.07$, calculate the resulting limit on the energy scale of inflation. Using that, find the maximum amount by which the scale factor can have expanded from reheating until today, assuming there are only Standard Model degrees of freedom.

References

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