Contents

8 Inflation: perturbations

8.1 The evolution of perturbations

8.1.1 The equations of motion

In the previous chapter, we discussed background evolution during inflation. Let us now see how perturbations are generated during inflation and how they evolve. In order to do a consistent calculation, we would have to consider perturbations both in the inflaton field and the spacetime metric. Instead of delving deep into cosmological perturbation theory, we will go for a simplified treatment where we neglect perturbations in the metric. (This calculation, properly interpreted, will give the right result to leading order in the slow-roll parameters.)

We split the inflaton field into a background part that depends only on time and a perturbation that depends also on space:

$$\varphi(t,\boldsymbol{x}) = \bar\varphi(t) + \delta\varphi(t,\boldsymbol{x}) \ . \tag{8.1}$$

This split is not unique, as we could add a time-dependent part to the perturbation and subtract it from the background. This can be fixed by for example demanding that the spatial average of $\delta\varphi(t,\boldsymbol{x})$ vanishes. This still leaves open the question of how the hypersurface of constant t on which this averags is taken is chosen (the spacetime is no longer exactly homogeneous and isotropic, so there is no preferred time slicing). This is related to the *gauge freedom* of cosmological perturbation theory, which we will not discuss further.

In chapter 7, we wrote down the equation of motion of the scalar field,

$$\ddot{\bar\varphi} + 3H\dot{\bar\varphi} = -V'(\bar\varphi) \ , \tag{8.2}$$

The equation of motion for the full field (neglecting perturbations in the metric) is similar,

$$\ddot\varphi - \frac{1}{a^2}\nabla^2\varphi + 3H\dot\varphi = -V'(\varphi) \ , \tag{8.3}$$

where the new addition is the spatial Laplacian $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Note the a^{-2} factor, which corresponds to the fact that the measure of proper length is $a(t)\mathrm{d}x^i$, not $\mathrm{d}x^i$. Using the decomposition (8.1) and expanding $V'(\varphi) = V'(\bar{\varphi}) + V''(\bar{\varphi})\delta\varphi + \mathcal{O}(\delta\varphi^2)$, we get to first order in $\delta\varphi$,

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \left(-\frac{1}{a^2}\nabla^2 + V''(\bar{\varphi})\right)\delta\varphi = 0 \ . \tag{8.4}$$

Although we have neglected metric perturbations, this expression is (in the coordinate system called the *spatially flat gauge*) correct during slow-roll to leading order in the slow-roll parameters.

As the equation of motion is linear, it is easily solved with a Fourier transformation. Let us assume that the universe is spatially flat ($K = 0$) – recall that inflation quickly drives the spatial curvature to small values. We can then write

$$\delta\varphi(t, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k \delta\varphi_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ , \tag{8.5}$$

Because the universe expands, the variable $\boldsymbol{k}$, called the *comoving momentum* or *comoving wavenumber*, is not the physical momentum, which is given by $\boldsymbol{k}/a$. With the scale factor normalised to unity today, the comoving momentum of a Fourier mode is the physical momentum it has today.

Spatial flatness is crucial here. If space were curved, plane waves would not form a complete set of basis functions, and we would have to use more complicated functions. (There would also be an additional scale present, given by the spatial curvature term K/a^2.)

Different Fourier modes decouple, so (8.4) reduces to

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = 0 \ , \tag{8.6}$$

where we have denoted $m^2(\bar{\varphi}) \equiv V''(\bar{\varphi})$; note that it is possible to have $m^2(\bar{\varphi}) < 0$.

8.1.2 Fourier decomposition

As a short interjection, let us give a few results regarding Fourier transform and Fourier series. We will consider infinite Euclidean spatial sections, but often it is useful to consider a finite box, so we will want to interconvert between the two. Following Liddle & Lyth [1] we have, for any function $g(t, \boldsymbol{x})$

$$\begin{aligned} g(t, \boldsymbol{x}) &= \frac{1}{(2\pi)^{3/2}} \int g(t, \boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \mathrm{d}^3k \\ g(t, \boldsymbol{k}) &= \frac{1}{(2\pi)^{3/2}} \int g(t, \boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \mathrm{d}^3x \ . \end{aligned} \tag{8.7}$$

To take the limit of infinite box size, $L^3 \to \infty$, we replace

$$\begin{aligned} \left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{k}} \quad &\to \quad \int \mathrm{d}^3k \\ \left(\frac{L}{2\pi}\right)^3 g_{\boldsymbol{k}}(t) \quad &\to \quad \frac{1}{(2\pi)^{3/2}} g(t, \boldsymbol{k}) \\ \left(\frac{L}{2\pi}\right)^3 \delta_{\boldsymbol{k}\boldsymbol{k}'} \quad &\to \quad \delta^3(\boldsymbol{k} - \boldsymbol{k}') \ . \end{aligned} \tag{8.8}$$

It is usually easiest to work with the series and convert to the integral at the end, to avoid dealing with products of delta functions. Formally, this corresponds to considering some cubic region ("box") of the universe, in comoving coordinates, with some comoving volume L^3, and assuming periodic boundary conditions. The box is just a physically irrelevant convenient mathematical device. In the end we can take the limit $L^3 \to \infty$ and replace the Fourier series with a Fourier integral.

8.1.3 Gaussian perturbations

Simplest models of inflation predict, and observations show, that cosmological perturbations are (in the linear regime) close to *Gaussian*. Possible deviations from Gaussianity are a topical subject in cosmology at the moment. No deviations in the primordial perturbations have been found, and the non-Gaussian contribution relative to the Gaussian contribution has to be less than 10^{-4}, according to observations by the Planck satellite. (Non-linear structure formation does destroy the Gaussianity of the initial perturbations on small scales.) Let us discuss a generic Gaussian perturbation $g(\boldsymbol{x})$, where the set of Fourier coefficients $\{g_{\boldsymbol{k}}\}$ given in (8.7) is the result of a Gaussian random process. In cosmology, we can only predict the probability distribution from which the perturbations are drawn (since they originate in a quantum process), not the particular realisation that corresponds to our universe. This causes some limitations on the comparison between theory and observation, as will see when we discuss the CMB in chapter 10.

Cosmological perturbations are real, so we have $g_{-\boldsymbol{k}} = g^*_{\boldsymbol{k}}$. We can write $g_{\boldsymbol{k}}$ in terms of its real and imaginary part,

$$g_{\boldsymbol{k}} = \alpha_{\boldsymbol{k}} + i\beta_{\boldsymbol{k}} \,. \tag{8.9}$$

To know a random process means to know the *probability distribution* $\mathrm{Prob}(g_{\boldsymbol{k}})$. The *expectation value* of a quantity which depends on $g_{\boldsymbol{k}}$ as $f(g_{\boldsymbol{k}})$ is given by

$$\langle f(g_{\boldsymbol{k}})\rangle \equiv \int f(g_{\boldsymbol{k}})\mathrm{Prob}(g_{\boldsymbol{k}})d\alpha_{\boldsymbol{k}}d\beta_{\boldsymbol{k}} \,, \tag{8.10}$$

where the integral is over the complex plane, i.e.

$$\int_{-\infty}^{\infty} d\alpha_{\boldsymbol{k}} \int_{-\infty}^{\infty} d\beta_{\boldsymbol{k}} \ .$$

We now define what we mean by *Gaussian perturbations* (or by a *Gaussian random process*, a process that produces such perturbations). We restrict to perturbations with zero mean, which is the relevant situation in cosmology. Such perturbations $g(\boldsymbol{x})$ satisfy two properties:

1. The probability distribution of an individual Fourier component is Gaussian[1]:

$$\begin{aligned}
\mathrm{Prob}(g_{\boldsymbol{k}}) &= \frac{1}{2\pi s_{\boldsymbol{k}}^2}\exp\left(-\frac{1}{2}\frac{|g_{\boldsymbol{k}}|^2}{s_{\boldsymbol{k}}^2}\right) \\
&= \frac{1}{\sqrt{2\pi}s_{\boldsymbol{k}}}\exp\left(-\frac{1}{2}\frac{\alpha_{\boldsymbol{k}}^2}{s_{\boldsymbol{k}}^2}\right) \times \frac{1}{\sqrt{2\pi}s_{\boldsymbol{k}}}\exp\left(-\frac{1}{2}\frac{\beta_{\boldsymbol{k}}^2}{s_{\boldsymbol{k}}^2}\right) \,.
\end{aligned} \tag{8.11}$$

[1] We consider only Gaussian distributions with zero mean.

From this distribution we get (**Exercise:** Show this.) the variance:

$$\langle |g_{\boldsymbol{k}}|^2 \rangle = 2s_{\boldsymbol{k}}^2 \,. \tag{8.12}$$

The distribution has one free parameter for each value of $\boldsymbol{k}$, the real positive number $s_{\boldsymbol{k}}$ that gives the width (determines the variance) of the distribution.

2. The probabilities of different Fourier modes are independent (i.e. they are not correlated),

$$\langle g_{\boldsymbol{k}} g^*_{\boldsymbol{k}'} \rangle = 0 \qquad \text{for} \quad \boldsymbol{k} \neq \boldsymbol{k}' \,. \tag{8.13}$$

In addition, the distribution is taken to be *statistically isotropic* in space. This means that the probability distribution is independent of the direction of the Fourier mode $\boldsymbol{k}$:

$$s_{\boldsymbol{k}} = s(k) \,. \tag{8.14}$$

Like Gaussianity, this is a prediction of typical models of inflation (it follows from the symmetry of the background spacetime), and seems to be agreement with the data. There appear to be some anomalies in the CMB which may point to a small violation of this symmetry, but the issue remains unsettled.

We can combine (8.12) and (8.13) into a single equation,

$$\langle g_{\boldsymbol{k}} g^*_{\boldsymbol{k}'} \rangle = 2\delta_{\boldsymbol{k}\boldsymbol{k}'} s_{\boldsymbol{k}}^2 = \delta_{\boldsymbol{k}\boldsymbol{k}'} \langle |g_{\boldsymbol{k}}|^2 \rangle \tag{8.15}$$

Going from Fourier space back to coordinate space, we find

$$\langle g(\boldsymbol{x}) \rangle = \left\langle \sum_{\boldsymbol{k}} g_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \right\rangle = \sum_{\boldsymbol{k}} \langle g_{\boldsymbol{k}} \rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0 \tag{8.16}$$

The expectation value of the perturbation is zero, since it represents a deviation from the background value. The square of the perturbation can be written as

$$g(\boldsymbol{x})^2 = \sum_{\boldsymbol{k}} g^*_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} \tag{8.17}$$

since $g(\boldsymbol{x})$ is real. The typical amplitude of the perturbation is described by the variance, the expectation value of this square,

$$\langle g(\boldsymbol{x})^2 \rangle = \sum_{\boldsymbol{k}\boldsymbol{k}'} \langle g^*_{\boldsymbol{k}} g_{\boldsymbol{k}'} \rangle e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{x}} = \sum_{\boldsymbol{k}} \langle |g_{\boldsymbol{k}}|^2 \rangle = 2 \sum_{\boldsymbol{k}} s_{\boldsymbol{k}}^2 \,. \tag{8.18}$$

Going from the Fourier series to the Fourier integral, we can write the variance as[2]

$$\begin{aligned} \langle g(\boldsymbol{x})^2 \rangle &= \sum_{\boldsymbol{k}} \langle |g_{\boldsymbol{k}}|^2 \rangle \equiv \left(\frac{2\pi}{L} \right)^3 \sum_{\boldsymbol{k}} \frac{1}{4\pi k^3} \mathcal{P}_g(k) \\ &\to \quad \frac{1}{4\pi} \int \frac{\mathrm{d}^3 k}{k^3} \mathcal{P}_g(k) = \int_0^\infty \frac{\mathrm{d}k}{k} \mathcal{P}_g(k) = \int_{-\infty}^{\infty} \mathcal{P}_g(k) \mathrm{d} \ln k \,, \end{aligned} \tag{8.19}$$

[2] Note that the result has no $\boldsymbol{x}$-dependence. Even though the function $g(\boldsymbol{x})^2$ varies from place to place, its expectation value is the same everywhere.

where we have defined the *power spectrum* $\mathcal{P}_g(k)$ as

$$\mathcal{P}_g(k) \equiv \left(\frac{L}{2\pi}\right)^3 4\pi k^3 \langle |g_{\boldsymbol{k}}|^2 \rangle = \frac{L^3}{2\pi^2} k^3 \langle |g_{\boldsymbol{k}}|^2 \rangle \ . \tag{8.20}$$

The power spectrum of g gives the contribution of a logarithmic scale interval to the variance of $g(\boldsymbol{x})$. For Gaussian perturbations, the power spectrum gives the complete statistical description, and all statistical quantities can be calculated from it.

8.1.4 Solutions

Let us now solve the equation of motion (8.6) for the field perturbations. During inflation, H and m^2 change slowly. Thus, to first order in the slow-roll parameters, they are constant. The general solution of (8.6) is then

$$\delta\varphi_{\boldsymbol{k}}(t) = a^{-3/2} \left[A_{\boldsymbol{k}} J_{-\nu}\left(\frac{k}{aH}\right) + B_{\boldsymbol{k}} J_{\nu}\left(\frac{k}{aH}\right) \right] , \tag{8.21}$$

where $A_{\boldsymbol{k}}$ and $B_{\boldsymbol{k}}$ are constants, and J_ν is the Bessel function of order ν, with

$$\nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \ . \tag{8.22}$$

To leading order, H is constant, and the scale factor is

$$a(t) \propto e^{Ht} \ . \tag{8.23}$$

In the slow-roll approximation, the inflaton mass is negligible, $|m^2| \ll H^2$, since

$$\frac{|m^2|}{H^2} = 3M_{\rm Pl}^2 \frac{|V''|}{V} = 3|\eta| \ll 1 \, . \tag{8.24}$$

We can thus drop m^2/H^2 in (8.22), so

$$\nu = \frac{3}{2} \, . \tag{8.25}$$

Bessel functions of half-integer order are the spherical Bessel functions, which can be expressed in terms of trigonometric functions. The solution (8.21) reduces to

$$\delta\varphi_{\boldsymbol{k}}(t) = A_{\boldsymbol{k}} w_k(t) + B_{\boldsymbol{k}} w_k^*(t) \ , \tag{8.26}$$

where the constants $A_{\boldsymbol{k}}$ and $B_{\boldsymbol{k}}$ have been redefined to absorb some numerical constants, and

$$w_k(t) \equiv \left(i + \frac{k}{aH}\right) \exp\left(\frac{ik}{aH}\right) \ . \tag{8.27}$$

Well before Hubble exit, $k \gg aH$, the exponent is large, and the solution oscillates rapidly. After Hubble exit, $k \ll aH$, the solution stops oscillating and approaches the constant value $i(A_{\boldsymbol{k}} - B_{\boldsymbol{k}})$. As the equation for the field perturbation is linear, we need extra information to fix the constants of integration in (8.26), i.e. the initial conditions. They are given by quantum mechanical vacuum fluctuations.

8.2 The generation of perturbations

It may sound odd to discuss the generation of perturbations. This implies that we consider the state of a system that is homogeneous and isotropic at some initial time, but where the behaviour is nevertheless different at different positions at a later time. This may seem impossible, because then we would have to a have a rule that would say where the perturbations are going to be, which would distinguish one position from another. Therefore it would seem that perturbations have to be given as an initial condition, and cannot be calculated from first principles. In a deterministic theory, this is true. However, quantum theory offers a way out of this impasse. It is is indeterministic, and there is no rule that will tell what the outcome of a quantum process will be, only the probability of various outcomes (i.e. statistical distributions) are calculable. To discuss quantum behaviour of the inflaton field, we need to use quantum field theory in an inflating FLRW universe. To warm up, let us first consider quantum field theory of a scalar field in Minkowski space.

8.2.1 Vacuum fluctuations in Minkowski space

The field equation for a massive free (i.e. $V(\varphi) = \frac{1}{2}m^2\varphi^2$) real scalar field in Minkowski space is

$$\ddot{\varphi} - \nabla^2\varphi + m^2\varphi = 0\,, \tag{8.28}$$

or equivalently

$$\ddot{\varphi}_{\boldsymbol{k}} + E_k^2\varphi_{\boldsymbol{k}} = 0\,, \tag{8.29}$$

where $E_k^2 = k^2 + m^2$. We recognise (8.29) as the equation for a harmonic oscillator. Thus each Fourier component of the field behaves as an independent harmonic oscillator.

In the quantum mechanical treatment of the harmonic oscillator the creation and annihilation operators are introduced, which raise and lower the occupation number of the system. It is also useful to do that here.

We have a different pair of creation and annihilation operators $\hat{a}^\dagger_{\boldsymbol{k}}$, $\hat{a}_{\boldsymbol{k}}$ for every Fourier mode $\boldsymbol{k}$. We denote the ground state of the system by $|0\rangle$, and call it the *vacuum*. Particles are quanta of the oscillations of the field. The vacuum $|0\rangle$ is the state with no particles, more precisely the state annihilated by the annihilation operator:

$$\hat{a}_{\boldsymbol{k}}|0\rangle = 0\ . \tag{8.30}$$

The vacuum is has unit norm, $\langle 0|0\rangle = 1$. Operating on the vacuum with the creation operator $\hat{a}^\dagger_{\boldsymbol{k}}$ adds one quantum with momentum $\boldsymbol{k}$ and energy E_k to the system, i.e. creates one particle. We denote this state with one particle with momentum $\boldsymbol{k}$ by $|1_{\boldsymbol{k}}\rangle$. Thus

$$\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle = |1_{\boldsymbol{k}}\rangle\ , \tag{8.31}$$

and the state is normalised as $\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}'}\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}$. This particle has a well-defined momentum $\boldsymbol{k}$, and therefore it is completely unlocalised, as dictated by the Heisenberg uncertainty principle.

We denote the hermitian conjugate of the vacuum state by $\langle 0|$. Thus

$$\langle 0|\hat{a}_{\boldsymbol{k}} = \langle 1_{\boldsymbol{k}}| \qquad \text{and} \qquad \langle 0|\hat{a}^\dagger_{\boldsymbol{k}} = 0\ . \tag{8.32}$$

The commutation relations of the creation and annihilation operators are

$$[\hat{a}^\dagger_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}] = [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}] = 0 \ , \qquad [\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}] = \delta_{\boldsymbol{k}\boldsymbol{k}'} \ . \tag{8.33}$$

In quantum physics, observables are described by operators, not functions. We can then calculate expectation values for these observables using the operators. So instead of the classical field

$$\varphi(t, \boldsymbol{x}) = \sum \varphi_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{8.34}$$

we have the *field operator*

$$\hat{\varphi}(t, \boldsymbol{x}) = \sum \hat{\varphi}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ . \tag{8.35}$$

The field operator can be written in terms of the creation and annihilation operators as[3]

$$\hat{\varphi}_{\boldsymbol{k}}(t) = w_k(t)\hat{a}_{\boldsymbol{k}} + w^*_k(t)\hat{a}^\dagger_{-\boldsymbol{k}} \tag{8.36}$$

and

$$w_k(t) = L^{-3/2}\frac{1}{\sqrt{2E_k}} e^{-iE_k t} \tag{8.37}$$

is the mode function, a solution of the field equation (8.29). (The normalisation has been fixed to get the right commutation relations, (8.39).) We are using the Heisenberg picture, i.e. we have time-dependent operators and the quantum states are time-independent. Note that since the operator $\hat{\varphi}(t, \boldsymbol{x})$ is Hermitian (corresponding to a real field), $\hat{\varphi}(t, \boldsymbol{x})^\dagger = \hat{\varphi}(t, \boldsymbol{x})$, the corresponding Fourier components satisfy $\hat{\varphi}_{\boldsymbol{k}}(t)^\dagger = \hat{\varphi}_{-\boldsymbol{k}}(t)$. So the Fourier component operators are not Hermitian.

In quantum mechanics of point particles, we have two conjugate variables, position $\hat{x}$ and momentum $\hat{p}$. In quantum field theory, the canonical momentum corresponding to the field is just given by the time derivative of the field. Combining (8.36) and (8.37), we have

$$\dot{\hat{\varphi}}_{\boldsymbol{k}}(t) = -iE_k \left(w_k(t)\hat{a}_{\boldsymbol{k}} - w^*_k(t)\hat{a}^\dagger_{-\boldsymbol{k}} \right) \ . \tag{8.38}$$

We can now calculate the commutator between the field operator and the corresponding conjugate momentum. A straightforward calculation with the rules (8.33) gives

$$[\hat{\varphi}_{\boldsymbol{k}}(t), \dot{\hat{\varphi}}_{\boldsymbol{k}'}(t)] = iL^{-3}\delta_{\boldsymbol{k},-\boldsymbol{k}'} \ . \tag{8.39}$$

This is analogous to the canonical commutation relation $[\hat{x}, \hat{p}] = i$ of quantum mechanics of point particles (**Exercise:** Show that demanding the canonical commutation relation (8.39) fixes the normalisation to be the one given in (8.37).)

The Hamiltonian density of the scalar field in Minkowski space is

$$\hat{\mathcal{H}} \ = \ \frac{1}{2}\dot{\hat{\varphi}}^2 - \frac{1}{2}\sum_i \partial_i\hat{\varphi}\partial_i\hat{\varphi} + V(\hat{\varphi}) \ , \tag{8.40}$$

[3] We skip the detailed derivation of the field operator, which belongs to a course of quantum field theory. See e.g. Peskin & Schroeder, section 2.3 (note the different normalisations of operators and states, related to doing Fourier integrals rather than sums, and considerations of Lorentz invariance).

The Hamiltonian is the spatial integral of the Hamiltonian density,

$$\hat{H} = \int \mathrm{d}^3x \hat{\mathcal{H}} \ . \tag{8.41}$$

Since the Hamiltonian depends on the field velocity operator, it does not commute with the field operator,

$$[\hat{H}, \hat{\varphi}] \neq 0 \ . \tag{8.42}$$

As a result, the Hamiltonian and the field operator do not share a complete set of eigenstates. So, in general an eigenstate of the Hamiltonian is not an eigenstate of the field operator. Eigenstates of the Hamiltonian operator are the energy eigenstates, and the vacuum has the lowest energy. Since the vacuum is not an eigenstate of the field operator, the eigenvalues of the field operator are not well defined, instead we have only a distribution of values. In other words, the scalar field has *vacuum fluctuations*.

The vacuum fluctuations of the field are Gaussian (we skip the proof), and are thus completely completely characterised by their variance, which we can express with the power spectrum as (note that $\langle\hat{\varphi}\rangle = 0$)

$$\langle\hat{\varphi}(\boldsymbol{x})^2\rangle = \int_0^\infty \frac{dk}{k}\mathcal{P}_\varphi(k) \ . \tag{8.43}$$

For the vacuum state $|0\rangle$, the expectation value of $|\varphi_{\boldsymbol{k}}|^2$ is

$$\begin{aligned}
\langle 0|\hat{\varphi}_{\boldsymbol{k}}\hat{\varphi}^\dagger_{\boldsymbol{k}}|0\rangle \ &= \\
&|w_k|^2\langle 0|\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle + w_k^2\langle 0|\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}|0\rangle + (w_k^*)^2\langle 0|\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle + |w_k|^2\langle 0|\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}|0\rangle \\
&= \ |w_k|^2\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}}\rangle \ = \ |w_k|^2
\end{aligned} \tag{8.44}$$

since all but the first term give 0, and the states are normalised so that $\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}'}\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}$. Therefore the power spectrum is, using the definition (8.20),

$$\mathcal{P}_\varphi(k) = L^3\frac{k^3}{2\pi^2}|w_k|^2 \ . \tag{8.45}$$

From (8.37) we have $|w_k|^2 = 1/(2L^3E_k)$, so we get the final result

$$\mathcal{P}_\varphi(k) = \frac{k^3}{4\pi^2 E_k} \ . \tag{8.46}$$

We will next see that in the case of inflation, the mode functions are different because space is expanding, but the reasoning remains the same.

8.2.2 Vacuum fluctuations during inflation

In inflation, the background field is treated classically, and only the perturbations around the mean value of the field are quantised. In fact, if we were to take into account perturbations of the metric in a coordinate-independent manner, we would see that the variables that are quantised are a linear combination of the scalar field perturbations and metric perturbations. Thus in inflation, part of the spacetime metric is quantised. Inflation may thus be called the first quantum gravity scenario whose non-trivial predictions[4] have been successfully confronted with observations. However, just like the background scalar field, the background metric is not quantised. How to quantise the metric in general, and not just small perturbations, remains one of the most studied and most difficult questions in physics. In this course, we just treat the field perturbation during inflation the same way that we treated the field in Minkowski space. That is, the Fourier modes of the field perturbation are written as

$$\delta\hat{\varphi}_{\boldsymbol{k}}(t) = w_k(t)\hat{a}_{\boldsymbol{k}} + w_k^*(t)\hat{a}_{-\boldsymbol{k}}^\dagger \ , \tag{8.47}$$

where the mode function $w_k(t)$ satisfies the classical equation of motion (8.6), with the normalisation fixed by the canonical commutation relation,

$$[\delta\hat{\varphi}_{\boldsymbol{k}}(t), \delta\dot{\hat{\varphi}}_{\boldsymbol{k}'}(t)] = i(aL)^{-3}\delta_{\boldsymbol{k},-\boldsymbol{k}'} \ , \tag{8.48}$$

where the only difference from the Minkowski space commutator (8.39) is the change $L \to aL$ on the right-hand side.

Taking the solution of (8.4) given in section 8.1.4, under the approximations $H =$ constant and $m^2/H^2 = 3\eta = 0$ and fixing the normalisation with (8.48), we get the solution

$$w_k(t) = L^{-3/2}\frac{H}{\sqrt{2k^3}}\left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right) \ , \tag{8.49}$$

where the time-dependence is given by $a(t) \propto e^{Ht}$, where the prefactor in (8.49) has been chosen so that the normalisation agrees with the Minkowski space mode function (8.37) for $k \gg aH$ (up to a slowly varying phase), with the lengths scaled by a. (**Exercise:** Show this.) This choice is called the *Bunch–Davies vacuum*. The motivation is that when we consider distance and time scales much smaller than the Hubble scale, spacetime curvature does not matter and things should behave as in Minkowski space. However, other choices of initial state are possible, and would lead to different predictions for the power spectrum.

The calculation of the power spectrum of inflaton fluctuations is the same as in Minkowski space, with the same result,

$$\mathcal{P}_{\delta\varphi}(k) = L^3\frac{k^3}{2\pi^2}|w_k|^2 \ . \tag{8.50}$$

Well before Hubble exit, $k \gg aH$, and on timescales $\ll H^{-1}$, the field operator $\delta\hat{\varphi}_{\boldsymbol{k}}(t)$ agrees with the Minkowski space field operator and we have the same kind of initial $\delta\varphi$ vacuum fluctuations as in Minkowski space. However, the time evolution of the perturbations is different. Well after Hubble exit, $k \ll aH$, the mode function approaches a constant

$$w_k(t) \ \to \ L^{-3/2}\frac{iH}{\sqrt{2k^3}} \ , \tag{8.51}$$

so the vacuum fluctuations freeze and the power spectrum becomes constant,

$$\mathcal{P}_{\delta\varphi}(k) \ = \ L^3\frac{k^3}{2\pi^2}|w_k|^2 \ = \ \left(\frac{H}{2\pi}\right)^2 \ . \tag{8.52}$$

[4] Trivial predictions are those of the type "if we look here, we will see nothing new".

We have calculated the power spectrum of the inflaton field perturbations by using the quantum mechanical expectation value of the square of the field perturbation. We now identify this with the expectation value of a probability distribution of a classical variable, i.e. we assume that the quantum mechanical fluctuations become classical. Some part of this process is understood (it can be shown that the quantum mechanical expectation values become equal to those of a classical stochastic distribution, or "squeezed"), but the emergence of (at least the appearance of) classical reality from a quantum system remains an unsolved problem. In particle physics appeal is often made to the Copenhagen interpretation according to which states become classical when they are measured, but for cosmology this is inadequate. We simply assume that we can replace an expectation value of a quantum state with the ensemble average of a classical distribution.

For our purposes, quantum mechanics generates the initial perturbations and solves the problem of how perturbations can emerge from a state which is homogeneous and isotropic. As a remnant of the indeterministic origin of the perturbations, we cannot predict the specific member of the ensemble which is realised in the universe, we can only calculate the statistical distribution of perturbations. As noted, this distribution is Gaussian, so all Fourier modes $\delta\varphi_{\boldsymbol{k}}$ are independent random variables (except for the reality condition $\delta\varphi_{-\boldsymbol{k}} = \delta\varphi^*_{\boldsymbol{k}}$) with a Gaussian probability distribution.

8.2.3 The comoving curvature perturbation

Relating the inflationary prediction for the power spectrum of the field perturbation to the power spectrum of the density perturbation in the late universe requires a number of extra steps. We will discuss this further in the next chapter, let us now just outline some main points. Generally, the field perturbation $\delta\varphi_{\boldsymbol{k}}$ is related to the *comoving curvature perturbation* $\mathcal{R}_{\boldsymbol{k}}$, which is a measure of how much the field curves spacetime. The advantage of using $\mathcal{R}_{\boldsymbol{k}}$ is that it is constant on super-Hubble scales, and is more general than the inflaton field perturbation. The perturbation $\mathcal{R}_{\boldsymbol{k}}$ is conserved (on super-Hubble scales) not only during inflation, but during reheating, when the inflaton decays into particles, and afterwards, so it can be used even when the field perturbations are gone (i.e. the field is in its vacuum state after having transferred its energy to the particle bath). In the late universe, we can thus relate $\mathcal{R}_{\boldsymbol{k}}$ to the density perturbation of the gas formed by those particles, which eventually form galaxies and other structures.

The result (8.52) was obtained treating H as a constant. However, H does change, albeit slowly, during inflation. To take into account evolution we use for each scale k the value of H which is representative for the evolution of that particular scale through the Hubble radius. That is, we choose the value of H at Hubble exit[5],

[5] A more precise calculation, where the evolution of $H(t)$ is taken into account gives a correction to the amplitude of $\mathcal{P}_{\mathcal{R}}(k)$ that is first order in slow-roll parameters and a correction to the spectral index n_s that is second order in the slow-roll parameters. Note that H is assumed to be constant only for each k mode during the time it crosses the Hubble radius. The equations of motion of the different modes are independent, so in principle H could be very different for modes that exit at very different times without violating our assumptions.

so that $aH = k$. Thus the power spectrum is

$$\mathcal{P}_{\delta\varphi}(k) \;=\; L^3 \frac{k^3}{2\pi^2}|w_k|^2 \;=\; \left(\frac{H}{2\pi}\right)^2_{aH=k} \,, \tag{8.53}$$

where the subscript notation signifies that the value of H for each k is to be taken at Hubble exit of that particular scale.

Since we have only one quantity that has fluctuations, the inflaton field, and the perturbations are treated in linear theory, the perturbations of any other quantity are related to the inflaton field fluctuation by linear equations. So the distribution of the perturbations inherits the property of homogeneity and isotropy from the symmetry of the background on which they are created and evolve. Perturbations generated by inflation are statistically homogeneous and isotropic, i.e. the power spectrum depends only on the magnitude k of $\boldsymbol{k}$, not on the direction.

In particular, for the comoving curvature perturbation we have (we skip the precise definition and the calculation and just give the result)

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\bar{\varphi}}}\right)^2 \mathcal{P}_{\delta\varphi}(k) = \left(\frac{H}{\dot{\bar{\varphi}}}\frac{H}{2\pi}\right)^2_{aH=k} \,. \tag{8.54}$$

This the main result for quantum fluctuations during inflation. The problem has been completely reduced to the evolution of the background scalar field and the background Hubble parameter. We just need to specify the inflation potential and calculate how the background evolves, and plug it in (8.54) to get complete information about the perturbations. That, in turn, is the starting point for calculating structure formation and the CMB anisotropy. Turning this around, observations of large-scale structure and the CMB can be used obtain information about quantum processes in the primordial universe. Note that the power spectrum depends only on k. Statistical homogeneity and isotropy of the perturbations, inherited from the symmetry of the background, is a strong feature of inflation. ('Feature' may be more a approriate term than 'prediction', because it is possible to construct models where, for example, space expands anisotropically during inflation. However, that requires untypical assumptions, such as having only a short period of inflation, so that the anisotropy is not washed away, or inflation driven by a vector field instead of a scalar field.)

8.3 The primordial spectrum in slow-roll inflation

So, inflation generates primordial perturbations $\mathcal{R}_{\boldsymbol{k}}$ with the power spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\varphi}}\frac{H}{2\pi}\right)^2_{aH=k} \,, \tag{8.55}$$

(In this section, we drop the overbar from the background values.) Let's now get back to the inflaton potential and the presentation of the dynamics of slow-roll inflation in terms of the two slow-roll variables. With the slow-roll equations

$$H^2 = \frac{V}{3M_{\rm Pl}^2} \qquad \text{and} \qquad 3H\dot{\varphi} = -V' \,,$$

equation (8.55) becomes

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{12\pi^2}\frac{1}{M_{\rm Pl}^6}\frac{V^3}{V'^2} = \frac{1}{24\pi^2}\frac{1}{M_{\rm Pl}^4}\frac{V}{\varepsilon} \,, \tag{8.56}$$

where ε is the slow-roll parameter.

According to observations of CMB and large-scale structure, the amplitude of the primordial power spectrum is [2]

$$\mathcal{P}_{\mathcal{R}}(k) = 2.1 \times 10^{-9} \tag{8.57}$$

i.e. $\mathcal{P}_{\mathcal{R}}(k)^{1/2} = 4.6 \times 10^{-5}$ on the scale $k = 0.05$ Mpc^{-1}; the scale-dependence is weak (we discuss it below). This gives a constraint on inflation,

$$\left(\frac{V}{\varepsilon}\right)^{1/4} \approx 24^{1/4}\sqrt{\pi}\sqrt{4.6\times 10^{-5}}M_{\rm Pl} \approx 0.027 M_{\rm Pl} = 6.4\times 10^{16}\ \text{GeV} \,, \tag{8.58}$$

so we get an upper limit on the energy scale of inflation,

$$V^{1/4} = 0.027\varepsilon^{1/4}M_{\rm Pl} \,, \tag{8.59}$$

which, as $\varepsilon < 1$, gives $V^{1/4} < 0.027M_{\rm Pl}$. As we will soon discuss, we have observational limit $\varepsilon < 2.3 \times 10^{-3}$, so $V^{1/4} < 0.027M_{\rm Pl}$, so the upper bound tightens into $V^{1/4} < 5.9 \times 10^{-3}M_{\rm Pl} = 1.4 \times 10^{16}$ GeV. This puts a limit on the Hubble scale during inflation. From $H^2 = V/(3M_{\rm Pl}^2)$, the constraint on V translates into $H < 4.8 \times 10^{13}$ GeV, or in terms of length, $H^{-1} > 4.1 \times 10^{-30}$ m.

Exercise: From the limit on the energy scale of inflation, find the maximum amount by which the scale factor can have expanded from reheating until today, assuming there are only Standard Model degrees of freedom.

Since during slow-roll inflation V and V' change slowly while a wide range of scales k exit the Hubble radius, we expect $\mathcal{P}_{\mathcal{R}}(k)$ to be a slowly varying function of k. We describe this small variation with the *spectral index* n_s of the primordial spectrum, defined as[6]

$$n_s(k) - 1 \equiv \frac{\mathrm{d}\ln\mathcal{P}_{\mathcal{R}}}{\mathrm{d}\ln k} \,. \tag{8.60}$$

If the spectral index is independent of k, we say that the spectrum is *scale-free*. In this case the primordial spectrum is a *power-law*

$$\mathcal{P}_{\mathcal{R}}(k) = A^2\left(\frac{k}{k_*}\right)^{n_s-1} \,, \tag{8.61}$$

where the pivot scale k_* is some chosen reference scale (for most of the data analysis of of the Planck satellite team, $k_* = 0.05$ Mpc^{-1}), and and A is the amplitude at the pivot scale.

If the power spectrum is constant, corresponding to $n_s = 1$, we say that the spectrum is *scale-invariant*, a special case of a scale-free spectrum. The scale-invariant spectrum is also called the *Harrison–Zel'dovich* spectrum.

[6] The -1 is in the definition for historical reasons, related to other ways of defining the power spectrum of perturbations.

If $n_s \neq 1$, the spectrum is called *tilted*. A tilted spectrum is called *red* if $n_s < 1$ (more power on large scales) and *blue* if $n_s > 1$ (more power on small scales). If $\mathrm{d}n_s/\mathrm{d}k \neq 0$, it is said that there is a *running spectral index.*

Using (8.56) and (8.60), we can calculate the spectral index for slow-roll inflation. Since $\mathcal{P}_\mathcal{R}(k)$ is evaluated from (8.56) when $k = aH$, we have

$$\frac{\mathrm{d}\ln k}{\mathrm{d}t} = \frac{\mathrm{d}\ln(aH)}{\mathrm{d}t} = \frac{\dot{a}}{a} + \frac{\dot{H}}{H} = (1-\varepsilon)H \ ,$$

where we used the fact that in the slow-roll approximation $\dot{H} = -\varepsilon H^2$ in the last step. Thus

$$\frac{\mathrm{d}}{\mathrm{d}\ln k} = \frac{1}{1-\varepsilon}\frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{1-\varepsilon}\frac{\dot{\varphi}}{H}\frac{\mathrm{d}}{\mathrm{d}\varphi} = -\frac{M_{\mathrm{Pl}}^2}{1-\varepsilon}\frac{V'}{V}\frac{\mathrm{d}}{\mathrm{d}\varphi} \approx -M_{\mathrm{Pl}}^2\frac{V'}{V}\frac{\mathrm{d}}{\mathrm{d}\varphi} \,. \tag{8.62}$$

Let us first calculate the scale dependence of the slow-roll parameters:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\ln k} = -M_{\mathrm{Pl}}^2\frac{V'}{V}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[\frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{V'}{V}\right)^2\right] = M_{\mathrm{Pl}}^4\left[\left(\frac{V'}{V}\right)^4 - \left(\frac{V'}{V}\right)^2\frac{V''}{V}\right] = 4\varepsilon^2 - 2\varepsilon\eta \tag{8.63}$$

and, in a similar manner we get (**Exercise:** show this.),

$$\frac{\mathrm{d}\eta}{\mathrm{d}\ln k} = 2\varepsilon\eta - \xi \ , \tag{8.64}$$

where we have defined a third slow-roll parameter

$$\xi \equiv M_{\mathrm{Pl}}^4\frac{V'}{V^2}V''' \,. \tag{8.65}$$

The parameter ξ is typically second-order small in the sense that $\sqrt{|\xi|}$ is of the same order of magnitude as ε and η.

We can now calculate the spectral index:

$$\begin{aligned} n_s - 1 = \frac{1}{\mathcal{P}_\mathcal{R}}\frac{\mathrm{d}\mathcal{P}_\mathcal{R}}{\mathrm{d}\ln k} &= \frac{\varepsilon}{V}\frac{\mathrm{d}}{\mathrm{d}\ln k}\left(\frac{V}{\varepsilon}\right) = \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\ln k} - \frac{1}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\ln k} \\ &= \ -M_{\mathrm{Pl}}^2\frac{V'}{V}\cdot\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\varphi} - 4\varepsilon + 2\eta \ = \ -6\varepsilon + 2\eta \ . \end{aligned} \tag{8.66}$$

Slow-roll requires $\varepsilon \ll 1$ and $|\eta| \ll 1$, so the spectrum is predicted to be close to scale invariant. This agrees well with observations. Note how, as in the case of dark matter, things fall into place automatically. In order to have negative pressure, a scalar field has to roll slowly. In slow-roll the background changes slowly, so the perturbations are close to scale-invariant, without needing to add new ingredients or tune anything.

Assuming that at late times the universe is described by the ΛCDM model, the current constraint on the spectral index from CMB data by the Planck satellite is, assuming a power-law spectrum [2],

$$n_s = 0.9649 \pm 0.0042 \ . \tag{8.67}$$

The precise value of the mean and the error bars depend on the data included in the analysis. The value is also model-dependent, and with a different cosmological model (different dark energy model, the presence of cosmic strings, and so on), the preferred value of the spectral index can change slightly. However, in all but the most exotic models it remains close to scale-invariant, and in most models less than unity.

From the results of the running of ε and η, we get the running of the spectral index (**Exercise:** show this.):

$$\alpha_s \equiv \frac{\mathrm{d}n_s}{\mathrm{d}\ln k} = 16\varepsilon\eta - 24\varepsilon^2 - 2\xi \ . \tag{8.68}$$

The running is second order in slow-roll parameters, so it's expected to be even smaller than the deviation from scale invariance. The observational range is

$$\alpha_s = -0.0045 \pm 0.0067 \ . \tag{8.69}$$

So there is no observational evidence for the running, nor for any other deviation from scale invariances. (The value of the spectral index also changes slightly when running is included; the change is within the error bars of (8.67).)

In addition to scalar perturbations in the inflaton field, inherited by other matter in reheating, inflation also produces gravitational waves. These are not waves produced by the motion of matter, they are born from vacuum fluctuations like the scalar perturbations. We will skip the details of the treatment. It can be shown that the power spectrum of gravitational waves (we skip the definition) produced by inflation is

$$\mathcal{P}_t(k) = \frac{8}{M_{\mathrm{Pl}}^2}\left(\frac{H}{2\pi}\right)^2_{aH=k} \ .$$

The tensor power spectrum is usually given in terms of the *tensor-to-scalar ratio*, which is

$$r \equiv \frac{\mathcal{P}_t(k)}{\mathcal{P}_{\mathcal{R}}(k)} = 16\varepsilon \ . \tag{8.70}$$

and the tensor spectral index

$$n_t \equiv \frac{\mathrm{d}\ln \mathcal{P}_t}{\mathrm{d}\ln k} = -2\varepsilon \ , \tag{8.71}$$

where we have written them to in terms of the slow-roll parameters to first order. (**Exercise:** Derive the expressions in terms of ϵ.) Note that combining (8.70) and (8.71) leads to the consistency condition

$$r = -8n_t \ . \tag{8.72}$$

This condition is important, because it does not depend on the values of the slow-roll parameters: it is a model-independent prediction shared by all models of slow-roll single-field inflation[7]. Gravitational waves from inflation have not been detected. From observations of the Planck satellite and the Keck/BICEP telescope, we have the upper bound [3]

$$r < 0.036 \ . \tag{8.73}$$

[7] With a minimal coupling to gravity: if the field equation of motion is more complicated than (8.3), the predictions may change.

Hundreds of inflationary models have been proposed, and while many have been ruled out because their observations do not agree with the above limit, many viable models remain [4]. CMB experiments have measured the CMB temperature anisotropy over about three orders of magnitude in k, from the largest possible scale down to Mpc scales, so the perturbations from about $\ln(10^3) = 7$ e-folds of inflation have been measured. Recall that for high energy-scale inflation, the number of e-folds until the end of inflation when the largest observable modes are generated is about 50 to 60. On scales smaller than those that have been probed, the CMB anisotropy is expected to be negligible, so we expect there is nothing interesting to see in the CMB temperature anisotropies. (We'll discuss this when we get to the CMB in chapter 10.) It is possible to probe smaller scales by observations large-scale structure and from deviations of the CMB spectrum (not the anisotropies) from the blackbody shape.

Example: Consider the simple inflation model

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 \; . \tag{8.74}$$

In chapter 7 we already calculated the slow-roll parameters for this model,

$$\varepsilon = \eta = 2\left(\frac{M_{\rm Pl}}{\varphi}\right)^2 \; , \tag{8.75}$$

and we immediately see that $\xi = 0$, because $V''' = 0$. We thus have

$$\begin{aligned}
\mathcal{P}_\mathcal{R} &= \frac{1}{96\pi^2}\frac{m^2}{M_{\rm Pl}^2}\left(\frac{\varphi}{M_{\rm Pl}}\right)^4 && (8.76)\\
n_s &= 1 - 6\varepsilon + 2\eta = 1 - 8\left(\frac{M_{\rm Pl}}{\varphi}\right)^2 && (8.77)\\
\alpha_s &= 16\varepsilon\eta - 24\varepsilon^2 - 2\xi = -32\left(\frac{M_{\rm Pl}}{\varphi}\right)^4 && (8.78)\\
r &= 16\varepsilon = 32\left(\frac{M_{\rm Pl}}{\varphi}\right)^2 \; . && (8.79)
\end{aligned}$$

To get the numbers, we need the values of φ when the relevant cosmological scales left the Hubble radius. We know that the number of inflation e-foldings after that should be about $N = 60$, depending on the preheating history. We have

$$N(\varphi) = \frac{1}{M_{\rm Pl}^2}\int_{\varphi_{\rm end}}^{\varphi}\frac{V}{V'}d\varphi = \frac{1}{M_{\rm Pl}^2}\int\frac{\varphi}{2}d\varphi = \frac{1}{4M_{\rm Pl}^2}\left(\varphi^2 - {\varphi_{\rm end}}^2\right) \; , \tag{8.80}$$

and we estimate $\varphi_{\rm end}$ from $\varepsilon(\varphi_{\rm end}) = 2M_{\rm Pl}^2/{\varphi_{\rm end}}^2 = 1 \quad \Rightarrow \quad \varphi_{\rm end} = \sqrt{2}M_{\rm Pl}$ to get

$$\varphi^2 = {\varphi_{\rm end}}^2 + 4M_{\rm Pl}^2 N = 2M_{\rm Pl}^2 + 4M_{\rm Pl}^2 N \approx 4M_{\rm Pl}^2 N \, . \tag{8.81}$$

Thus

$$\left(\frac{M_{\rm Pl}}{\varphi}\right)^2 = \frac{1}{4N} \; , \tag{8.82}$$

so we get

$$
\begin{aligned}
\mathcal{P}_{\mathcal{R}} &= \frac{N^2}{6\pi^2}\frac{m^2}{M_{\rm Pl}^2} = \frac{600}{\pi^2}\frac{m^2}{M_{\rm Pl}^2} && (8.83)\\
n_s &= 1-\frac{2}{N} = 0.97 \\
\alpha_s &= -\frac{2}{N^2} = -0.0006 && (8.84)\\
r &= \frac{8}{N} = 0.13\ , && (8.85)
\end{aligned}
$$

where we have input $N = 60$. For $\mathcal{P}_{\mathcal{R}}$ we have, according to (8.57) $\mathcal{P}_{\mathcal{R}} = 2.1\times10^{-9}$, which gives

$$m \approx \frac{8}{N}10^{14}\ \text{GeV} \approx 1\times10^{13}\ \text{GeV} \approx 6\times10^{-6}M_{\rm Pl}\ , \tag{8.86}$$

for $N = 60$. We get $V^{1/4} = (2Nm^2M_{\rm Pl}^2)^{1/4} \approx 2\times10^{16}$ GeV as the energy scale for the period when the perturbations seen in the CMB were generated. Potential energy at the end of inflation is

$$V_{\rm end}^{1/4} = \left(\frac{1}{2}m^2{\varphi_{\rm end}}^2\right)^{1/4} = \sqrt{\frac{m}{M_{\rm Pl}}}M_{\rm Pl} \approx 2\times10^{-3}M_{\rm Pl} \approx 6\times10^{15}\ \text{GeV}\ . \tag{8.87}$$

Because of the high energy scale, r is well in excess of the observational upper bound. Therefore, the simple $m^2\varphi^2$ model is now ruled out, although it fitted the observations well until the Planck data.

References

[1] A. R. Liddle and D. H. Lyth, *Cosmological inflation and large scale structure.* 2000.

[2] Planck collaboration, N. Aghanim et al., *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6, [1807.06209].

[3] BICEP, Keck collaboration, P. A. R. Ade et al., *Improved Constraints on Primordial Gravitational Waves using Planck, WMAP, and BICEP/Keck Observations through the 2018 Observing Season*, *Phys. Rev. Lett.* **127** (2021) 151301, [2110.00483].

[4] J. Martin, C. Ringeval and V. Vennin, *Encyclopædia Inflationaris*, *Phys. Dark Univ.* **5-6** (2014) 75–235, [1303.3787].