Contents

4 Thermodynamics in the expanding universe

4.1 Phase space density

As we look out to space we see the history of the universe unfolding in front of our telescopes. However, at redshift $z = 1090$ our line of sight hits the *last scattering surface*, from which the cosmic microwave background (CMB) radiation originates. This corresponds to $t = 370\,000 \ldots 380\,000$ years. Before that the universe was opaque, so we cannot see further back in time using electromagnetic radiation. (However, it is possible to see to earlier times using gravitational waves.) However, the isotropy of the CMB indicates that matter was distributed almost homogeneously and isotropically in the early universe, and the spectrum of the CMB shows that this matter, the primordial soup of particles, was in thermal equilibrium. Therefore we can use thermodynamics to calculate the history of the early universe and obtain testable predictions. We will now derive the thermodynamics of the primordial soup starting from statistical physics. We only deal with the statistical physics of a gas of particles: thermodynamics of the gravitational degrees of freedom is poorly understood, and will not be relevant for our discussion. Also, the interactions responsible for thermal equilibrium are those of non-gravitational physics. The only role of gravity here is to determine the expansion of space.

From elementary quantum mechanics we are familiar with the model of a particle in a box. Consider a cubic box, with edge length L and volume $V = L^3$, with periodic boundary conditions. Solving the Schrödinger equation gives us the energy and momentum eigenstates, with possible momentum values

$$\vec{p} = \frac{h}{L}(n_1\hat{x} + n_2\hat{y} + n_3\hat{z}) \qquad (n_i = 0, \pm1, \pm2, \ldots). \tag{4.1}$$

The state density in momentum space (number of states / $\Delta p_x \Delta p_y \Delta p_z$) is thus

$$\frac{L^3}{h^3} = \frac{V}{h^3}, \tag{4.2}$$

and the state density in phase space $\{\vec{x}, \vec{p}\}$ is $1/h^3$. If the particle has g internal degrees of freedom (such as spin), we have

$$\text{density of states} = \frac{g}{h^3} = \frac{g}{(2\pi)^3} \qquad \left(\hbar \equiv \frac{h}{2\pi} = 1\right). \tag{4.3}$$

This result is true even for relativistic momenta. The state density in phase space is independent of the volume V, so we can apply it to arbitrarily large systems (including an infinite universe).

For much of the early universe, we can ignore interaction energies between particles. Then the particle energy is

$$E(\vec{p}) = \sqrt{p^2 + m^2}\ , \tag{4.4}$$

where $p \equiv |\vec{p}|$ is the magnitude of the three-momentum (not to be confused with pressure, also denoted by p), and the states available for the particles are the free particle states discussed above.

Particles fall into two classes, *fermions* and *bosons*. Fermions obey the Pauli exclusion principle: no two fermions can be in the same state, i.e. they cannot have the same quantum numbers.

In thermodynamic equilibrium the *distribution function*, or the expectation value f of the occupation number of a state, depends only on the energy of the state. According to statistical physics, it is

$$f(\vec{p}) = \frac{1}{e^{(E-\mu)/T} \pm 1} \tag{4.5}$$

where $+$ is for fermions and $-$ is for bosons. (In the case of fermions, for which $f \leq 1$, f gives the probability that a state is occupied.) This equilibrium distribution has two parameters, the *temperature* T and the *chemical potential* μ. The temperature is related to the energy density ρ of the system and the chemical potential is related to the number density n of particles in the system. Note that since we use the relativistic formula for the particle energy E, which includes the mass m, the mass is also included' in the chemical potential μ. So in the nonrelativistic limit both E and μ differ from the corresponding quantities used in nonrelativistic statistical physics by m in such a way that $E - \mu$ and the distribution functions remain the same.

If there is no conserved particle number in the system (this is true for e.g. a photon gas), then $\mu = 0$ in equilibrium.

The particle density in phase space is the density of states times their occupation number,

$$\frac{g}{(2\pi)^3} f(\vec{p}). \tag{4.6}$$

We get the particle density in (ordinary) space by integrating over the momentum space. We thus have the following quantities:

$$\text{number density} \qquad n_i = \frac{g_i}{(2\pi)^3} \int f_i(\vec{p}) d^3p \tag{4.7}$$

$$\text{energy density} \qquad \rho_i = \frac{g_i}{(2\pi)^3} \int E_i(\vec{p}) f_i(\vec{p}) d^3p \tag{4.8}$$

$$\text{pressure} \qquad p_i = \frac{g_i}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E_i} f_i(\vec{p}) d^3p\ . \tag{4.9}$$

The index i here labels different particle species, which have different masses m_i and corresponding energies $E_i(\vec{p}) = \sqrt{p^2 + m_i^2}$. The above applies separately to each particle species.

4.2 Equilibrium distributions

We say that the species i is in *kinetic equilibrium* if it has the above distribution for some μ_i and T_i. We say that the system is in *thermal equilibrium* if species have the same temperature, $T_i = T$. We say that the system is in *chemical equilibrium* ("chemistry" here refers to reactions where particles change into other species) if the chemical potentials of different particle species are related according to reaction formulae. For example, if we have a reaction

$$i + j \quad \leftrightarrow \quad k + l \ , \tag{4.10}$$

then

$$\mu_i + \mu_j = \mu_k + \mu_l \ . \tag{4.11}$$

In particular, if the chemical potential of particle species i is μ_i, then the chemical potential of the corresponding antiparticle is $-\mu_i$. Via the reaction formulae all chemical potentials can be expressed in terms of the chemical potentials of conserved quantities, e.g. the baryon number chemical potential μ_B. So there are as many independent chemical potentials as there are independent conserved particle numbers.

As the universe expands, T and μ change in such a way that the energy continuity equation is satisfied and conserved quantum numbers remain constant. An expanding universe is not in equilibrium. However, in the early universe the particle interactions are so rapid compared to the expansion rate that the particle soup has time to settle close to local equilibrium. (And since the universe is homogeneous, the local values of thermodynamic quantities are also global values). From the numbers of fermions (electrons and nucleons) remaining in the present universe, we can conclude that in the early universe we had $T \gg |\mu|$ for them when $T \gg m$. We don't know the chemical potentials of the three neutrino species, but they are usually assumed to be small, too. If the temperature is much greater than the mass, $T \gg m$, in the *ultrarelativistic limit*, we can approximate $E = \sqrt{p^2 + m^2} \approx p$.

For $T \gg |\mu|$ and $T \gg m$, we approximate $\mu = 0$ and $m = 0$ to get the following formulae

$$n = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^2 dp}{e^{p/T} \pm 1} = \begin{cases} \dfrac{3}{4\pi^2}\zeta(3) g T^3 & \text{fermions} \\ \dfrac{1}{\pi^2}\zeta(3) g T^3 & \text{bosons} \end{cases} \tag{4.12}$$

$$\rho = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^3 dp}{e^{p/T} \pm 1} = \begin{cases} \dfrac{7}{8}\dfrac{\pi^2}{30} g T^4 & \text{fermions} \\ \dfrac{\pi^2}{30} g T^4 & \text{bosons} \end{cases} \tag{4.13}$$

$$p = \frac{g}{(2\pi)^3} \int_0^\infty \frac{\frac{4}{3}\pi p^3 dp}{e^{p/T} \pm 1} = \frac{1}{3}\rho \approx \begin{cases} 1.0505 nT & \text{fermions} \\ 0.9004 nT & \text{bosons} . \end{cases} \tag{4.14}$$

For the average particle energy we get

$$\langle E\rangle = \frac{\rho}{n} = \begin{cases} \dfrac{7\pi^4}{180\zeta(3)}T \approx 3.151T & \text{fermions} \\ \dfrac{\pi^4}{30\zeta(3)}T \approx 2.701T & \text{bosons}\,. \end{cases} \tag{4.15}$$

In the above, ζ is the Riemann zeta function, with $\zeta(3) \equiv \sum_{n=1}^{\infty} n^{-3} = 1.20206$.

If the chemical potential vanishes, $\mu = 0$, there are equal numbers of particles and antiparticles. If $\mu \neq 0$, we find for fermions in the ultrarelativistic limit $T \gg m$ (but keeping μ) the net particle number

$$\begin{aligned} n - \bar{n} &= \frac{g}{(2\pi)^3}\int_0^\infty dp\, 4\pi p^2 \left(\frac{1}{e^{(p-\mu)/T}+1} - \frac{1}{e^{(p+\mu)/T}+1}\right) \\ &= \frac{gT^3}{6\pi^2}\left[\pi^2\frac{\mu}{T} + \left(\frac{\mu}{T}\right)^3\right] \end{aligned} \tag{4.16}$$

and the total energy density[1]

$$\begin{aligned} \rho + \bar{\rho} &= \frac{g}{(2\pi)^3}\int_0^\infty dp\, 4\pi p^3 \left(\frac{1}{e^{(p-\mu)/T}+1} + \frac{1}{e^{(p+\mu)/T}+1}\right) \\ &= \frac{7}{8}g\frac{\pi^2}{15}T^4\left[1 + \frac{30}{7\pi^2}\left(\frac{\mu}{T}\right)^2 + \frac{15}{7\pi^4}\left(\frac{\mu}{T}\right)^4\right]\,. \end{aligned} \tag{4.17}$$

Note that the equations (4.16) and (4.17) are exact, not truncated series. (The difference $n - \bar{n}$ and the sum $\rho + \bar{\rho}$ lead to a nice cancellation between the two integrals. We don't get such an elementary form for the individual quantities n, $\bar{n}$, ρ, $\bar{\rho}$, nor for the sum $n + \bar{n}$ and the difference $\rho - \bar{\rho}$ when $\mu \neq 0$.)

In the nonrelativistic limit, $T \ll m$ and $T \ll m - \mu$, the typical kinetic energies are much below the mass m, so we can approximate $E = m + p^2/2m$. The second condition, $T \ll m - \mu$, leads to occupation numbers $\ll 1$, a *dilute* system. This second condition is usually satisfied in cosmology when the first one is. It is violated in systems of high density, such as white dwarf stars and neutron stars. We can then approximate

$$e^{(E-\mu)/T} \pm 1 \approx e^{(E-\mu)/T}\,, \tag{4.18}$$

so the boson and fermion expressions become equal[2], and we get (**Exercise**: Show this.)

$$\begin{aligned} n &= g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m-\mu}{T}} && (4.19) \\ \rho &= n\left(m + \frac{3T}{2}\right) && (4.20) \\ p &= nT \ll \rho && (4.21) \\ \langle E\rangle &= m + \frac{3T}{2} && (4.22) \\ n - \bar{n} &= 2g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m}{T}} \sinh\frac{\mu}{T}\,. && (4.23) \end{aligned}$$

[1] When the chemical potential is small, the contribution of the antiparticles to the energy density is often included in the definition of g, unlike here.

[2] This approximation leads to what is called Maxwell–Boltzmann statistics, whereas the previous exact formulae give Fermi–Dirac (for fermions) and Bose–Einstein (for bosons) statistics.

In the general case, where neither $T \ll m$ nor $T \gg m$, the integrals don't give elementary functions, so $n(T)$, $\rho(T)$, and so on need to be calculated numerically in the region $T \sim m$.[3]

Comparison of the ultrarelativistic and nonrelativistic limits above shows that the number density, energy density, and pressure of a particle species fall exponentially as the temperature falls below the mass of the particle. We have not so far made assumptions about the interactions that are responsible for maintaining equilibrium. In the cosmological case, these include annihilation and particle-antiparticle pair formation. At high temperatures, these reactions balance each other, but as the temperature falls below the mass, the thermal particle energies are not sufficient for pair production any more, so the reactions happen only in the annihilation direction. The process of particle-antiparticle annihilation takes place mainly (about 80%) during the temperature interval $T = m \to \frac{1}{6}m$, as shown in figure 1. It is not an instantaneous, but takes several Hubble times.

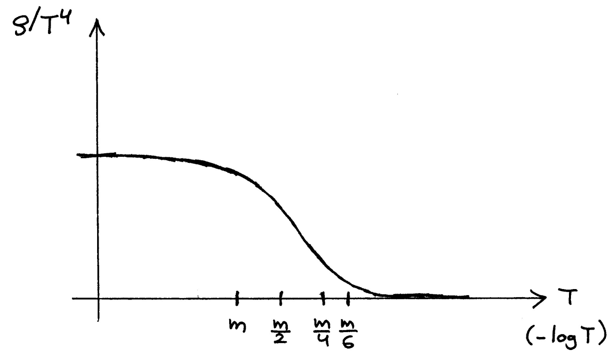

Figure 1: The fall of energy density of a particle species, with mass m, as a function of temperature (decreasing to the right).

4.3 Effective number of degrees of freedom

According to the Friedmann equation the expansion of the universe is governed by the total energy density

$$\rho(T) = \sum_i \rho_i(T) ,$$

where i runs over particle species. Because the energy density of relativistic species is much greater than that of nonrelativistic species (in thermal equilibrium and when we can neglect the chemical potential), it suffices to include only the relativistic species. We thus have

$$\rho(T) = \frac{\pi^2}{30} g_*(T) T^4 \tag{4.24}$$

where

$$g_*(T) = g_b(T) + \frac{7}{8} g_f(T) ,$$

and $g_b = \sum_i g_i$ sums over relativistic bosons and $g_f = \sum_i g_i$ sum over relativistic fermions. For pressure we have $p(T) \approx \frac{1}{3}\rho(T)$.

[3] If we use Maxwell–Boltzmann statistics, i.e. , drop the term ± 1 in the distribution function, the integrals give modified Bessel functions, and the error is often less than 10%.

The above is a simplification of the true situation: Since the annihilation takes a long time, there are long periods when the annihilation of some particle species is going on, and its contribution disappears gradually. Using the exact formula for ρ we define *the effective number of degrees of freedom* $g_*(T)$ as

$$g_*(T) \equiv \frac{30}{\pi^2}\frac{\rho}{T^4} . \tag{4.25}$$

We also define

$$g_{*p}(T) \equiv \frac{90}{\pi^2}\frac{p}{T^4} \approx g_*(T) . \tag{4.26}$$

When no annihilations are taking place, we have $g_{*p} = g_* =$ constant $\Rightarrow p = \frac{1}{3}\rho$. From the Friedmann equation it then follows that $\rho \propto a^{-4}$, so we have $\rho \propto T^4$ and $T \propto a^{-1}$. We will soon calculate the scale factor-temperature relation more precisely, including the effects of annihilation.

4.4 Redshift of momenta

The momentum of freely moving particles redshifts with the expansion of the universe as

$$p(t_2) = \frac{a(t_1)}{a(t_2)}p(t_1) . \tag{4.27}$$

Let us now show that it follows that ultrarelativistic non-interacting particles stay in kinetic equilibrium.

At time t_1 a phase space element $d^3p_1dV_1$ contains

$$dN = \frac{g}{(2\pi)^3}f(\vec{p}_1)d^3p_1dV_1 \tag{4.28}$$

particles, where

$$f(\vec{p}_1) = \frac{1}{e^{(p_1-\mu_1)/T_1} \pm 1}$$

is the distribution function at time t_1. At time t_2 these same dN particles are in a phase space element $d^3p_2dV_2$. How is the distribution function at t_2, given by

$$\frac{g}{(2\pi)^3}f(\vec{p}_2) = \frac{dN}{d^3p_2dV_2} ,$$

related to $f(\vec{p}_1)$? Since $d^3p_2 = (a_1/a_2)^3d^3p_1$ and $dV_2 = (a_2/a_1)^3dV_1$, we have

$$\begin{aligned} dN &= \frac{g}{(2\pi)^3}\frac{d^3p_1\,dV_1}{e^{(p_1-\mu_1)/T_1} \pm 1} && (dN \text{ evaluated at } t_1) \\ &= \frac{g}{(2\pi)^3}\frac{(\frac{a_2}{a_1})^3d^3p_2\,(\frac{a_1}{a_2})^3dV_2}{e^{(\frac{a_2}{a_1}p_2 - \mu_1)/T_1} \pm 1} && (\text{rewritten in terms of } p_2,\ dp_2, \text{ and } dV_2) \\ &= \frac{g}{(2\pi)^3}\frac{d^3p_2\,dV_2}{e^{(p_2-\mu_2)/T_2} \pm 1} && (\text{defining } \mu_2 \text{ and } T_2) , \end{aligned} \tag{4.29}$$

where $\mu_2 \equiv (a_1/a_2)\mu_1$ and $T_2 \equiv (a_1/a_2)T_1$. Thus distribution retains the thermal shape; the temperature and the chemical potential just redshift $\propto a^{-1}$.

Exercise. Show that for a non-relativistic particle species, the distribution function retains the thermal shape as the universe expands, with $T_2 = T_1(a(t_1)/a(t_2))^2 \propto a(t_2)^{-2}$ and $\mu(t_2) = m + (\mu(t_1) - m)T_2/T_1$.

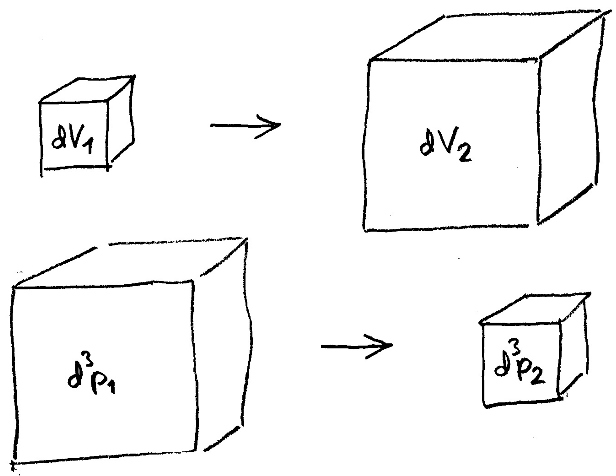

Figure 2: The expansion of the universe increases the volume element $\mathrm{d}V$ and decreases the momentum space element d^3p in such a way that the phase space element $\mathrm{d}^3p\mathrm{d}V$ stays constant.

4.5 Scale factor-temperature relation

The relation between the temperature T and the scale factor a follows from the conservation of entropy. According to the second law of thermodynamics the total entropy of the universe never decreases: it either stays constant or grows. Entropy production in various processes in the universe is insignificant compared to the total entropy of the universe[4], which is huge, and at all times dominated by the relativistic species. Thus it is an excellent approximation to treat the expansion of the universe as *adiabatic*, i.e. take the entropy to be constant. We find the entropy in terms of the other thermodynamical quantities by using the *fundamental equation of thermodynamics*

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \sum_i \mu_i \mathrm{d}N_i \ . \tag{4.30}$$

Dividing by $\mathrm{d}V$, we find the entropy density $s \equiv \mathrm{d}S/\mathrm{d}V$,

$$s = \frac{\rho + p - \sum_i \mu_i n_i}{T} \ . \tag{4.31}$$

We get the value of this quantity by summing up the contributions to $\rho + p - \sum_i \mu_i n_i$ from all particle species, using the exact expressions given earlier. If $T \gg |\mu_i|$, we have for a single relativistic species

$$s = \frac{\rho + p}{T} = \begin{cases} \frac{7\pi^2}{180} g T^3 & \text{fermions} \\ \frac{2\pi^2}{45} g T^3 & \text{bosons} \ . \end{cases} \tag{4.32}$$

[4]There may be exceptions to this in the very early universe, most notably the end of inflation, where essentially all of the entropy of the universe may have been produced. Recall that we discuss only the entropy of matter: the entropy of gravitational degrees of freedom remains poorly understood. Black holes are thought to have extremely large entropy.

We define the number of effective number of entropy degrees of freedom $g_{*s}(t)$ with the equation

$$s(T) \equiv \frac{2\pi^2}{45} g_{*s}(T) T^3 . \tag{4.33}$$

Adding up all relativistic species and allowing for the possibility that some of them may have a kinetic temperature T_i different from the temperature T of those species that remain in thermal equilibrium, we get

$$\begin{aligned} g_*(T) &= \sum_{\text{bos}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fer}} g_i \left(\frac{T_i}{T}\right)^4 \\ g_{*s}(T) &= \sum_{\text{bos}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fer}} g_i \left(\frac{T_i}{T}\right)^3 , \end{aligned} \tag{4.34}$$

and the sums are over all relativistic species of bosons and fermions. If some species are "semirelativistic", i.e. $m = \mathcal{O}(T)$, then $\rho(T)$ and $s(T)$ have to be calculated from the integral formulae of section 4.2. Non-relativistic species give negligible contribution to the entropy. As long as all species have the same temperature and $p \approx \frac{1}{3}\rho$, we have

$$g_{*s}(T) \approx g_*(T) . \tag{4.35}$$

We will see that this approximation breaks down in the real universe at around 1 s.

Now we have all we need to find the relation between the scale factor and temperature, taking into account annihilations. The entropy stays constant, i.e.

$$sa^3 = \text{constant} . \tag{4.36}$$

Using (4.33), we immediately get the relation between a and T:

$$g_{*s}(T) T^3 a(t)^3 = \text{constant} . \tag{4.37}$$

We will have much use for this formula.

4.6 Relation of time and temperature

So we know the relation between the scale factor and the temperature. The next step is to find the relation between the temperature T and time t in the early universe, and consider the evolution in terms of both. Spatial curvature can be neglected in the early universe, so the Friedmann equation is

$$3H^2 = 8\pi G_{\text{N}} \rho(T) = \frac{\pi^2}{30} g_*(T) \frac{T^4}{M_{\text{Pl}}^2} , \tag{4.38}$$

where we have written Newton's constant in terms of the Planck mass, $M_{\text{Pl}} \equiv 1/\sqrt{8\pi G_{\text{N}}} \approx 2.436 \times 10^{21}$ MeV. To integrate this equation exactly we would need to calculate numerically the function $g_*(T)$, taking into account all the annihilations. For most of the time, however, $g_*(T)$ changes slowly, so we can approximate $g_*(T) =$ constant. Then $T \propto a^{-1}$ and $H \propto a^{-2}$. Integrating $H^2 \propto a^{-4}$ gives $a \propto t^{1/2}$, as we saw in the previous chapter. So we have

$$a \propto T^{-1} \propto t^{1/2} .$$

We hence get the following relation between the age of the universe t (or equivalently the Hubble parameter H) and the temperature T:

$$t = \frac{1}{2}H^{-1} = \sqrt{\frac{45}{2\pi^2}}\frac{1}{\sqrt{g_*}}\frac{M_{\rm Pl}}{T^2} \approx \frac{1.51}{\sqrt{g_*}}\frac{M_{\rm Pl}}{T^2} \approx \frac{2.42}{\sqrt{g_*}}\left(\frac{\rm T}{\rm MeV}\right)^{-2} {\rm s}\ . \qquad (4.39)$$

The approximate result (4.39) will be sufficient for us as far as the time scale is concerned[5], but for the relation between a and T, we have to use the more exact result derived in section 4.5.

The distance to the horizon (i.e. proper comoving distance to $t = 0$, or $z = \infty$) is

$$D_{\rm hor}(t) = a(t)\int_0^t \frac{{\rm d}t'}{a(t')} = 2t = H^{-1}\ . \qquad (4.40)$$

In the radiation-dominated early universe, the distance to the horizon is equal to the Hubble length, so we can use the terms "horizon distance", "horizon" and "Hubble length" interchangeably. This is often also done for other eras, when the two are not equal. In particular, when there is a period of inflation at early times, the particle horizon will be much larger than the Hubble length at late times – we will come to this when we discuss inflation in the second part of the course.

4.7 Particle content

The primordial soup initially consists of all the different species of elementary particles. Their masses range from the heaviest known elementary particle, the top quark ($m = 173$ GeV) down to the lightest particles, the electron ($m = 511$ keV), neutrinos ($m < 0.12$ eV), and the photon ($m = 0$). In addition to the particles of the Standard Model, there are presumably other species that remain undiscovered. In particular, we will discuss dark matter particles in chapter 6.

There are two main features of in the evolution of the soup, both driven by the expansion of space, which leads to falling temperature and decreasing number density. One is that as the temperature falls below the masses of the various particles, they become nonrelativistic and *annihilate* at different times. The second is that as the interaction rate falls below the Hubble rate, the *decouple* from each other.

The particles of the Standard Model are listed in table 1. The limits on neutrino masses are from the Planck satellite experiment, the other values are from the Particle Data Group.[6] The internal degrees of freedom for quarks are 2 for spin, 2 for having both left- and right-handed components and 3 for colour. Electrons, muons and and taus don't have colour, but otherwise the counting is the same. In the Standard Model, there are only left-handed neutrinos, so they only have the spin degeneracy factor. Massless spin 1 particles like the photon only have 2 spin degrees of freedom, while massive spin 1 particles, the $W^\pm$ and Z, have 3 (note that W^+ and W^- are counted separately).

The effective number of degrees of freedom $g_*(T)$ (solid), $g_{*p}(T)$ (dashed) and $g_{*s}(T)$ are plotted in figure 1 as a function of temperature. In table 2 we list some important events in the early universe.

[5]Usually the error from ignoring the time-dependence of $g_*(T)$ is negligible, since the time scales of earlier events are so much shorter.

[6]Strictly speaking, the masses of the electron, muon and tau neutrinos are not defined, and the limits apply instead to the neutrino mass eigenstates.

Table 1: The particles in the Standard Model
Particle Data Group 2024 and the Planck collaboration [1, 2]

Quarks	t	172.57 ± 0.29 GeV	$\bar{t}$	spin $\frac{1}{2}$	$g = 2 \cdot 2 \cdot 3 = 12$	
	b	4.183 ± 0.007 GeV	$\bar{b}$	3 colours		
	c	1.273 ± 0.005 GeV	$\bar{c}$			
	s	93.5 ± 0.8 MeV	$\bar{s}$			
	d	4.70 ± 0.07 MeV	$\bar{d}$			
	u	2.16 ± 0.07 MeV	$\bar{u}$			
						72
Gluons	8 massless bosons			spin 1	$g = 2$	16
Leptons	τ^-	1776.93 ± 0.09 MeV	τ^+	spin $\frac{1}{2}$	$g = 2 \cdot 2 = 4$	
	μ^-	105.658 MeV	μ^+			
	e^-	510.999 keV	e^+			
						12
	ν_τ	< 0.12 eV	$\bar{\nu}_\tau$	spin $\frac{1}{2}$	$g = 2$	
	ν_μ	< 0.12 eV	$\bar{\nu}_\mu$			
	ν_e	< 0.12 eV	$\bar{\nu}_e$			
						6
Electroweak gauge bosons	$W^\pm$	80.3692 ± 0.0133 GeV		spin 1	$g = 3$	
	Z^0	91.1880 ± 0.0020 GeV				
	γ	0 $(< 1 \times 10^{-18}$ eV)			$g = 2$	
						11
Higgs boson	H^0	125.20 ± 0.11 GeV		spin 0	$g = 1$	1
						$g_f = 72 + 12 + 6 = 90$
						$g_b = 16 + 11 + 1 = 28$

For $T > m_t = 173$ GeV, all known particles are relativistic. Adding up their internal degrees of freedom we get

$$\begin{aligned} g_b &= 28 && \text{gluons } 8\times2\text{, photons 2, } W^\pm \text{ and } Z^0\ 3\times3\text{, Higgs 1} \\ g_f &= 90 && \text{quarks } 12\times6\text{, charged leptons } 6\times2\text{, neutrinos } 3\times2 \\ g_* &= 106.75\ . \end{aligned}$$

The electroweak crossover takes place at the temperature 160 GeV [3]. Sometimes this process is called the electroweak phase transition. However, in the Standard Model, it is a smooth crossover from one regime to another, and thermodynamic quantities remain continuous. In some extensions of the Standard Model, there is a phase transition, where the system is not in thermal equilibrium. This may have important cosmological consequences (in particular, it may determine the baryon-antibaryon asymmetry observed in the universe), depending on the way the electroweak phase transition happens. We will not discuss details of the electroweak

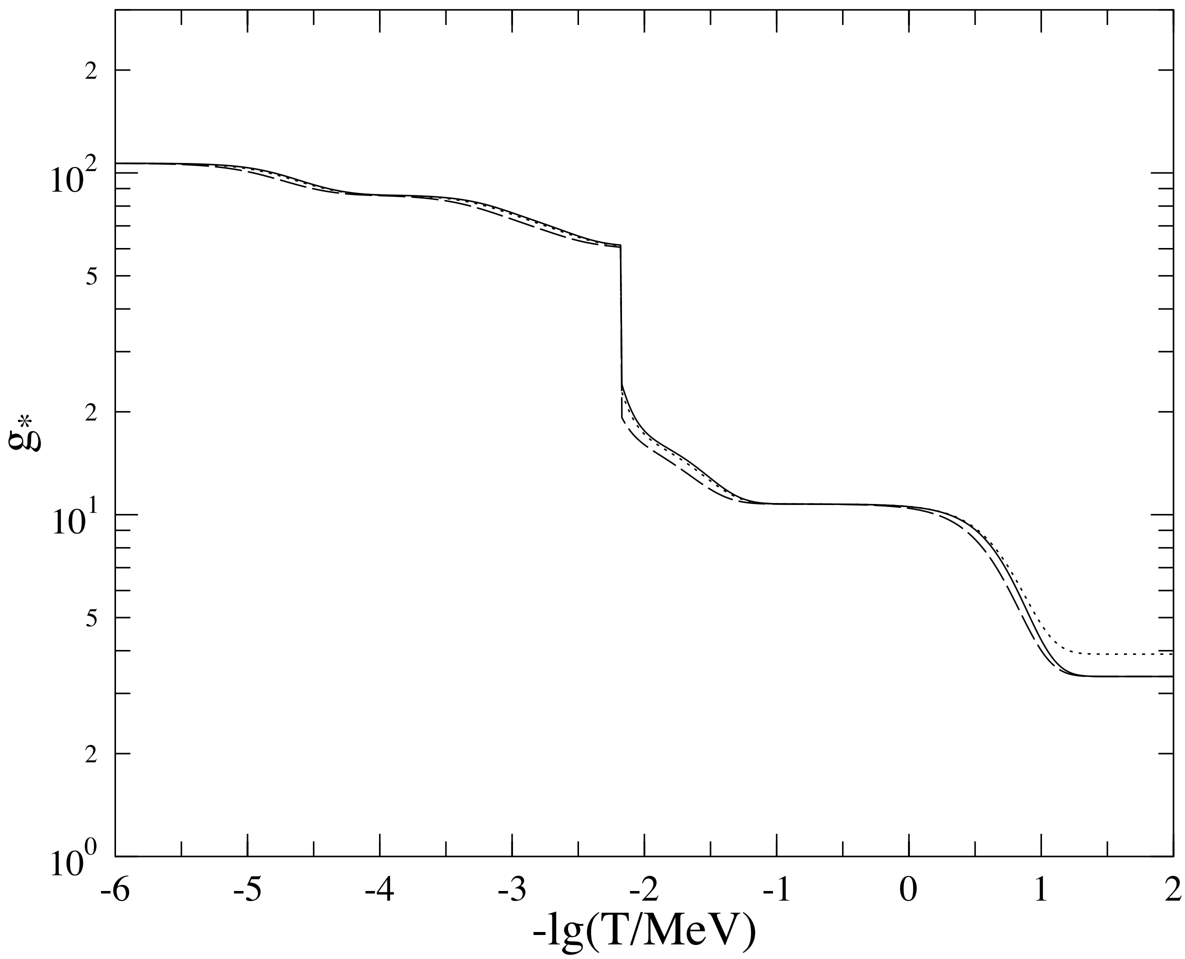

Figure 3: The functions $g_*(T)$ (solid), $g_{*p}(T)$ (dashed), and $g_{*s}(T)$ (dotted) for Standard Model particle content.

crossover, for our purposes it is enough to know that g_* is the same before and after the transition, at least in the Standard Model. If there are thus far unknown particle species beyond the Standard Model, g_* can have different values, especially at high temperatures.

Let us now follow the history of the universe starting at the time when the electroweak crossover has already happened. We have $T \sim 160$ GeV, $t \sim 10$ ps, and t quark annihilation is ongoing. (Recall that the transition from relativistic to non-relativistic behaviour is not complete until about $T \approx m/6 \approx 30$ GeV.) The Higgs boson annihilates next, and then the gauge bosons $W^{\pm}$ and Z^0. At $T \sim 10$ GeV, we have $g_* = 86.25$. Next the b and c quarks annihilate, followed by the τ lepton. If the s quark would also have had time to annihilate, we would reach $g_* = 51.25$.

4.8 QCD crossover

In the middle of the s quark annihilation, matter undergoes the *QCD transition* or *QCD crossover* (also called the quark–hadron transition). This takes place at $T = 154 \pm 9$ MeV, $t = 20 \ldots 30\mu$s [4]. The colour forces between quarks and gluons become important, so the formulae for the energy density for free particles no longer

Electroweak crossover	$T \sim 160$ GeV	$t \sim 10^{-11}$ s
QCD crossover	$T \sim 150$ MeV	$t \sim 30$ μs
Neutrino decoupling	$T \sim 1$ MeV	$t \sim 1$ s
Electron-positron annihilation	$T \sim m_e = 0.5$ MeV	$t \sim 1$ s
Big Bang Nucleosynthesis	$T \sim 50$–100 keV	$t \sim 3$–30 min
Matter-radiation equality	$T \sim 0.8$ eV $= 9000$ K	$t \sim 50\,000$ yr
Recombination + photon decoupling	$T \sim 0.3$ eV $= 3000$ K	$t \sim 380\,000$ yr

Table 2: Early universe events.

apply to them. The quarks and gluons form bound three-quark systems, called *baryons*, and quark-antiquark pairs, called *mesons*. (Together, these bound states of quarks are known as *hadrons*.) Baryons are fermions, mesons are bosons. After that, there are no more free quarks and gluons; the *quark-gluon plasma* has become a *hadron plasma*. The lightest baryons are the nucleons: the proton and the neutron. The lightest mesons are the pions $\pi^\pm$ and π^0.

There are many different species of baryons and mesons, but all except pions are non-relativistic below the QCD crossover temperature. Thus the only Standard Model particle species left in large numbers are pions, muons, electrons, neutrinos and photons. For pions, $g = 3$, so we have $g_* = 17.25$.

Table 3: History of $g_*(T)$

$T \sim 200$ GeV	all present	106.75	
$T < 170$ GeV	top annihilation	96.25	
$T \sim 160$ GeV	electroweak crossover	(no effect)	
$T < 125$ GeV	H^0	95.25	
$T < 80$ GeV	$W^\pm$, Z^0	86.25	
$T < 4$ GeV	bottom	75.75	
$T < 1$ GeV	charm, τ^-	61.75	
$T \sim 150$ MeV	QCD crossover	17.25	(u,d,g$\rightarrow \pi^{\pm,0}$, $37 \rightarrow 3$)
$T < 100$ MeV	$\pi^\pm$, π^0, μ^-	10.75	$e^\pm$, ν, $\bar{\nu}$, γ left
$T < 500$ keV	e^-	(7.25)	$2 + 5.25(4/11)^{4/3} = 3.36$

The above table gives the value $g_*(T)$ would have after the annihilation is over, assuming the next annihilation would not have begun yet. In reality the annihilations overlap in many cases. The temperature value on the left is (apart from the crossover temperatures) the approximate mass of the particle in question and indicates roughly when the annihilation begins. The temperature is much smaller when the annihilation ends. The top quark receives its mass in the electroweak crossover, so its annihilation does not begin before the crossover.

4.9 Neutrino decoupling and electron-positron annihilation

Soon after the QCD crossover, pions and muons annihilate and for $T = 20$ MeV $\rightarrow 1$ MeV, we have $g_* = 10.75$. Next the electrons annihilate, but to discuss the e^+e^- annihilation we need a bit more details.

So far we have assumed that all particle species have the same temperature, i.e. interactions keep them in thermal equilibrium. Neutrinos, however, feel only the weak interaction. The weak interaction is actually not that weak when particle energies are close to (or higher than) the masses of the W and Z bosons, which mediate the interaction. But as the temperature, and thus mean energy of particles, falls, the weak interaction becomes rapidly weaker.

A particle species falls out of chemical equilibrium when interactions become too weak to maintain it in touch with the other species as the universe expands. This happens when the *interaction rate* Γ becomes smaller than the expansion rate, $\Gamma < H$. The interaction rate Γ has units of 1/time, and it can be interpreted as the frequency of particle interactions. The limit $\Gamma < H$ can roughly be understood as saying that if particles on average have less than one interaction per Hubble time, the distribution cannot keep up with the expansion. The interaction rate can be written as $\Gamma = n\langle\sigma v\rangle$, where n is the number density of the particles, σ is the *interaction cross section*, v is the absolute value of the particle velocity and the brackets are average over the phase space. If the cross section is independent of velocity, we can take it out of the average. If the particles are ultrarelativistic, we can approximate $|v| = 1$, in which case we have simply $\Gamma = n\sigma$. The cross section has units of area, and it expresses the strength of the interaction[7].

For the weak interaction processes relevant for neutrinos, the cross section is $\sigma \sim G_{\rm F}^2 E^2 \sim G_{\rm F}^2 T^2$, where $G_{\rm F} \approx 1.17 \times 10^{-5}$ GeV^{-2} is the Fermi constant. The interaction rate is then $\Gamma = n\sigma v \sim G_{\rm F}^2 T^5$, where n is the number density and $v \approx 1$ is typical neutrino velocity. According to the Friedmann equation, $H \sim \sqrt{\rho/M_{\rm Pl}^2} \sim T^2/M_{\rm Pl}$. So we have $\Gamma/H \sim G_{\rm F}^2 M_{\rm Pl} T^3 \sim (T/\text{ MeV})^3$. So, neutrinos decouple close to $T \sim 1$ MeV, after which they move practically freely, without interactions.

Even though neutrinos are no longer in chemical equilibrium, they remain in thermal equilibrium as long as the temperature of the particle soup also evolves like $T \propto a^{-1}$, so $T_\nu = T$. However, annihilations will cause a deviation from $T \propto a^{-1}$. The next annihilation event is the electron-positron annihilation.

As the number of relativistic degrees of freedom falls, energy density and entropy are transferred from electrons and positrons to photons, but not to neutrinos, in the annihilation reactions

$$e^+ + e^- \quad \rightarrow \quad \gamma + \gamma \ .$$

The photons are thus heated relative to neutrinos (i.e. the photon temperature falls less rapidly). In the electron-positron annihilation, g_{*s} changes from

$$g_{*s} \ = \ g_* \ = \ \underset{\gamma}{2} + \underset{e^\pm}{3.5} + \underset{\nu}{5.25} \ = \ 10.75 \tag{4.41}$$

to

$$g_{*s} = 2 + 5.25 \left(\frac{T_\nu}{T}\right)^3 \ . \tag{4.42}$$

For time 1 before the annihilation and time 2 after it, we have from (4.37)

$$2a_2^3 T_2^3 + 5.25 a_2^3 T_{\nu 2}^3 = 10.75 a_1^3 T_1^3 \ . \tag{4.43}$$

[7]This terminology comes from particle physics. The idea is that if you consider a beam of classical particles randomly directed at a target with total area A, and classical particles take up an area σ, the probability of crossing a particle and hence interacting is σ/A.

Before the electron-positron annihilation, the neutrino temperature was the same as the temperature of the other species, so $a_1^3 T_1^3 = a_1^3 T_{\nu 1}^3 = a_2^3 T_{\nu 2}^3$, where we have used the fact that $T_\nu \propto a^{-1}$ throughout, since neutrinos are relativistic and they are not heated by the electron-positron annihilation. We thus have from (4.43)

$$10.75 = 2\left(\frac{T}{T_\nu}\right)^3 + 5.25 \ ,$$

from which we solve the neutrino temperature after e^+e^- annihilation[8],

$$T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T = 0.714\, T \tag{4.44}$$

$$g_{*s}(T) = 2 + 5.25 \cdot \frac{4}{11} = 3.909 \tag{4.45}$$

$$g_*(T) = 2 + 5.25 \left(\frac{4}{11}\right)^{\frac{4}{3}} = 3.363 \ . \tag{4.46}$$

These relations remain true for the photon+neutrino background as long as the neutrinos stay ultrarelativistic ($m_\nu \ll T$). The neutrinos are no longer in chemical or thermal equilibrium, but they are still in kinetic equilibrium, i.e. their distribution function has the thermal shape.

If the neutrino masses were small enough to be ignored, the above relation would apply even today, when the photon (the CMB) temperature is $T = T_0 = 2.725$ K $= 0.235$ meV, giving the neutrino background temperature $T_{\nu 0} = 0.714 \cdot 2.725$ K $= 1.95$ K $= 0.168$ meV. However, *neutrino oscillation* experiments have established that neutrinos have masses $\gtrsim 10$ meV [1][9], and there is the upper limit 0.12 eV on the sum of neutrino masses from cosmology [2]. Therefore, the neutrino background is non-relativistic today. As neutrinos become non-relativistic, they fall out of kinetic equilibrium, because the shape of the thermal distribution function is not preserved as the momenta redshift to the value $p \sim m$. Once neutrinos become very non-relativistic, with typical values of the momenta $p \ll m$, the distribution function again has the thermal shape, but with a different temperature scaling.

[8]To be more precise, neutrino decoupling was not complete when e^+e^- annihilation began, so some of the energy and entropy did leak to the neutrinos. Therefore the neutrino energy density after e^+e^- annihilation is about 1.3% higher (at a given T) than the above calculation gives. The neutrino distribution also deviates slightly from kinetic equilibrium.

[9]Specifically, the oscillations show that the mass differences between the neutrinos are of this order. The observations do not exclude the possibility that the lightest neutrino could be massless.

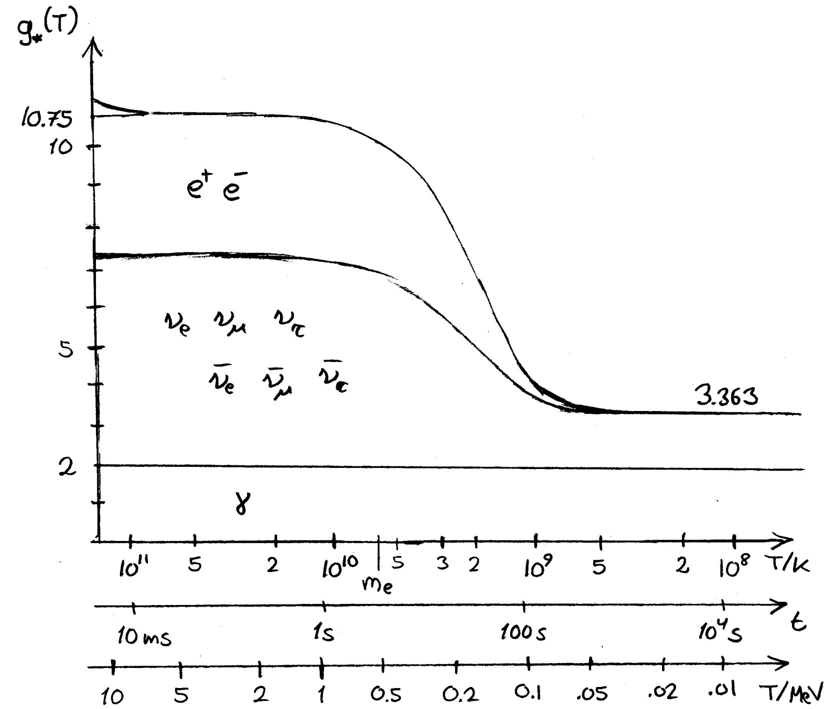

Figure 4: The evolution of the energy density, or rather, $g_*(T)$, and its different components through electron-positron annihilation. Since $g_*(T)$ is defined as $\rho/(\pi^2 T^4/30)$, where T is the photon temperature, the photon contribution appears constant. If we had plotted $\rho/(\pi^2 T_\nu^4/30) \propto \rho a^4$ instead, the neutrino contribution would appear constant, and the photon contribution would increase at the cost of the electron-positron contribution.

4.10 Matter

We noted that the early universe is dominated by the relativistic particles, and we can forget the nonrelativistic particles when we are considering the dynamics of the universe. We followed one species after another becoming nonrelativistic and disappearing from the picture, until only photons (the CMB) and neutrinos were left, and the neutrinos had stopped interacting.

We now return to look in more detail what happens to nucleons and electrons. They annihilated with their antiparticles when the temperature fell below their respective rest masses. For nucleons, the annihilation began immediately after they were formed in the QCD crossover. There were however slightly more particles than antiparticles, and this small excess of particles was left over. (This has to be the case because we observe electrons and nucleons today – we'll be more quantitative in chapter 6.) This means that the chemical potential μ_B associated with baryon number differs from zero (it is positive). Baryon number is a conserved quantity in the eras we are considering (though not before the electroweak crossover). Baryon number resides today in nucleons (protons and neutrons; since the proton is lighter than the neutron, free neutrons have decayed into protons, but there are neutrons in atomic nuclei) because they are the lightest baryons. The universe is electrically neutral, and the negative charge lies in the electrons, the lightest particles with negative charge. Therefore the number of electrons equals the number of protons.

The number density, energy density and pressure of the electrons and the nucleons by the equations written down in section 4.2. But what is the value of the chemical potential μ that appears in them? For each species, we get $\mu(T)$ from the conserved quantities[10]. The baryon number resides in the nucleons,

$$n_\mathrm{b} = n_\mathrm{N} - n_{\bar{\mathrm{N}}} = n_\mathrm{p} + n_\mathrm{n} - n_{\bar{\mathrm{p}}} - n_{\bar{\mathrm{n}}} \,. \tag{4.47}$$

Let us define the parameter η, the baryon-photon ratio today,

$$\eta \equiv \frac{n_\mathrm{b}(t_0)}{n_\gamma(t_0)} \,. \tag{4.48}$$

From observations we know that $\eta \approx 6 \times 10^{-10}$. (We will take a closer look at the observational value in the next chapter.) Since baryon number is conserved, $n_\mathrm{b} V \propto n_\mathrm{b} a^3$ stays constant, so

$$n_\mathrm{b} \propto a^{-3} \,. \tag{4.49}$$

After electron-positron annihilation, we have $n_\gamma \propto a^{-3}$, so we get

$$n_\mathrm{b}(T) = \eta n_\gamma = \eta \frac{2\zeta(3)}{\pi^2} T^3 \qquad \text{for } T \ll m_\mathrm{e} \,. \tag{4.50}$$

We can put (4.49) and (4.50) together and replace a^{-3} using the relation (4.34) between the temperature and the scale factor to obtain

$$n_\mathrm{b}(T) = \eta \frac{2\zeta(3)}{\pi^2} \frac{g_{*s}(T)}{g_{*s}(T_0)} T^3 \,. \tag{4.51}$$

[10] In general, the way to find how the thermodynamical parameters evolve in the expanding FLRW universe is to use the conservation laws of the conserved number densities, entropy conservation and the energy continuity equation to find how the number densities and energy densities evolve. The other thermodynamical parameters then evolve so as to satisfy these requirements.

For $T < 10$ MeV we have

$$n_{\bar{N}} \ll n_N \qquad \text{and} \qquad n_N \equiv n_n + n_p = n_b \,.$$

In the next chapter, we will discuss big bang nucleosynthesis, i.e. how protons and neutrons form atomic nuclei. Approximately one quarter of all nucleons (all neutrons and roughly the same number of protons) form nuclei (with mass number $A > 1$) and three quarters remain as free protons. Let us denote by n_p^* and n_n^* the total number densities of protons and neutrons including those in nuclei (and also those in atoms), whereas we use n_p and n_n for the number densities of *free* protons and neutrons, which are not bound to each other or electrons. We thus write

$$n_N^* \equiv n_n^* + n_p^* = n_b \,.$$

In the same manner, for $T < 10$ keV we have

$$n_{e^+} \ll n_{e^-} \qquad \text{and} \qquad n_{e^-} = n_p^* \,.$$

At this time ($T \sim 10$ keV $\to 1$ eV) the universe contains a relativistic photon and neutrino background ("radiation") and nonrelativistic free electrons, protons, and nuclei ("matter"). Since $\rho \propto a^{-4}$ for radiation and $\rho \propto a^{-3}$ for matter, the energy density in radiation falls eventually below the energy density in matter—the universe becomes *matter-dominated*.

The above discussion only takes into account the known particle species. Today there is much observational evidence for the existence of *dark matter*, which presumably consists of either black holes (formed in the early universe, not from stellar collapse) or some yet undiscovered species of particles. The most popular candidate is *cold dark matter* (CDM). CDM particles interact weakly with normal matter, and hence decouple early. At early times, when they are in thermal equilibrium, they would slightly increase the number of degrees of freedom; after they decouple, they have no effect on the discussion above. They become nonrelativistic early and dominate the matter density of the universe today (there is five times as much mass in CDM as there is in baryons). So CDM causes the universe to become matter-dominated earlier than if the matter consisted of nucleons and electrons only. The CDM will be important later when we discuss the formation of structures in the universe. The time of matter-radiation equality t_{eq} is calculated in an exercise at the end of this chapter.

4.11 Recombination

Radiation (photons) and matter (electrons, protons, and nuclei) remained in thermal equilibrium as long as there were lots of free electrons. When the temperature became low enough the electrons and nuclei combined to form neutral atoms, an event known as *recombination*[11], and the density of free electrons fell sharply. The *photon mean free path* grew rapidly and became longer than the horizon distance. Thus the universe became *transparent*. Photons and matter decoupled, i.e. their interactions were no longer able to maintain them in thermal equilibrium with each other. After this, by T we refer to the photon temperature. Today, these photons are

[11] This is the first time when nuclei and electrons combine, so the term *re*combination, adopted from chemistry, is somewhat of a misnomer.

the CMB, and $T = T_0 = 2.725$ K. (After photon decoupling, the matter temperature at first fell faster than the photon temperature, but structure formation then heated up the matter to different temperatures in different places.)

The relevant interaction here is not the weak interaction, as in the case of the neutrinos, but instead electromagnetic interaction between photons and electrons. The interaction rate is $\Gamma \sim n_e \sigma_T$, where $\sigma_T = \frac{8\pi}{3}\alpha^2/m_e^2 \approx 2 \times 10^{-3}$ MeV^{-2} is the Thomson cross-section, and $\alpha \approx 1/137$ is the electromagnetic coupling constant. (The $1/m^2$ factor shows that interactions between photons and nuclei are not important, as they are suppressed by the large masses of the nuclei.) Finding the photon decoupling era is a bit more involved than in the neutrino case, as the evolution of the electron number density is more complicated.

To simplify the discussion, let us ignore other nuclei than protons (over 90%, by number, of the nuclei are protons, and almost all the rest are ^{4}He nuclei). We denote the number density of *free* protons by n_p, free electrons by n_e, and hydrogen atoms by n_H. Since the universe is electrically neutral, $n_\mathrm{p} = n_\mathrm{e}$. The conservation of baryon number gives $n_\mathrm{b} = n_\mathrm{p} + n_\mathrm{H}$. We have

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\frac{\mu_i - m_i}{T}} . \tag{4.52}$$

As long as the reaction

$$p + e^- \leftrightarrow \mathrm{H} + \gamma \tag{4.53}$$

is in chemical equilibrium the chemical potentials are related by $\mu_\mathrm{p} + \mu_\mathrm{e} = \mu_\mathrm{H}$ (since $\mu_\gamma = 0$). Using this we get the relation

$$n_\mathrm{H} = \frac{g_\mathrm{H}}{g_\mathrm{p} g_\mathrm{e}} n_\mathrm{p} n_\mathrm{e} \left(\frac{m_\mathrm{e} T}{2\pi}\right)^{-3/2} e^{B/T}, \tag{4.54}$$

between the number densities. Here $B = m_\mathrm{p} + m_\mathrm{e} - m_\mathrm{H} = 13.6$ eV is the *binding energy* of hydrogen. The numbers of internal degrees of freedom are $g_\mathrm{p} = g_\mathrm{e} = 2$, $g_\mathrm{H} = 4$. Outside the exponent we have approximated $m_\mathrm{H} \approx m_\mathrm{p}$. Defining the *fractional ionisation*

$$x \equiv \frac{n_\mathrm{p}}{n_\mathrm{b}} , \tag{4.55}$$

equation (4.54) becomes

$$\frac{1-x}{x^2} = \frac{4\sqrt{2}\,\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{T}{m_\mathrm{e}}\right)^{3/2} e^{B/T} . \tag{4.56}$$

This is the *Saha equation* for ionisation in thermal equilibrium. When $B \ll T \ll m_\mathrm{e}$, the right-hand side is $\ll 1$, so $x \sim 1$, and almost all protons and electrons are free. As temperature falls, $e^{B/T}$ grows, but since both η and $(T/m_\mathrm{e})^{3/2}$ are $\ll 1$, the temperature needs to fall to $T \ll B$ before the whole expression becomes large ($\gtrsim 1$).

The ionisation fraction at first follows the equilibrium result (4.56) closely, but as this equilibrium fraction begins to fall rapidly, it starts to lag behind the equilibrium value. As the number densities of free electrons and protons fall, it becomes more difficult for them to find each other, and they are no longer able to maintain chemical equilibrium for the reaction (4.53). To find the correct ionisation evolution, $x(t)$,

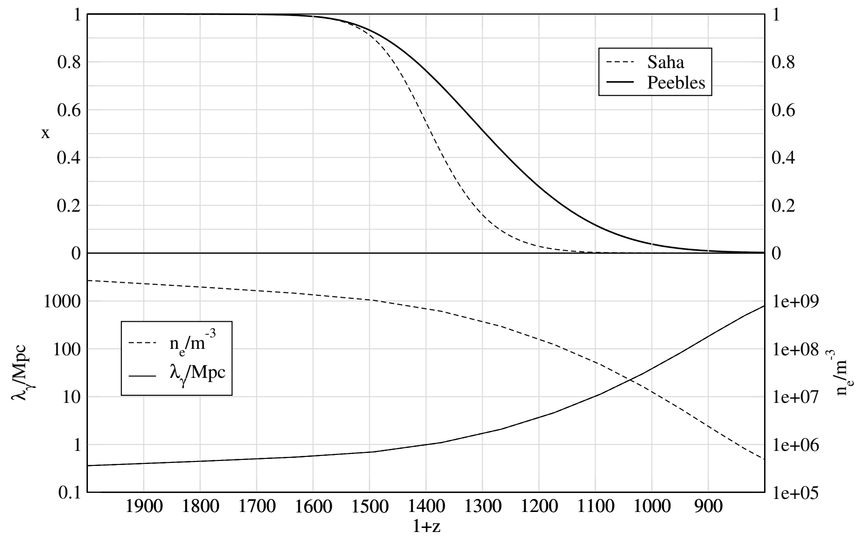

Figure 5: Recombination. In the top panel the dashed curve gives the equilibrium ionisation fraction as given by the Saha equation. The solid curve is the true ionisation fraction, calculated using the actual reaction rates (original calculation by Peebles). You can see that the equilibrium fraction is followed at first, but then the true fraction lags behind. The bottom panel shows the free electron number density n_e and the photon mean free path λ_γ. The latter is given in comoving units, i.e. , the distance is scaled to the corresponding present distance. This figure is for $\eta = 8.22 \times 10^{-10}$. (Figure by R. Keskitalo.)

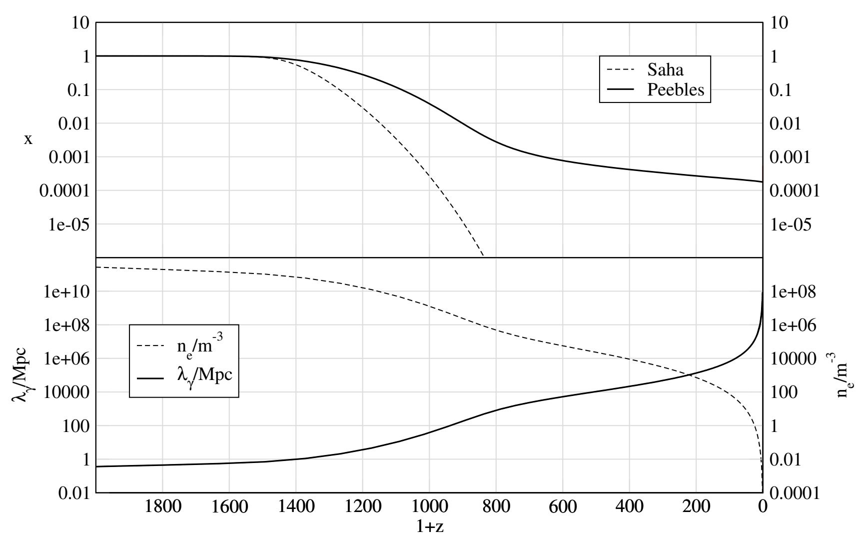

Figure 6: Same as figure 5, but with a logarithmic scale for the ionisation fraction, and the redshift scale extended to present time ($z = 0$ or $1 + z = 1$). You can see that a residual ionisation $x \sim 10^{-4}$ remains. This figure does not include reionisation, which happened around $z \sim 10$. (Figure by R. Keskitalo.)

requires then a more complicated calculation involving the reaction cross section of this reaction. See figures 5 and 6.

Although the equilibrium formula is thus not enough to give us the true ionisation evolution, its benefit is twofold:

1. It tells us when recombination begins. While the equilibrium ionisation changes only very slowly, it is easy to stay in equilibrium. Thus things won't start to happen until the equilibrium fraction begins to change a lot.

2. It gives the initial conditions for the more complicated calculation that gives the true evolution.

A similar situation holds for many other events in the early universe, such as big bang nucleosynthesis that we will discuss in chapter 5.

Recombination is not instantaneous. Let us define the recombination temperature T_{rec} as the temperature where $x = 0.5$. We have $T_{\text{rec}} = T_0(1 + z_{\text{rec}})$ since $1 + z = a^{-1}$ and the photon temperature falls as $T \propto a^{-1}$. (Since $\eta \ll 1$, the energy release in recombination is negligible compared to ρ_γ; and after photon decoupling photons travel freely maintaining kinetic equilibrium with $T \propto a^{-1}$.)

We get (for $\eta \sim 10^{-9}$)

$$\begin{aligned} T_{\text{rec}} &\sim 0.3 \text{ eV} \\ z_{\text{rec}} &\sim 1300 \ . \end{aligned}$$

You might have expected that $T_{\text{rec}} \sim B$. Instead we found $T_{\text{rec}} \ll B$. The main reason for this is that $\eta \ll 1$. This means that there are very many photons for each hydrogen atom. Even when $T \ll B$, the high-energy tail of the photon distribution contains photons with energy $E > B$ so that they can ionise a hydrogen atom.

The photon decoupling takes place somewhat later, at $T_{\text{dec}} \equiv (1 + z_{\text{dec}})T_0$, when the ionisation fraction has fallen enough. We define the photon decoupling time as the time when the photon mean free path exceeds the Hubble distance. The numbers are roughly

$$\begin{aligned} T_{\text{dec}} &\sim 3000 \text{ K} \sim 0.26 \text{ eV} \\ z_{\text{dec}} &\sim 1100 \ . \end{aligned}$$

Because of the decoupling, the recombination reaction cannot anymore keep the ionisation fraction on the equilibrium track, and we are left with a residual ionisation of $x \sim 10^{-4}$.

A long time later (at $z \approx 30$) the first stars form, and their radiation *reionises* the gas in interstellar space. The gas has now such a low density, however, that the universe remains transparent.

Exercise: Transparency of the universe. We say the universe is transparent when the photon mean free path λ_γ is larger than the Hubble length $l_H = H^{-1}$, and opaque when $\lambda_\gamma < l_H$. The photon mean free path is determined mainly by the scattering of photons by free electrons, so $\lambda_\gamma = 1/(\sigma_T n_e)$, where $n_e = x n_e^*$ is the number density of free electrons, n_e^* is the total number density of electrons, and x is the ionisation fraction. The cross section for photon-electron scattering is independent of energy for $E_\gamma \ll m_e$ and is then called the Thomson cross section, $\sigma_T = \frac{8\pi}{3}(\alpha/m_e)^2$, where α is the fine-structure constant. In recombination x

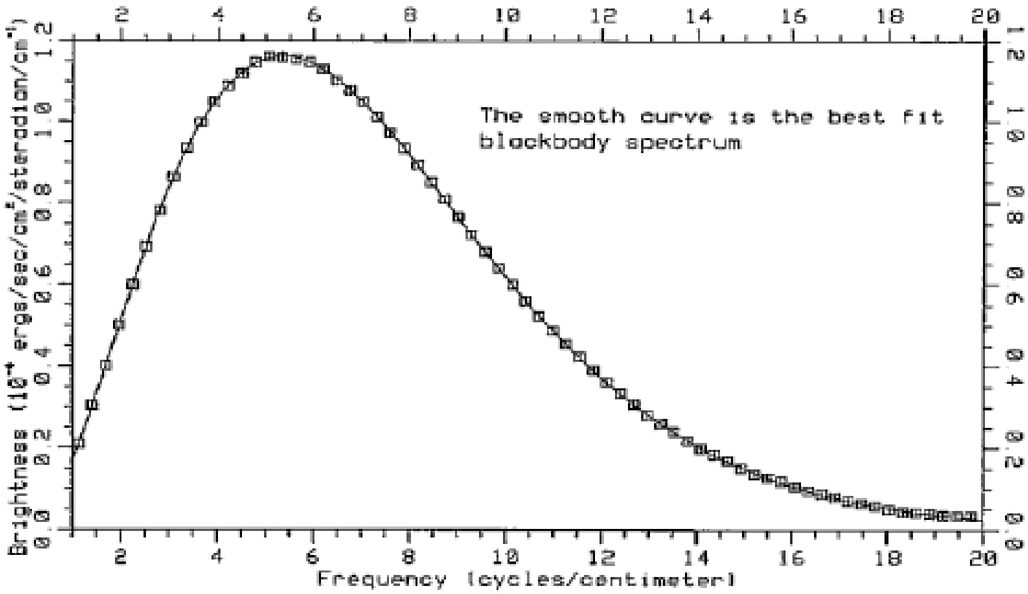

Figure 7: The CMB frequency spectrum as measured by the FIRAS instrument on the COBE satellite [5]. This first spectrum from FIRAS was based on just 9 minutes of measurements. The CMB temperature estimated from it was $T = 2.735 \pm 0.060$ K. The final result from FIRAS is $T = 2.725 \pm 0.002$ K (95% confidence interval) [6]. Using data from other experiments as well, the best current value is $T_0 = 2.72548 \pm 0.00057$ K [7].

falls from 1 to 10^{-4}. Show that the universe is opaque before recombination and transparent after recombination. (Assume recombination takes place instantly at $z = 1300$. You can assume a matter-dominated universe—see below for parameter values.) The interstellar matter gets later reionised (to $x \sim 1$) by the light from the first stars. What is the earliest redshift when this can happen without making the universe opaque again? (You can assume that most ($\sim$ all) matter has remained interstellar.) Calculate for $\Omega_{m0} = 1.0$ and $\Omega_{m0} = 0.3$ (note that Ω_m includes non-baryonic matter). Use $\Omega_\Lambda = 0$, $h = 0.7$ and $\eta = 6 \times 10^{-10}$.

The photons in the cosmic background radiation have thus travelled almost without scattering through space all the way since we had $T = T_{\rm dec} \sim 1090\, T_0$.[12] When we look at this cosmic background radiation we thus see the universe (its faraway parts near our horizon) as it was at that early time. Because of the redshift, these photons which were then largely in the visible part of the spectrum, have now become microwave photons, so this radiation is now the CMB. It retains the thermal equilibrium distribution, although it has not been in thermal equilibrium since last scattering. This was confirmed to high accuracy by the FIRAS (Far InfraRed Absolute Spectrophotometer) instrument on the COBE (Cosmic Background Explorer) satellite in 1989. John Mather received the 2006 Physics Nobel Prize for this measurement of the CMB frequency (photon energy) spectrum, see figure 7.[13]

[12]The probability for a photon to have one or more scatterings between decoupling and today is about 10% – in the second part of the course, we'll discuss how we know this.

[13]He shared the prize with George Smoot, who got it for the discovery of the CMB anisotropy with the DMR instrument on the same satellite. We will discuss the CMB anisotropy in the second part of the course.

We shall now, for a while, stop the detailed discussion of recombination and photon decoupling. The universe is about 380 000 years old at decoupling. Next, gravitationally bound structures start to form as overdense regions evolve and collapse under gravity. We will discuss structure formation in the second part of the course, before that let us discuss big bang nucleosynthesis and dark matter.

4.12 The Dark Ages

How would the universe after recombination appear to an observer with human eyes? At first we would see a uniform glow, since the wavelengths of many of the CMB photons are in the visible range, though the peak is in the infrared. (It would also feel rather hot, 3000 K). As time goes on, this glow gets dimmer and dimmer as the photons redshift towards the infrared, and after a few million years it gets completely dark, as photons even deep into the tail of the Planck distribution are redshifted into the infrared. There are no stars yet. This era is called the *Dark Ages* of the universe. It lasts dozens of millions of years, during which the universe becomes colder. In the dark, however, masses are gathering together. And then, one by one, the first stars light up. It seems that the star formation rate peaked between redshifts $z = 1$ and $z = 2$. Thus the universe at a few billion years was brighter than it is today, since the brightest stars are short-lived, and the galaxies were closer to each other back then.

4.13 The radiation and neutrino backgrounds

While the starlight is more visible to the naked human eye than the CMB, its energy density and number density in the universe is much smaller. Thus the photon density is essentially given by the CMB. The number density of CMB photons today is

$$n_{\gamma 0} = \frac{2\zeta(3)}{\pi^2} T_0^3 = 410.7 \text{ photons/cm}^3 \ . \tag{4.57}$$

This corresponds to the mean value of the measured temperature $2.72548 \pm 0.00057 K$. The 68% range is between 410.46 and 410.98 photons/cm^3. The photon energy density is

$$\rho_{\gamma 0} = \frac{\pi^2}{15} T_0^4 = 4.645 \times 10^{-31} \text{ kg/m}^3 \ , \tag{4.58}$$

where this number, again, corresponds to the mean value, and the 68% range is from 4.6410 to 4.6488 $\times 10^{-31}$ kg/m^3. The critical density today is

$$\rho_{c0} = \frac{3H_0^2}{8\pi G_{\rm N}} = h^2 \cdot 1.87834 \times 10^{-26} \text{ kg/m}^3 \ . \tag{4.59}$$

Recall that the proton mass is 10^{-27} kg. The photon density parameter is

$$\Omega_{\gamma 0} \equiv \frac{\rho_{\gamma 0}}{\rho_{c0}} = 2.47 \times 10^{-5} h^{-2} \ . \tag{4.60}$$

While relativistic, neutrinos contribute another radiation component, with the energy density

$$\rho_\nu = \frac{7N_\nu}{8} \frac{\pi^2}{15} T_\nu^4 \ . \tag{4.61}$$

After e^+e^- annihilation this gives (recalling the relation (4.44) between the photon and neutrino temperature)

$$\rho_\nu = \frac{7N_\nu}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\rho_\gamma \ , \tag{4.62}$$

where N_ν is the number of neutrino species.

When the number of (light) neutrino species was not yet known from colliders, cosmology was used to constrain it. Big bang nucleosynthesis is sensitive to the expansion rate in the early universe, and that depends on the energy density. Observations of CMB and large-scale structure require $N_\nu = 2.99 \pm 0.17$ [2] (this limit is somewhat dependent on the precise assumptions about the cosmological model). Any new particle species that would be relativistic around big bang nucleosynthesis ($T \sim 50$ keV – 1 MeV) and would thus contribute to the expansion rate through its energy density, but which would not interact directly with nuclei and electrons, would have the same effect. The presence of such unknown particles at big bang nucleosynthesis is thus rather constrained.

If we take (4.62) to define N_ν, but then take into account the extra contribution to ρ_ν from energy leakage during e^+e^--annihilation (and some other small effects), we get (as a result of years of hard work by many theorists) [8]

$$N_\nu = 3.043 \ . \tag{4.63}$$

(This does not mean that there are 3.043 neutrino species, but that the total energy density in neutrinos is 3.043 times as much as the energy density one neutrino species would contribute had it decoupled completely before e^+e^- annihilation.)

If neutrinos were still relativistic today, the neutrino density parameter would be

$$\Omega_{\nu 0} = \frac{7N_\nu}{22}\left(\frac{4}{11}\right)^{\frac{1}{3}}\Omega_{\gamma 0} = 1.71 \times 10^{-5}h^{-2} \ , \tag{4.64}$$

so the total radiation density parameter would be

$$\Omega_{r0} = \Omega_{\gamma 0} + \Omega_{\nu 0} = 4.18 \times 10^{-5}h^{-2} \sim 10^{-4} \ . \tag{4.65}$$

We thus confirm the claim in chapter 3 that the radiation component can be ignored in the Friedmann equation, except in the early universe. The combination $\Omega_i h^2$ is denoted by ω_i, so we have

$$\omega_\gamma = 2.47 \times 10^{-5} \tag{4.66}$$

$$\omega_\nu = 1.71 \times 10^{-5} \tag{4.67}$$

$$\omega_r = \omega_\gamma + \omega_\nu = 4.18 \times 10^{-5} \, . \tag{4.68}$$

As noted earlier, neutrinos have masses in range $\sim 10 \ldots 100$ meV, and are non-relativistic today. Therefore they count as matter, not radiation, so the above result for the neutrino energy density does not apply. However, they were still relativistic, and so counted as radiation, at the time of recombination and matter-radiation equality. While the neutrinos are relativistic, the neutrino energy density is

$$\rho_\nu = \Omega_{\nu 0}\rho_{c0}a^{-4} \tag{4.69}$$

using $\Omega_{\nu 0}$ from (4.64), even though $\Omega_{\nu 0}$ does not give the present density of neutrinos.

Today, even though the photon and neutrino backgrounds do not dominate the energy density of the universe, they do dominate the entropy density.

Exercise: Matter–radiation equality. The present density of matter is $\rho_{m0} = \Omega_{m0}\rho_c$ and the present density of radiation is $\rho_{r0} = \rho_{\gamma 0} + \rho_{\nu 0}$ (we assume we can neglect neutrino masses). What was the age of the universe $t_{\rm eq}$ when $\rho_m = \rho_r$? (Assume spatial flatness; note that in these early times—but not today—you can ignore vacuum energy term in the Friedmann equation.) Give numerical value (in years) for the cases $\Omega_{m0} = 0.1$, 0.3, and 1.0, and $H_0 = 70$ km/s/Mpc. What was the temperature at that time, $T_{\rm eq}$?

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