Due on October 5 by 14.15.

1. **Temperature of non-relativistic species.** Show that for a non-relativistic particle species, the distribution function retains the thermal shape as the universe expands, with $T \propto a^{-2}$ and $\mu - m \propto a^{-2}$.

2. **Matter–radiation equality.** Today the energy density of matter is $\rho_{m0} = \Omega_{m0}\rho_{c0}$ and the energy density of radiation is $\rho_{r0} = \rho_{\gamma 0} + \rho_{\nu 0}$, where $\rho_{\gamma 0} = AT_0^4$ is the contribution of the microwave background ($T_0 = 2.725$ K) and $\rho_{\nu 0} = (21/8)AT_{\nu 0}^4$ is the contribution of the neutrino background (we assume neutrinos are massless). Here $A = \pi^2/15$ and $T_{\nu 0} = (4/11)^{1/3}T_0$.

   a) What was the age of the universe $t_{\rm eq}$ when $\rho_m = \rho_r$? (Note that at these early times –but not today– you can ignore the curvature and vacuum terms in the Friedmann equation; you don't have to make other assumptions about the values of $\Omega_0$ or $\Omega_{\Lambda 0}$, since the answer does not depend on them.) Give the numerical value (in years) for the cases $\Omega_{m0} = 0.1$, 0.3, and 1.0, assuming $H_0 = 70$ km/s/Mpc.

   b) What is the temperature $T_{\rm eq} \equiv T(t_{\rm eq})$? Give the numerical value (in K) in the three different cases.

3. **Transparency of the universe.** We say that the universe is transparent when the photon mean free path $\lambda_\gamma$ is larger than the Hubble length $l_H = H^{-1}$, and opaque when $\lambda_\gamma < l_H$. The photon mean free path is determined mainly by the scattering of photons by free electrons, so $\lambda_\gamma = 1/(\sigma_T n_e)$, where $n_e = x n_e^*$ is the number density of free electrons, $n_e^*$ is the total number density of electrons, and $x$ is the ionization fraction. The cross section for photon-electron scattering is independent of energy for $E_\gamma \ll m_e$ and is then given by the Thomson cross section, $\sigma_T = \frac{8\pi}{3}(\alpha/m_e)^2$, where $\alpha$ is the fine-structure constant. In recombination $x$ falls from 1 to $10^{-4}$. Assume instant recombination at $1 + z = 1300$.

   a) Show that the universe is opaque before recombination and transparent after recombination. (Neglect radiation; see below for the parameter values.)

   b) Interstellar baryonic matter gets later reionised (to $x \sim 1$) by light from the first stars. What is the earliest redshift when this can happen without making the universe opaque again? (You can assume that most ($\sim$ all) matter has remained interstellar.)

   Calculate for $\Omega_{m0} = 1.0$ and $\Omega_{m0} = 0.3$ (note that $\Omega_m$ includes nonbaryonic matter). Use $\Omega_K = 0$ (so $\Omega_{\Lambda 0} = 1 - \Omega_{m0}$), $h = 0.7$ and $\eta = 6 \times 10^{-10}$.

4. **Hubble length at the QCD transition.**

   a) What is the Hubble length $H^{-1}$ in meters during the QCD crossover, at $T = 150$ MeV?

   b) To what comoving scale does that distance correspond to today, at $T = 2.7$ K?