**Cosmology I** Homework 4 Fall 2026

Due on Monday September 28 by 14.15.

1. **Age of the positively curved universe.** Find the age-redshift relation for a positively curved universe ($K > 0$) with only matter (no radiation or vacuum energy). Calculate $t_0$ if $H_0 = 70$ km/s/Mpc and a) $\Omega_0 = 1.1$, b) $\Omega_0 = 2$.

2. **The ΛCDM model.** Suppose that we have $H_0 = 70$ km/s/Mpc, $\Omega_{m0} = 0.3$ and $\Omega_{\Lambda 0} = 0.7$, so that $\Omega = \Omega_m + \Omega_\Lambda = 1$ and the universe is spatially flat.

   a) Find the age of the universe today and at redshift $z = 1090$.

   b) When is the matter density equal to the vacuum energy density? (Give both $t$ and $z$.)

   c) The scale factor has an inflection point, where $\ddot{a} = 0$, at which the expansion starts to accelerate. When does this happen, in $t$ and in $z$?

   (Hint: Use the substitution $x^{3/2} = b \sinh \phi$ for the integral $\int \frac{x^{1/2} dx}{\sqrt{b^2 + x^3}}$ .)

3. **Dynamical dark energy.** Consider a non-flat ($\Omega_K \neq 0$) universe that contains matter, radiation, and dark energy (with positive energy density) with the equation of state (3.88).

   a) Find $H(z)$.

   b) What are the different possibilities for the fate of the universe? How do they depend on the density parameters?

4. **Thermal distributions in the non-relativistic limit.** Derive the following equations for non-relativistic Maxwell-Boltzmann statistics ($T \ll m$ and $T \ll |m - \mu|$).

$$
\begin{aligned}
n &= g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-\frac{m-\mu}{T}} \\
\rho &= n \left( m + \frac{3T}{2} \right) \\
p &= nT \\
\langle E \rangle &= m + \frac{3T}{2} \\
n - \bar{n} &= 2g \left( \frac{mT}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{m}{T}} \sinh \frac{\mu}{T} \ .
\end{aligned}
$$