Due on October 14 by 14.00. These are the last exercises.

1. **Neutrino chemical potential.** In the standard analysis, neutrino chemical potential is assumed to be negligible. Neutrino chemical potential affects BBN in two ways: 1) The energy density of neutrinos is larger, as shown in eq. (4.17), leading to faster expansion of the universe. 2) The chemical potential of the electron neutrino is reflected in the neutron and proton chemical potentials and thus in the n/p ratio. The chemical potential redshifts as the neutrino temperature. Define $\xi_i \equiv \mu_{\nu_i}/T_\nu$. Suppose that we have $\xi_e = 0.1$ (around the current upper limit), $\xi_\mu = \xi_\tau = 0$. Use a simplified nucleosynthesis model where the reactions $n+\nu_e \leftrightarrow p+e^-$, etc. are in equilibrium down to $T = 0.8$ MeV, when the neutrinos decouple instantaneously, after which neutrons decay (into protons) with a lifetime of 878 s, and nucleosynthesis, where all neutrons go into ^{4}He nuclei, happens instantaneously at $T = 70$ keV. Use the $T \propto t^{-1/2}$ expansion law, which applies after electron annihilation, for the whole period. How much ^{4}He is produced in this case? Is it more or less than in the standard ($\xi_e = 0$) case?

2. **WIMP miracle.** Consider a WIMP with a constant thermally averaged self-annihilation cross section times velocity, $\langle\sigma v\rangle = \sigma_0$, mass m and $g = 2$. Assume that the WIMP decouples instantaneously when $\Gamma = H$, that $m \gg T_\mathrm{d}$ and $g_*(T_\mathrm{d}) = g_{*S}(T_\mathrm{d}) = 100$, and that $\eta = 6\times10^{-10}$. Find the dark matter energy density today ρ_dm0 relative to the baryon energy density today ρ_b0, as a function of σ_0 and m. Give the numerical value for $\sigma_0 = 10^{-38}$ cm^2 and $m = 100$ GeV.

3. **Baryon-symmetric universe.** Consider a universe with zero baryon number. Use $g = 4$ (this includes protons and neutrons), $m_N = 0.94$ GeV, $\langle\sigma v\rangle = m_{\pi^0}^{-2}$ with $m_{\pi^0} = 0.135$ GeV and $g_*(T_d) = 10.75$.

   a) Find the decoupling temperature.

   b) In analogy with the WIMP calculation above, find the present number density of nucleons plus antinucleons left over from annihilation, relative to photons. Compare it to the value of η in the real universre.

4. **Bonus problem: Tremaine–Gunn limit.** (The points from this question are extra: they only increase your number of points, not the maximum number of points. So it is possible to get more than 100% of the points.) Neutrinos are non-relativistic today. Let us suppose they dominate the mass of galaxies, so we ignore other forms of matter there. We know the mass of a galaxy (within a certain radius) from its rotation velocity. The mass could come from a small number of heavier neutrinos or a large number of lighter neutrinos, but the available phase space (you don't have to assume a thermal distribution) limits the total number of neutrinos whose velocity is below the escape velocity. This leads to a lower limit of the neutrino mass m_ν. Let r be the radius of the galaxy and v its rotation velocity at this distance.

   Find the minimum m_ν needed for neutrinos to dominate the galaxy mass, assuming that only one species is massive. (In reality, we know that at least two neutrino species have non-zero mass.) You can assume that the neutrino distribution is spherically symmetric, and that the escape velocity within radius r equals the escape velocity at r. Give the numerical value for the case $v = 200$ km/s and $r = 10$ kpc. (We know that neutrinos are only a small part of dark matter, but the reasoning applies to any fermions.)