Due on October 7 by 14.00.

1. **Matter–radiation equality.** The present energy density of matter is $\rho_{m0} = \Omega_{m0}\rho_{c0}$ and the present energy density of radiation is $\rho_{r0} = \rho_{\gamma0} + \rho_{\nu0}$, where $\rho_{\gamma0} = AT_0^4$ is the contribution of the microwave background ($T_0 = 2.725$K) and $\rho_{\nu0} = (21/8)AT_{\nu0}^4$ is the contribution of the neutrino background (we assume neutrinos are massless). Here $A = \pi^2/15$ and $T_{\nu0} = (4/11)^{1/3}T_0$.

   a) What was the age of the universe $t_{\rm eq}$ when $\rho_m = \rho_r$? (Note that at these early times –but not today– you can ignore the curvature and vacuum terms in the Friedmann equation; you don't need to make other assumptions about the values of Ω_0 or $\Omega_{\Lambda0}$, since the answer does not depend on them.) Give the numerical value (in years) for the cases $\Omega_{m0} = 0.1$, 0.3 and 1.0, assuming $H_0 = 70$ km/s/Mpc.

   b) What is the temperature $T_{\rm eq} \equiv T(t_{\rm eq})$? Give the numerical value (in K) in the three different cases.

2. **Thermal distributions in the relativistic limit.** Derive the following equations in the limit $T \gg m, T \gg |\mu|$.

$$n = \begin{cases} \frac{3}{4\pi^2}\zeta(3)gT^3 & \text{fermions} \\ \frac{1}{\pi^2}\zeta(3)gT^3 & \text{bosons} \end{cases}$$

$$\rho = \begin{cases} \frac{7}{8}\frac{\pi^2}{30}gT^4 & \text{fermions} \\ \frac{\pi^2}{30}gT^4 & \text{bosons} \end{cases}$$

$$p = \frac{1}{3}\rho$$

$$\langle E\rangle = \begin{cases} \frac{7\pi^4}{180\zeta(3)}T & \text{fermions} \\ \frac{\pi^4}{30\zeta(3)}T & \text{bosons}\ . \end{cases}$$

3. **Thermal distributions in the non-relativistic limit.** Derive the following equations for non-relativistic Maxwell-Boltzmann statistics ($T \ll m$ and $T \ll |m - \mu|$).

$$\begin{aligned} n &= g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m-\mu}{T}} \\ \rho &= n\left(m + \frac{3T}{2}\right) \\ p &= nT \\ \langle E\rangle &= m + \frac{3T}{2} \\ n - \bar{n} &= 2g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m}{T}} \sinh\frac{\mu}{T}\ . \end{aligned}$$

4. **Transparency of the universe.** We say that the universe is transparent when the photon mean free path λ_γ is larger than the Hubble length $l_H = H^{-1}$, and opaque when $\lambda_\gamma < l_H$. The photon mean free path is determined mainly by the scattering of photons by free electrons, so that $\lambda_\gamma = 1/(\sigma_T n_e)$, where $n_e = xn_e^*$ is the number density of free electrons, n_e^* is the total number density of electrons, and x is the ionization fraction. The cross section for photon-electron scattering is independent of energy for $E_\gamma \ll m_e$ and is then called the Thomson cross

section, $\sigma_T = \frac{8\pi}{3}(\alpha/m_e)^2$, where α is the fine-structure constant. In recombination x falls from 1 to 10^{-4}. Assume instant recombination at $1 + z = 1300$.

a) Show that the universe is opaque before recombination and transparent after recombination. (You can assume a matter-dominated universe; see below for parameter values.)

b) Interstellar baryonic matter gets later reionized (to $x \sim 1$) by light from the first stars. What is the earliest redshift when this can happen without making the universe opaque again? (You can assume that most ($\sim$ all) matter has remained interstellar.)

Calculate for $\Omega_{m0} = 1.0$ and $\Omega_{m0} = 0.3$ (note that Ω_m includes nonbaryonic matter). Use $\Omega_\Lambda = 0$, $h = 0.7$ and $\eta = 6 \times 10^{-10}$.