**COSMOLOGY I** Homework 2 Fall 2024

Due on Monday September 16 by 14.00.

From now on, the exercise sessions are on Thursday and Friday at 12.15-14.00. This week we also start the assistants' help session, on Tuesday at 12.15-13.00, this week in Jenni's office in Physicum D315.

1. **Practice with natural units.**

   (a) The Planck mass is defined as $M_{\rm Pl} \equiv \frac{1}{\sqrt{8\pi G_{\rm N}}}$, where $G_{\rm N}$ is Newton's gravitational constant. Give the Planck mass in units of kg, J, eV, K, $\rm m^{-1}$, and $\rm s^{-1}$.

   (b) The energy density of the cosmic microwave background is $\rho_\gamma = \frac{\pi^2}{15}T^4$ and its photon density is $n_\gamma = \frac{2}{\pi^2}\zeta(3)T^3$, where $\zeta$ is Riemann's zeta function ($\zeta(3) \approx 1.20$). What is the energy density in units of kg/m$^3$ and the photon density in units of m$^{-3}$, i) today, when $T = 2.725$ K, ii) when the temperature was $T = 1$ MeV? What was the average photon energy in eV, and what was the wavelength and frequency of such an average photon?

   (c) Suppose the mass of an average galaxy is $m_G = 10^{11}M_\odot$ and the galaxy density in the universe is $n_G = 3\times10^{-3}$ Mpc$^{-3}$. What is the galactic contribution to the average mass density of the universe, in kg/m$^3$?

   (d) The critical density of the universe today is $\rho_{\rm c0} \equiv \frac{3H_0^2}{8\pi G_{\rm N}}$, where $H_0$ is the Hubble constant; let us adopt the value 70 km/s/Mpc. What is the critical density in units of kg/m$^3$, GeV$^4$, GeV/m$^3$, and $M_\odot$/Mpc$^3$? What fraction of the critical density is contributed by the microwave background (today), by starlight (see problem 1.2), and by galaxies?

2. **Redshift in Newtonian cosmology.** Continuing from problem 1.3, consider the case of critical density ($K = 0$). Denote the distance of galaxy $G$ from the origin by $r_G$.

   (a) Solve for $r_G(t)$.

   (b) An observer at the origin sees the light from the galaxies redshifted due to the Doppler effect. For electromagnetic radiation, when the source is moving away from the observer at speed $v$, we have $\lambda_{\rm obs}/\lambda_{\rm em} = \sqrt{\frac{1+v}{1-v}}$. Show that at short distances we obtain the Hubble law: $z = Hr_G$. What approximations do you have to make?

3. **Curved space.**

   (a) Consider the spatial part of the Robertson–Walker metric (at time $t$ when $a(t) = a$) in the two cases $K > 0$ and $K < 0$. What is the volume of the spherical region whose proper distance from the origin is between $s$ and $s + {\rm d}s$? (It is convenient to use the $\chi$ coordinate here.) What is the deviation from the Euclidean ($K = 0$) result when $s = 0.1R_{\rm curv}$, $s = R_{\rm curv}$, $s = 3R_{\rm curv}$, and $s = 10R_{\rm curv}$? (Recall that the curvature radius is $R_{\rm curv} \equiv a/\sqrt{|K|}$.)

   (b) Show that the volume of a hypersphere with curvature radius $R_{\rm curv}$ is $2\pi^2R_{\rm curv}^3$.