COSMOLOGY I Homework 1 Fall 2024

Due on Monday September 9 by 14.00. The exercises are returned on the course Moodle page. You will get help for the exercises in the exercise sessions on Friday at 12-14 and 14-16. The help sheet at the bottom of the course webpage may come in handy. It will also be available in the course exam (it may be updated during the course, but material will not be removed).

1. **Nuclear cosmochronometers.** The uranium isotopes 235 and 238 have half-lives $t_{1/2}(235) = 0.704 \times 10^9$ a and $t_{1/2}(238) = 4.47 \times 10^9$ a. The ratio of their present abundance on Earth is ${}^{235}U/{}^{238}U = 0.00725$.

   (a) When were they equal in abundance?

   (b) The heavy elements were produced in supernova explosions. If the element production ratio is ${}^{235}U/{}^{238}U = 1.3 \pm 0.2$, what does this tell us about the age of the Earth and the age of the Universe?

2. **Olbers' paradox.**

   (a) Let's assume that the universe is infinite, eternal and unchanging (and has Euclidean geometry). For simplicity, let's also assume that all stars are the same size as the sun, and are distributed evenly in space. Show that the line of sight meets the surface of a star in every direction, sooner or later.

   (b) Let's put in some numbers: The luminosity density of the universe is of the order $10^8 L_\odot/\text{Mpc}^3$. With the above assumption we have then a number density of stars $n_* = 10^8\ \text{Mpc}^{-3}$. The radius of the sun is $r_\odot = 7 \times 10^8$ m. Define $r_{1/2}$ so that stars closer than $r_{1/2}$ cover 50% of the sky. Calculate $r_{1/2}$.

   (c) Let's assume instead that stars have a finite age: they all appeared $t_\odot = 4.6 \times 10^9$ a ago. What fraction f of the sky do they now cover? What is the energy density of starlight in the universe, in kg/m^3? (The luminosity, or radiated power, of the sun is $L_\odot = 3.8 \times 10^{26}$ W.)

   (d) Calculate $r_{1/2}$ and f for galaxies, using $n_G = 3 \times 10^{-3}\ \text{Mpc}^{-3}$, $r_G = 10$ kpc, and $t_G = 10^{10}$ a.

3. **Newtonian cosmology.** Let us use Euclidean geometry and Newtonian gravity, where the expansion of the universe is interpreted as the motion of galaxies instead of the expansion of space. Consider a spherical group of galaxies in otherwise empty space, and treat the galaxies as a homogeneous cloud of particles. Let the mass density of the cloud be $\rho(t)$. Assume that each galaxy moves according to Hubble's law $\dot{\mathbf{r}}(t, \mathbf{r}) = H(t)\mathbf{r}$. The expansion of the cloud slows down due to its gravity.

   (a) Choose some reference time $t = t_0$ and define $a(t) \equiv r(t)/r(t_0)$. Show that $a(t)$ is the same function for every galaxy, regardless of the value of $r(t_0)$. Note that $\rho(t) = \rho(t_0)a(t)^{-3}$.

   (b) Rewrite your differential equation for $H(t)$ as a differential equation for $a(t)$.

   (c) Denote the total energy (kinetic + potential) of a galaxy per unit mass by κ. Show that $K \equiv -2\kappa/r(t_0)^2$ has the same value for each galaxy, regardless of the value of $r(t_0)$.

   (d) Relate $H(t)$ to $\rho(t_0)$, K, and $a(t)$. Whether the expansion continues forever or stops and turns into collapse depends on how large H is in relation to ρ. Find out the critical value of H (corresponding to the escape velocity of the galaxies) separating these two possibilities. Turn the relation around to give the *critical density* that corresponds to the critical value of H. What is this critical density (in kg/m^3) for $H_0 = 70$ km/s/Mpc?