Due on December 9 by 14.00. These are the last exercises.

1. **Gaussian random variables.** Show that if α is a real Gaussian random variable with $\langle\alpha\rangle = 0$, then
$$\langle|\alpha|^4\rangle = 3\langle|\alpha|^2\rangle^2 \ ,$$
and that, if a is a complex Gaussian random variable, with $\langle a\rangle = 0$, then
$$\langle|a|^4\rangle = 2\langle|a|^2\rangle^2 \ .$$

2. **Cosmic variance.** Assume that the CMB multipole coefficients a_{lm} are independent Gaussian random variables with zero mean (constrained by $a_{l-m} = a^*_{lm}$). Calculate the cosmic variance $\langle(\hat{C}_l - C_l)^2\rangle$, where $\hat{C}_l$ is the observed angular power spectrum. (The relations given in the previous problem may be helpful.)

3. **Positions of the acoustic peaks.**

   a) Calculate the sound horizon at decoupling,
$$r_s = (1+z)^{-1}\int_0^{t_{\rm dec}} \mathrm{d}t\frac{c_s(t)}{a(t)} \ ,$$
   in terms of $z_{\rm dec}$, ω_m, and ω_b. Assume constant speed of sound, $c_s = c_s(t_{\rm dec})$, but include the effect of radiation and matter components in the expansion law. Neglect neutrino masses.

   b) What is the separation ℓ_A between the acoustic peaks in the CMB angular power spectrum C_ℓ for the cases $\Omega_\Lambda = 0$ and $\Omega_\Lambda = 1 - \Omega_m$?

   c) Give the numerical values of r_s and ℓ_A for $z_{\rm dec} = 1090$, $h = 0.7$, $\Omega_{m0} = 0.3$ and $\omega_b = 0.02$. Give also the numerical value of the comoving sound horizon.

4. **The effect of varying sound speed.** Same as the previous problem, but now take into account the evolution of the sound speed.