COSMOLOGY II Homework 2 Fall 2024

Due on November 11 by 14.00. This week's exercise session is exceptionally on Friday at 14-16.

1. **Small-field inflation.** Suppose that the inflaton potential is

$$V(\varphi) \approx V_0 \left[1 - \lambda \left(\frac{\varphi}{M_{\rm Pl}} \right)^4 + \ldots \right] .$$

The omitted terms (which are responsible for keeping $V(\varphi)$ nonnegative) are assumed to be negligible in the region of interest (the slow-roll section). Assume further that

$$\lambda \left(\frac{\varphi}{M_{\rm Pl}} \right)^4 \ll 1$$

in the slow-roll section, so that we can approximate $V(\varphi) \approx V_0$, except when calculating derivatives.

a) Find $\varphi_{\rm end}$ from the condition $\varepsilon = 1$.
b) Find $N(\varphi)$. (When φ is sufficiently deep in the slow-roll section, so that $|\eta| \ll 1$, you can make an approximation where the term that depends on $\varphi_{\rm end}$ is dropped from $N(\varphi)$. Justify this.)
c) Find φ, $\eta(\varphi)$, and $\varepsilon(\varphi)$ in terms of N and give the values for $N(\varphi) = 50$.
d) Find n_s and r, for $N = 50$. How do these values compare to the observational limits?
e) Find how much the value of φ changes during the last 50 e-folds of inflation.
f) Show from the above assumptions that the condition $\varepsilon \ll |\eta|$ holds in the slow-roll region.

2. **Inflaton perturbations.** Consider equation (8.6) for the inflaton perturbation. Approximate that $H = \text{constant}$ ($\Rightarrow a = e^{Ht}$) and $V''(\bar{\varphi}) = 0$.

a) Show that

$$\delta\varphi_{\boldsymbol{k}}(t) = C(\boldsymbol{k}) \eta^{\frac{3}{2}} H_{\frac{3}{2}}(k\eta) ,$$

where

$$H_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}} e^{-ix} \left(1 + \frac{1}{ix} \right)$$

is a Hankel function, and $\eta \equiv -H^{-1} e^{-Ht}$, is a solution.

b) Show that η is the conformal time, and that the solution agrees with the solution (8.27) given in the lecture notes.

c) What happens to the perturbation (its amplitude, distance scale and oscillation time scale) as inflation proceeds?

3. **Inflaton perturbations as vacuum fluctuations.**

a) Starting from the canonical commutation relation for the quantised perturbation, show that with the correct the normalisation factor, $\delta\varphi_{\boldsymbol{k}}(t)$ of the previous problem is

$$\delta\varphi_{\boldsymbol{k}}(t) = L^{-3/2} \frac{H}{\sqrt{2k^3}} \left(i + \frac{k}{aH} \right) \exp\left(\frac{ik}{aH} \right) .$$

b) Show that on time and length scales much smaller than the Hubble scale, $t \ll H^{-1}$, $k \gg aH$, this becomes (up to a slowly varying phase factor) the Minkowski space mode function

$$w_k(t) = (aL)^{-3/2} \frac{1}{\sqrt{2E_k}} e^{-iE_k t} .$$

c) What is E_k?

4. **Energy scale of inflation and expansion.** Given the limit on the energy scale of inflation given after eq. (8.59), find the maximum amount by which the scale factor can have expanded from reheating until today, assuming there are only Standard Model degrees of freedom.